GENERALIZED TOTAL COLORINGS OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let \( P \) and \( Q \) be additive hereditary properties of graphs. A \((P, Q)\)-total coloring

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of a simple graph $G$ is a coloring of the vertices $V(G)$ and edges $E(G)$ of $G$ such that for each color $i$ the vertices colored by $i$ induce a subgraph of property $\mathcal{P}$, the edges colored by $i$ induce a subgraph of property $\mathcal{Q}$ and incident vertices and edges obtain different colors. In this paper we present some general basic results on $(\mathcal{P}, \mathcal{Q})$-total colorings. We determine the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of paths and cycles and, for specific properties, of complete graphs. Moreover, we prove a compactness theorem for $(\mathcal{P}, \mathcal{Q})$-total colorings.

Keywords: hereditary properties, generalized total colorings, paths, cycles, complete graphs.

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1. Introduction

We denote the class of all finite simple graphs by $\mathcal{I}$ (see [1]). A graph property $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$.

We use the following standard notations for specific hereditary properties:

- $\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
- $\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\}$,
- $\mathcal{D}_k = \{G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k\}$,
- $\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}$,
- $\mathcal{J}_k = \{G \in \mathcal{I} : \chi'(G) \leq k\}$,
- $\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices}\}$,
- $\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,

where $\chi(G)$ is the chromatic number, $\chi'(G)$ the chromatic index and $\Delta(G)$ the maximum degree of the graph $G = (V,E)$.

A total coloring of a graph $G$ is a coloring of the vertices and edges (together called the elements of $G$) such that all pairs of adjacent or incident elements obtain distinct colors. The minimum number of colors of a total coloring of $G$ is called the total chromatic number $\chi''(G)$ of $G$.

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties. Then a $(\mathcal{P}, \mathcal{Q})$-total coloring of a graph $G$ is a coloring of the vertices and edges of $G$ such that for any color $i$ all vertices of color $i$ induce a subgraph of property $\mathcal{P}$, all edges of color $i$ induce a subgraph of property $\mathcal{Q}$ and
vertices and incident edges are colored differently. The minimum number of colors of a \((P, Q)\)-total coloring of \(G\) is called the \((P, Q)\)-total chromatic number \(\chi''_{P, Q}(G)\) of \(G\).

If \(G\) contains edges then \(\chi''_{P, Q}(G)\) is only defined if \(K_2 \in Q\) and therefore \(\emptyset_1 \subseteq Q\). Since \(\emptyset \subseteq P\) for all additive hereditary properties we obtain \(\chi''_{P, Q}(G) \leq |V| + |E|\) which guarantees the existence of \((P, Q)\)-total chromatic numbers.

\((P, Q)\)-total colorings are \textit{generalized total colorings} since \(\chi''_{\emptyset_1, \emptyset_1}(G) = \chi''(G)\) for all graphs \(G\).

\textit{Generalized} \(P\)-\textit{vertex colorings} and \(P\)-\textit{chromatic numbers} \(\chi_P(G)\) as well as \textit{generalized} \(Q\)-\textit{edge colorings} and \(Q\)-\textit{chromatic indices} \(\chi'_Q(G)\) are analogously defined (see [3, 9] for some results). Evidently, these are generalizations of proper vertex colorings and proper edge colorings since \(\chi_G(G) = \chi(G)\) and \(\chi'_{\emptyset_1}(G) = \chi'(G)\).

The \(P\)-chromatic number and the \(Q\)-chromatic index of \(G\) provide general lower and upper bounds for \(\chi''_{P, Q}(G)\).

\begin{theorem}
\begin{enumerate}
\item[(a)] \(\max\{\chi_P(G), \chi'_Q(G)\} \leq \chi''_{P, Q}(G) \leq \chi_P(G) + \chi'_Q(G)\),
\item[(b)] \(\chi_P(G) \leq \chi''_{P, Q}(G) \leq \chi_P(G) + 1\) if \(G \in Q\),
\item[(c)] \(\chi'_Q(G) \leq \chi''_{P, Q}(G) \leq \chi'_Q(G) + 1\) if \(G \in P\),
\item[(d)] \(\chi''_{P, Q}(G) = 1\) iff \(G \in \emptyset\),
\item[(e)] \(\chi''_{P, Q}(G) = 2\) iff \(G \in (P \cap Q) \setminus \emptyset\),
\item[(f)] \(\chi''_{P, Q}(G) \geq 3\) iff \(G \in I \setminus (P \cap Q)\).
\end{enumerate}
\end{theorem}

\textbf{Proof.} Since a \((P, Q)\)-total coloring induces a \(P\)-\textit{vertex coloring} and a \(Q\)-\textit{edge coloring} it follows that \(\chi_P(G) \leq \chi''_{P, Q}(G)\) and \(\chi'_Q(G) \leq \chi''_{P, Q}(G)\). A \(P\)-\textit{vertex coloring} of \(G\) with \(\chi_P(G)\) colors and a \(Q\)-\textit{edge coloring} with \(\chi'_Q(G)\) additional colors induce a \((P, Q)\)-total coloring of \(G\) with \(\chi_P(G) + \chi'_Q(G)\) colors.

