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# CYCLICALLY $k$-PARTITE DIGRAPHS AND $k$-KERNELS

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**Abstract**

Let $D$ be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively.

A $(k,l)$-kernel $N$ of $D$ is a $k$-independent set of vertices (if $u,v \in N$ then $d(u,v) \geq k$) and $l$-absorbent (if $u \in V(D) - N$ then there exists $v \in N$ such that $d(u,v) \leq l$). A $k$-kernel is a $(k,k-1)$-kernel. A digraph $D$ is cyclically $k$-partite if there exists a partition $\{V_i\}_{i=0}^{k-1}$ of $V(D)$ such that every arc in $D$ is a $V_iV_{i+1}$-arc (mod $k$). We give a characterization for an unilateral digraph to be cyclically $k$-partite through the lengths of directed cycles and directed cycles with one obstruction, in addition we prove that such digraphs always have a $k$-kernel. A study of some structural properties of cyclically $k$-partite digraphs is made which bring interesting consequences, e.g., sufficient conditions for a digraph to have $k$-kernel; a generalization of the well known and important theorem that states *if every cycle of a graph $G$ has even length, then $G$ is bipartite* (*cyclically* 2*-partite*), we prove that if every cycle of a graph $G$ has length $\equiv 0$ (mod $k$) then $G$ is cyclically $k$-partite; and a generalization of another well known result about bipartite digraphs, *a strong digraph $D$ is bipartite if and only if every directed cycle has even length*, we prove that an unilateral digraph $D$ is bipartite if and only if every directed cycle with at most one obstruction has even length.

**Keywords:** digraph, kernel, $(k,l)$-kernel, $k$-kernel, cyclically $k$-partite.

**2010 Mathematics Subject Classification:** 05C20.

## 1. Introduction

For general concepts and notation we refer the reader to [1, 2] and [5], particularily we will use the notation of [5] for walks, if $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i \mathscr{C} x_j$ will denote the subwalk of $\mathscr{C}$ $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$, if $x_i = x_0$ we will simply write $\mathscr{C} x_j$, analogously if $x_j = x_n$. Union of walks will be denoted by concatenation or with $\cup$.

Several classes of $k$-partite graphs and digraphs have been extensively studied as they are a natural generalization of bipartite graphs and digraphs; $k$-partite tournaments (e.g. [1]), which have been studied for hamiltonicity and pancyclism, and cyclically $k$-partite digraphs stand out for their multiple properties. Cyclically $k$-partite digraphs have received attention for their connection with matrix theory (e.g. [4]) in the study of the properties of cyclic matrices and some special cases of diagonal matrices since the digraph associated with an irreducible matrix with imprimitivity index $k$ is exactly a $k$-partite digraph. Our aim is to find structural properties of cyclically $k$-partite graphs and digraphs wich are of general interest and that we can use to state sufficient conditions for the existence of $k$-kernels in some families of digraphs.

In [8], M. Kwaśnik introduces the concept of $(k, l)$-kernel in a digraph generalizing the concept of kernel of a digraph in the Berge's sense which is a $(2, 1)$-kernel. As a special case of $(k, l)$-kernels we consider the $k$-kernels; we define a $k$-kernel to be a $(k, k-1)$-kernel. Under this definition a kernel is a 2-kernel.

There are not many of results concerning the existence of $k$-kernels nor $(k, l)$-kernels in large families of digraphs, many of the existing results come from the study of products of digraphs and how the $k$-kernels are preserved (like the work of Włoch and Włoch, in particular with Szumny in [12, 13]) or the superdigraphs or certain families of digraphs ([7]). We begin with some of the classical results in Kernel Theory that we will use as platform for the results we propose.

Since every (directed) cycle of odd length does not has a kernel, sufficient conditions for the existence of kernels in digraphs have been found imposing conditions on the cycles of a