If \(G \in Q\) or \(G \in P\), respectively, then all edges or all vertices can obtain the same additional color which implies \(\chi''_{P, Q}(G) \leq \chi_P(G) + 1\) or \(\chi''_{P, Q}(G) \leq \chi'_Q(G) + 1\), respectively.

If \(G\) has no edges then \(G \in \emptyset \subseteq P\) and therefore all vertices can obtain the same color which implies \(\chi''_{P, Q}(G) = 1\). If \(G\) has edges then \(G \notin \emptyset\) and therefore at least two colors are needed to color a vertex and an incident edge which implies \(\chi''_{P, Q}(G) \geq 2\).
It holds $\chi''_{\mathcal{P}, \mathcal{Q}}(G) = 2$ if and only if $G$ contains edges and for each non-trivial component of $G$ all vertices as well as all edges can be colored with one color each, that is, if and only if $G \in (\mathcal{P} \cap \mathcal{Q}) \setminus \mathcal{O}$.

Obviously, if $G \notin \mathcal{P} \cap \mathcal{Q}$ then $\chi''_{\mathcal{P}, \mathcal{Q}}(G) \geq 3$.

The following monotonicity and additivity results are obvious.

**Lemma 1.** If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, then $\chi''_{\mathcal{P}_2, \mathcal{Q}_2}(G) \leq \chi''_{\mathcal{P}_1, \mathcal{Q}_1}(G)$.

**Proof.** If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ then each $(\mathcal{P}_1, \mathcal{Q}_1)$-total coloring is a $(\mathcal{P}_2, \mathcal{Q}_2)$-total coloring. □

It follows $\chi''_{\mathcal{P}, \mathcal{Q}}(G) \leq \chi''_{\mathcal{Q}, \mathcal{O}_1}(G) = \chi''(G)$ since $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O}_1 \subseteq \mathcal{Q}$, that is, the total chromatic number is an upper bound for the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of a graph $G$.

**Lemma 2.** If $H \subseteq G$, then $\chi''_{\mathcal{P}, \mathcal{Q}}(H) \leq \chi''_{\mathcal{P}, \mathcal{Q}}(G)$.

**Proof.** The restriction of a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ to the elements of $H$ is a $(\mathcal{P}, \mathcal{Q})$-total coloring. □

The following lemma implies that one can restrict oneself to connected graphs, in general.

**Lemma 3.** If $G$ and $H$ are disjoint, then $\chi''_{\mathcal{P}, \mathcal{Q}}(G \cup H) = \max\{\chi''_{\mathcal{P}, \mathcal{Q}}(G), \chi''_{\mathcal{P}, \mathcal{Q}}(H)\}$.

**Proof.** $(\mathcal{P}, \mathcal{Q})$-total colorings of $G$ and of $H$ provide a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G \cup H$ since $G$ and $H$ are disjoint which implies $\chi''_{\mathcal{P}, \mathcal{Q}}(G \cup H) \leq \max\{\chi''_{\mathcal{P}, \mathcal{Q}}(G), \chi''_{\mathcal{P}, \mathcal{Q}}(H)\}$. Lemma 2 implies equality. □

If one of the properties is the class $\mathcal{I}$ of all finite simple graphs then the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of $G$ attains one of two possible values by Theorem 1:

1. $\chi'_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P}, \mathcal{I}}(G) \leq \chi'_{\mathcal{P}}(G) + 1$, $\chi'_{\mathcal{Q}}(G) \leq \chi''_{\mathcal{I}, \mathcal{Q}}(G) \leq \chi'_{\mathcal{Q}}(G) + 1$.

If $\mathcal{P} = \mathcal{Q} = \mathcal{I}$ then $\chi''_{\mathcal{I}, \mathcal{I}}(G) = 1$ if $G \notin \mathcal{O}$ and $\chi''_{\mathcal{I}, \mathcal{I}}(G) = 2$ otherwise by Theorem 1.

If $G \in \mathcal{Q}$ then $\chi''_{\mathcal{P}, \mathcal{Q}}(G)$ and therefore $\chi''_{\mathcal{P}, \mathcal{I}}(G)$ for all graphs $G$ can be determined as follows.
Theorem 2. If \( G \in \mathcal{Q} \), then

\[
\chi''_{\mathcal{P}, \mathcal{Q}}(G) = \begin{cases} 
\chi_{\mathcal{P}}(G) & \text{if } G \in \mathcal{O} \text{ or } \chi_{\mathcal{P}}(G) \geq 3, \\
\chi_{\mathcal{P}}(G) + 1 & \text{if } G \in \mathcal{P} \setminus \mathcal{O} \text{ or } \chi_{\mathcal{P}}(G) = 2.
\end{cases}
\]

Proof. By Theorem 1, \( \chi_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P}, \mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1 \).

If \( \chi_{\mathcal{P}}(G) = 1 \) then \( G \in \mathcal{P} \) which implies \( \chi''_{\mathcal{P}, \mathcal{Q}}(G) = 1 \) for \( G \in \mathcal{O} \) and \( \chi''_{\mathcal{P}, \mathcal{Q}}(G) = 2 \) for \( G \in \mathcal{P} \setminus \mathcal{O} \) by Theorem 1.

If \( \chi_{\mathcal{P}}(G) = 2 \) then \( G \notin \mathcal{P} \) and therefore \( \chi''_{\mathcal{P}, \mathcal{Q}}(G) \geq 3 \) by Theorem 1. On the other hand, \( \chi''_{\mathcal{P}, \mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1 = 3 \).

If \( \chi_{\mathcal{P}}(G) \geq 3 \) then \( \chi''_{\mathcal{P}, \mathcal{Q}}(G) \geq \chi_{\mathcal{P}}(G) \). Consider a \( \mathcal{P} \)-vertex coloring of \( G \) with \( \chi_{\mathcal{P}}(G) \) colors. Each edge can be colored with a color different to the colors of its end-vertices. This is a \( (\mathcal{P}, \mathcal{Q}) \)-total coloring of \( G \) with \( \chi_{\mathcal{P}}(G) \) colors since \( H \in \mathcal{Q} \) for all \( H \subseteq G \).

2. \( \mathcal{P} = \mathcal{O} \) or \( \mathcal{Q} = \mathcal{O}_1 \)

Since \( \mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I} \) and \( \mathcal{O}_1 \subseteq \mathcal{Q} \subseteq \mathcal{I} \), Lemma 1 provides the following bounds:

(2) \( \chi''_{\mathcal{I}, \mathcal{I}}(G) \leq \chi''_{\mathcal{P}, \mathcal{P}}(G) \leq \chi''_{\mathcal{P}, \mathcal{Q}}(G) \leq \chi''_{\mathcal{O}, \mathcal{O}_1}(G) = \chi''(G) \),

(3) \( \chi''_{\mathcal{I}, \mathcal{I}}(G) \leq \chi''_{\mathcal{P}, \mathcal{Q}}(G) \leq \chi''_{\mathcal{O}, \mathcal{O}_1}(G) = \chi''(G) \),

(4) \( \chi''_{\mathcal{P}, \mathcal{P}}(G) \leq \chi''_{\mathcal{O}, \mathcal{O}_1}(G) \leq \chi''_{\mathcal{O}, \mathcal{O}_1}(G) = \chi''(G) \),

(5) \( \chi''_{\mathcal{I}, \mathcal{I}}(G) \leq \chi''_{\mathcal{O}, \mathcal{O}_1}(G) \leq \chi''_{\mathcal{P}, \mathcal{P}}(G) \).

\((\mathcal{O}, \mathcal{I})\)- and \((\mathcal{I}, \mathcal{O}_1)\)-total coloring are certain \([r, s, t]\)-colorings which also are generalizations of ordinary colorings.

Given non-negative integers \( r, s, \) and \( t \) with \( \max\{r, s, t\} \geq 1 \), an \([r, s, t]\)-coloring of a finite and simple graph \( G \) with vertex set \( V(G) \) and edge set \( E(G) \) is a mapping \( c \) from \( V(G) \cup E(G) \) to the color set \( \{0, 1, \ldots, k - 1\} \), \( k \in \mathbb{N} \), such that \( |c(v_i) - c(v_j)| \geq r \) for every two adjacent vertices \( v_i, v_j \), \( |c(e_i) - c(e_j)| \geq s \) for every two adjacent edges \( e_i, e_j \), and \( |c(v_i) - c(e_j)| \geq t \) for all pairs of incident vertices and edges, respectively. The \([r, s, t]\)-chromatic number \( \chi_{r,s,t}(G) \) of \( G \) is defined to be the minimum \( k \) such that \( G \) admits an \([r, s, t]\)-coloring (see [10, 11]).

By this definition we obtain \( \chi''_{\mathcal{I}, \mathcal{I}}(G) = \chi_{0,0,1}(G) \), \( \chi''_{\mathcal{O}, \mathcal{I}}(G) = \chi_{1,0,1}(G) \), \( \chi''_{\mathcal{P}, \mathcal{O}_1}(G) = \chi_{0,1,1}(G) \) and \( \chi''_{\mathcal{O}, \mathcal{O}_1}(G) = \chi_{1,1,1}(G) \). The first three of these \([r, s, t]\)-chromatic numbers were determined in [10].
Theorem 3.

(a) \( \chi''_{\mathcal{O},\mathcal{T}}(G) = \chi_{1,0,1}(G) = \begin{cases} \chi(G) & \text{if } \chi(G) \neq 2, \\ 3 = \chi(G) + 1 & \text{if } \chi(G) = 2, \end{cases} \)

(b) \( \chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi_{0,1,1}(G) = \Delta(G) + 1. \)

**Proof.** (a) By Theorem 2 we obtain for \( \mathcal{P} = \mathcal{O} \) that \( \chi''_{\mathcal{O},\mathcal{T}}(G) = \chi_\mathcal{O}(G) = \chi(G) \) if \( G \in \mathcal{O} \) or \( \chi(G) \geq 3 \) and \( \chi''_{\mathcal{O},\mathcal{T}}(G) = \chi(G) + 1 \) if \( \chi(G) = 2. \)

(b) If \( \chi'(G) = \Delta(G) \) then \( \chi''_{\mathcal{O},\mathcal{O}_1}(G) \geq \Delta(G) + 1 \) since an additional color is necessary to color a vertex of maximum degree. If \( \chi'(G) = \Delta(G) + 1 \) then \( \chi''_{\mathcal{O},\mathcal{O}_1}(G) \geq \chi'(G) = \Delta(G) + 1 \) by Theorem 1.

On the other hand, we have \( \chi''_{\mathcal{O},\mathcal{O}_1}(G) \geq \Delta(G) + 1 \) since the edges can be colored with at most \( \Delta(G) + 1 \) colors by Vizing’s Theorem and at each vertex there is a missing edge color which can be used to color this vertex.

To illustrate the results we consider as examples paths \( P_n \), cycles \( C_n \) and complete graphs \( K_n \).

**Examples 1.**

1. Theorem 3 implies \( \chi''_{\mathcal{O},\mathcal{T}}(P_1) = \chi''_{\mathcal{O},\mathcal{O}_1}(P_1) = 1, \chi''_{\mathcal{O},\mathcal{T}}(P_2) = 2 \) and \( \chi''_{\mathcal{O},\mathcal{T}}(P_n) = \chi''_{\mathcal{O},\mathcal{O}_1}(P_n) = 3 \) for \( n \geq 3. \)

2. We have \( \chi_\mathcal{O}(C_n) = \chi(C_n) = \chi_{\mathcal{O}_1}(C_n) = \chi'(C_n) \) and \( \chi(C_n) = 2 \) if \( n \) is even and \( \chi(C_n) = 3 \) if \( n \) is odd. Moreover, we have \( \chi''_{\mathcal{O},\mathcal{T}}(C_n) = \chi''_{\mathcal{O},\mathcal{O}_1}(C_n) = 3 \) by Theorem 3. Therefore, the lower and upper bounds of (1) are attained for cycles \( C_n \).

3. Theorem 3 implies \( \chi''_{\mathcal{O},\mathcal{O}_1}(K_n) = n \) and \( \chi''_{\mathcal{O},\mathcal{T}}(K_n) = \begin{cases} n & \text{if } n \neq 2, \\ n + 1 & \text{if } n = 2. \end{cases} \)

If \( n \) is odd then \( n = \chi''_{\mathcal{O},\mathcal{O}_1}(K_n) = \chi''_{\mathcal{O},\mathcal{T}}(K_n) \).

In Theorems 4 and 5 we also consider complete graphs of even order.

**Theorem 4.** \( \chi''_{\mathcal{O},\mathcal{Q}}(K_n) = \begin{cases} n & \text{if } n \text{ odd or } (n \geq 4 \text{ even and } \mathcal{O}_1 \subset \mathcal{Q}), \\ n + 1 & \text{if } n = 2 \text{ or } (n \text{ even and } \mathcal{Q} = \mathcal{O}_1). \end{cases} \)

**Proof.** The case that \( n \) is odd is considered in the above example and the case \( n = 2 \) is obvious.
If $n$ is even and $Q = O_1$ then $\chi''_{O_1, Q}(K_n) = \chi''(K_n) = n + 1$.

If $n \geq 4$ is even and $O_1 \neq Q$ then $P_3 \in Q$. We partition the elements of $K_n$ with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in $n$ color classes as follows:

Class $F_i$, $0 \leq i \leq n-1$, contains the vertex $v_i$, the edges $v_{i-1}v_{i+1}, v_{i-2}v_{i+2}, \ldots, v_{i-y+1}v_{i+y-1}$ as well as the edges $v_{i+n/2}v_{i+n/2+1}, v_{i+n/2-1}v_{i+n/2+2}, \ldots, v_{i+y+1}v_{i-y}$ where $y = \lceil n/4 \rceil$ and the indices are reduced modulo $n$ (see Figure 1).

![Figure 1. Color class $F_i$ of $K_n$ for $n = 8$ and $n = 10$.](image)

In each of the color classes $F_i$ the vertex $v_{i+y}$ is unmatched. Therefore, we can add the edge $v_{i+y}v_{i-[n/4]}$ in each $F_i$, $0 \leq i \leq n/2 - 1$ (represented as a dashed line in Figure 1).

Each vertex and each edge of $K_n$ is contained in exactly one of these color classes. The induced subgraphs of this partition consist of $K_1$, $K_2$, and $P_3$. Therefore, this is an $(O, Q)$-total coloring of the complete graph $K_n$ with $n$ colors.

**Theorem 5.** $\chi''_{P, O_1}(K_n) = \begin{cases} n & \text{if } P \neq O \text{ or } n \text{ odd,} \\ n + 1 & \text{if } P = O \text{ and } n \text{ even.} \end{cases}$

**Proof.** The case that $n$ is odd is treated in the above example, the case $P = O$ and $n$ even in Theorem 4.

If $n$ is even and $P \neq O$ then $K_2 \in P$. First note that $\chi''_{P, O_1}(K_n) \geq \chi''_{I, O_1}(K_n) = n$ by Lemma 1 and Theorem 3.

In the following we provide a $(P, O_1)$-total coloring of $K_n$ with $n$ colors which implies $\chi''_{P, O_1}(K_n) = n$.

For $n = 2$ and $n = 4$ see Figure 2.
If $n \geq 6$ then there exists an edge coloring of $K_n$ with $n - 1$ colors such that there are $n/2$ independent edges with pairwise distinct colors. This can be seen as follows. Consider a drawing of $K_n - v \cong K_{n-1}$ with vertex set $\{v_0, \ldots, v_{n-2}\}$ as a regular $(n - 1)$-gon. Color parallel edges of $K_{n-1}$ with one color and the edges $vv_i$, $0 \leq i \leq n - 2$, with the missing color at $v_i$. If $n \equiv 2 \pmod{4}$ then the edges $v_0v_1, v_2v_3, \ldots, v_{n-2}v_{n-1}$ are independent with mutually distinct colors. If $n \equiv 0 \pmod{4}$ then the edges $v_0v_1, v_2v_4, v_3v_6, v_5v$ and if $n \geq 12$ also $v_7v_8, v_9v_{10}, \ldots, v_{n-3}v_{n-2}$ are independent with pairwise distinct colors.

Assign the color of each of these edges to its end-vertices and then replace the colors of all these edges by the $n$th color (see Figure 3 for an example).

The corresponding results concerning $(O, Q)$- and $(P, O_1)$-total colorings of paths and cycles are special cases of the following theorems.

**Theorem 6.** $\chi''_{P,O}(P_n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } P_n \in (P \cap Q) \setminus O, \\ 3 & \text{otherwise.} \end{cases}$
**Proof.** The result follows from Theorem 1 and from $\chi''\left(P_n\right) \leq \chi''(P_n) \leq 3$ (see Lemma 1).

**Theorem 7.** $\chi''_{\mathcal{P}, \mathcal{Q}}(C_n) = \begin{cases} 
2 & \text{if } C_n \in \mathcal{P} \cap \mathcal{Q}, \\
4 & \text{if } (\mathcal{P} = \mathcal{O}, \mathcal{Q} = \mathcal{O}_1, n \not\equiv 0 \pmod{3}) \text{ or } n = 5, \\
3 & \text{if } \mathcal{P} = \mathcal{O}, P_4 \notin \mathcal{Q} \text{ or } (n = 5, \mathcal{P} = \mathcal{Q} = \mathcal{O}_1), \\
\end{cases}$

**Proof.** If $C_n \in \mathcal{P} \cap \mathcal{Q}$ then $\chi''_{\mathcal{P}, \mathcal{Q}}(C_n) = 2$ by Theorem 1 and if $C_n \notin \mathcal{P} \cap \mathcal{Q}$ then $3 \leq \chi''_{\mathcal{P}, \mathcal{Q}}(C_n) \leq 4$ by Theorem 1, Lemma 1, and the fact that $\chi''(C_n) \leq 4$.

If $n \equiv 0 \pmod{3}$ then $\chi''(C_n) = 3$ and therefore $\chi''_{\mathcal{P}, \mathcal{Q}}(C_n) = 3$.

Let $n \not\equiv 0 \pmod{3}$. If $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$ then $\chi''_{\mathcal{O}, \mathcal{O}_1}(C_n) = 4$. If $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$ then color the successive vertices $v_0, v_1, \ldots, v_{n-1}$ of $C_n$ by colors $1, 2, 3, 1, 2, 3, \ldots$ if $n \equiv 1 \pmod{3}$ and by colors $1, 2, 3, 1, 2, 3, \ldots$ if $n \equiv 2 \pmod{3}$, $n \geq 8$, and the edges with the at their end-vertices missing color of $\{1, 2, 3\}$. This is an $(\mathcal{O}, \mathcal{Q})$-total coloring of $C_n$ since $P_3 \in \mathcal{Q}$. If $n = 5$ then color the vertices with colors $1, 2, 1, 2, 3$ (unique up to permutation) and the edges again with the at their end-vertices missing color of the set $\{1, 2, 3\}$. This is an $(\mathcal{O}, \mathcal{Q})$-total coloring of $C_5$ if $P_3 \in \mathcal{Q}$. If $P_3 \notin \mathcal{Q}$ then $\chi''_{\mathcal{O}, \mathcal{Q}}(C_5) = 4$.

By switching the colors of vertices and edges one obtains $\chi''_{\mathcal{P}, \mathcal{O}_1}(C_n) = 3$ if $\mathcal{P} \supset \mathcal{O}$ with the exception of $\chi''_{\mathcal{O}, \mathcal{O}_1}(C_5) = 4$ if $P_3 \notin \mathcal{P}$.

If $\mathcal{P} \supset \mathcal{O}$ and $\mathcal{Q} \supset \mathcal{O}_1$ then color the elements $v_0, v_0 v_1, v_1, v_1 v_2, \ldots$ successively with colors $1, 2, 3, 1, 2, 3, \ldots$ if $n \not\equiv 2 \pmod{3}$ and with colors $1, 2, 3, 1, 2, 3, \ldots$ if $n \equiv 2 \pmod{3}$ to obtain a $(\mathcal{P}, \mathcal{Q})$-total coloring of $C_n$ with 3 colors.

3. **Total Acyclic Colorings ($\mathcal{P} = \mathcal{Q} = \mathcal{D}_1$)**

Total acyclic colorings are $(\mathcal{D}_1, \mathcal{D}_1)$-total colorings where $\mathcal{D}_1$ contains the 1-degenerate graphs which are the acyclic graphs. The $\mathcal{D}_1$-vertex chromatic number is the *vertex arboricity* $a(G) = \chi_{\mathcal{D}_1}(G)$ and the $\mathcal{D}_1$-edge chromatic number is the *(edge) arboricity* $a'(G) = \chi'_{\mathcal{D}_1}(G)$.

We mention some known results on the vertex and edge arboricity: $\chi_{\mathcal{D}_1}(G) = \chi'_{\mathcal{D}_1}(G) = 1$ if and only if $G$ is acyclic, $\chi_{\mathcal{D}_1}(C_n) = \chi'_{\mathcal{D}_1}(C_n) = 2$, $\chi_{\mathcal{D}_1}(K_n) = \chi'_{\mathcal{D}_1}(K_n) = \lceil n/2 \rceil$, $\chi_{\mathcal{D}_1}(K_m,n) = 1$ if $m = 1$ or $n = 1$. 
\[ \chi_{D_1}(K_{m,n}) = 2 \] if \( m \neq 1 \neq n \), \( \chi'_{D_1}(K_{m,n}) = \lfloor mn / (m+n-1) \rfloor \) (see [13], e.g.).

We denote induced subgraphs \( H \) of \( G \) by \( H \leq G \). Proved upper bounds are \( \chi_{D_1}(G) \leq \max_{H \leq G} \{ \lceil \delta(H) / 2 \rceil + 1 \} \) [7] which implies \( \chi_{D_1}(G) \leq \lfloor \Delta(G) / 2 \rfloor + 1 \) and \( \chi'_{D_1}(G) \leq \lfloor \Delta(G) / 2 \rfloor + 1 \). The latter is an implication of

\[ \chi'_{D_1}(G) = \max \frac{|E(H)|}{|V(H)|} \] when \( |V(H)| > 1 \)

which is due to Nash-Williams [13]. Moreover, \( \chi_{D_2}(G) \leq \chi'_{D_1}(G) \) (see [5]).

Observe that we have an analogous situation for ordinary colorings: \( \chi(G) \leq \Delta(G) + 1 \), \( \chi'(G) \leq \Delta(G) + 1 \) (Vizing [14]) and \( \chi(G) \leq \chi'(G) \) (Brooks [4]).

Theorem 1 implies that \( \chi_{D_2}''_{D_1}(G) = 1 \) if and only if \( G \in \mathcal{O} \) and \( \chi_{D_2}''_{D_1}(G) = 2 \) if and only if \( G \in \mathcal{O} \setminus \mathcal{O} \) (acyclic graphs with edges). For cycles \( C_n \), we have \( \chi_{D_2}''_{D_1}(C_n) = 3 \) by Theorem 7 since \( C_n \notin \mathcal{D}_1 \).

**Theorem 8.** \( \chi_{D_2}''_{D_1}(K_1) = 1 \), \( \chi_{D_2}''_{D_1}(K_2) = 2 \), \( \chi_{D_2}''_{D_1}(K_n) = [n/2] + 2 \) for \( n \geq 3 \).

**Proof.** The results for \( n = 1 \) and \( n = 2 \) follow from Theorem 1.

Let \( n \geq 3 \). Each color class of a \((D_1, D_1)\)-total coloring of \( K_n \) with \( c \) colors contains \( 0, 1 \), or \( 2 \) vertices and at most \( n-1, n-2 \), or \( n-3 \) edges, respectively. If \( x_i \) denotes the number of color classes with \( i \) vertices we obtain \( x_0 + x_1 + x_2 = c \) (number of color classes), \( x_1 + 2x_2 = n \) (number of vertices) and \( (n-1)x_0 + (n-2)x_1 + (n-3)x_2 \geq \binom{n}{2} \) (number of edges). It follows \( (n-1)(c-1) - \lfloor n/2 \rfloor \) and therefore \( c \geq \lfloor n/2 + 1 + 1 / (n-1) \rfloor \). If \( n \) is even then \( c \geq n/2 + 2 \); if \( n \) is odd then \( 1/(n-1) \leq 1/2 \) and therefore \( c \geq \lceil n/2 \rceil + 1 = \lfloor n/2 \rfloor + 2 \) which implies \( \chi_{D_2}''_{D_1}(K_n) \geq \lfloor n/2 \rfloor + 2 \) if \( n \geq 3 \).

On the other hand, it holds \( \chi_{D_2}''_{D_1}(K_n) \leq \lceil n/2 \rceil + 2 \) which can be seen by the following partition of the elements of \( K_n \) in \( \lfloor n/2 \rfloor + 2 \) classes.

If \( n \) is even then class \( F_i, 0 \leq i \leq \frac{n}{2} - 1 \), contains vertices \( v_i \) and \( v_{i+n/2} \) and the \( n-3 \) edges of the path \( (v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+n/2-1}, v_{i-n/2+1}) \) where all indices are reduced modulo \( n \). The remaining edges \( v_0v_1, v_1v_2, \ldots, v_{n-1}v_0 \) induce a cycle which can be colored with two additional colors (see Figure 4, upper part).

If \( n \) is odd then class \( F_i, 0 \leq i \leq \frac{n-3}{2} \), contains vertices \( v_i \) and \( v_{i-(n-1)/2} \) and the \( n-3 \) edges of the path \( (v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+(n-1)/2}) \).
Moreover, the remaining elements of $K_n$ can be colored using two additional colors:

$\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil + 2.$

This conjecture is an analogy to the total coloring conjecture which says that $\chi''(G) \leq \Delta(G) + 2$ for all graphs $G$.

Since $m \leq 3n - 6$ for planar graphs $G$ of order $n \geq 3$ and size $m$ we obtain $\chi_{\mathcal{D}_1}(G) \leq \chi'_{\mathcal{D}_1}(G) \leq 3$ by (6) which implies $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 6$. We can improve this to $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 5$ but we do not know whether $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 4$ is true for all planar graphs. For outerplanar graphs $G$ it holds $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 3$. 

Figure 4. Color classes of $K_n$ if $n$ is even (above) or odd (below).
4. \((\mathcal{P}, \mathcal{Q})\)-Total Colorings of Infinite Graphs — A Compactness Theorem

All our considerations hold for arbitrary simple infinite graphs. Let us denote by \(\mathcal{I}^*\) the class of all simple infinite graphs. A graph property \(\mathcal{P}\) is any isomorphism-closed nonempty subclass of \(\mathcal{I}^*\).

In 1951, de Bruijn and Erdős [8] proved that an infinite graph \(G\) is \(k\)-colorable if and only if every finite subgraph of \(G\) is \(k\)-colorable. Analogous compactness theorems for generalized colorings were proved in [6]. They all have been based on the “Set Partition Compactness Theorem” (see [6]), where the key concept is that of a property being of finite character. A graph property \(\mathcal{P}\) is of finite character if a graph in \(\mathcal{I}^*\) has property \(\mathcal{P}\) if and only if each of its finite induced subgraphs has property \(\mathcal{P}\). It is easy to see that if \(\mathcal{P}\) is of finite character and a graph has property \(\mathcal{P}\) then so does every induced subgraph. A property \(\mathcal{P}\) is said to be induced-hereditary if \(G \in \mathcal{P}\) and \(H \leq G\) implies \(H \in \mathcal{P}\), that is, \(\mathcal{P}\) is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character. For example, the graph property of not containing a vertex of infinite degree is induced-hereditary but not of finite character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply hereditary) is induced-hereditary as well. The properties of being edgeless, of maximum degree at most \(k\), \(K_n\)-free, acyclic, complete, perfect, etc. are properties of finite character. Each additive hereditary graph property \(\mathcal{P}\) of finite character can be characterized (see, e.g., [12]) by the set of connected minimal forbidden graphs of \(\mathcal{P}\), which is defined as follows:

\[\mathbf{F}(\mathcal{P}) = \{G : G \text{ connected, } G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.\]

In the paper [6] also a compactness result for generalized colorings of hypergraphs has been presented. A simple hypergraph \(H = (X, E)\) is a hypergraph on a vertex set \(X\) where all hyperedges \(e \in E\) are different finite subsets of the vertex set \(X\). Let \(\mathcal{P}_1, \ldots, \mathcal{P}_m\) be properties of simple hypergraphs (i.e. classes of simple hypergraphs closed under isomorphism). A hypergraph \(H = (X, E)\) is \((\mathcal{P}_1, \ldots, \mathcal{P}_m)\)-colorable if the vertex set \(X\) of \(H\) can be partitioned into sets \(X_1, \ldots, X_m\) such that the induced subhypergraphs \(H[X_i] = (X_i, E(X_i))\) of \(H\), where \(E(X_i)\) consists of all hyperedges of \(H\) all of whose vertices belong to \(X_i\), has property \(\mathcal{P}_i\), \(i = 1, 2, \ldots, m\). A property
\( \mathcal{P} \) of hypergraphs is of \textit{finite vertex character} if a hypergraph has property \( \mathcal{P} \) if and only if every finite induced subhypergraph has property \( \mathcal{P} \). Then, using the Set Partition Compactness Theorem, it holds:

**Theorem 9.** Let \( H \) be a simple hypergraph and suppose \( \mathcal{P}_1, \ldots, \mathcal{P}_m \) are properties of hypergraphs of finite vertex character. Then \( H \) is \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \)-colorable if every finite induced subhypergraph of \( H \) is \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \)-colorable.

In particular, if \( \mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_m = \mathcal{O}_H \), where \( \mathcal{O}_H \) denotes the property of a hypergraph “to be hyperedgeless”, i.e., \( E = \emptyset \), we have a compactness theorem for the regular hypergraph coloring, since \( \mathcal{O}_H \) is of finite character.

Now we will use this result to prove the compactness theorem for \( (\mathcal{P}, \mathcal{Q}) \)-total colorings:

**Theorem 10.** Let \( G \in \mathcal{I}^* \) be a simple infinite graph and suppose \( \mathcal{P} \) and \( \mathcal{Q} \neq \mathcal{O} \) are additive properties of finite character. Then \( G \) is \( (\mathcal{P}, \mathcal{Q}) \)-totally \( k \)-colorable if and only if every finite induced subgraph of \( G \) is \( (\mathcal{P}, \mathcal{Q}) \)-totally \( k \)-colorable.

**Proof.** Let \( G = (V(G), E(G)) \) be a simple infinite graph and let \( \mathcal{P}, \mathcal{Q}, \mathcal{Q} \neq \mathcal{O} \) be additive hereditary properties of finite character. Let \( \mathbf{F}(\mathcal{P}) \) and \( \mathbf{F}(\mathcal{Q}) \) be the sets of minimal forbidden graphs of \( \mathcal{P} \) and \( \mathcal{Q} \), respectively. Let us define a hypergraph \( H(G) = (V^*, E^*) \) so that \( V^* = V(G) \cup E(G) \) and a set \( e \subset V^* \) is an hyperedge of \( H(G) \) if and only if

1. \( e = \{v, h\}, v \in V(G), h \in E(G), v \in h, \) or
2. \( G[e] \in \mathbf{F}(\mathcal{P}), e \subset V(G), \) or
3. \( G[e] \in \mathbf{F}(\mathcal{Q}), e \subset E(G). \)

By the definition of the hypergraph \( H(G) \) of \( G \), a graph \( G \) is \( (\mathcal{P}, \mathcal{Q}) \)-totally \( k \)-colorable if the hypergraph \( H(G) \) is regularly \( k \)-colorable. By Theorem 9, \( H(G) \) is regularly \( k \)-colorable if every finite induced subhypergraph of \( H(G) \) is regularly \( k \)-colorable. However, if every finite induced subgraph of \( G \) is \( (\mathcal{P}, \mathcal{Q}) \)-totally \( k \)-colorable, then obviously every finite induced subhypergraph of \( H(G) \) is regularly \( k \)-colorable.

**References**


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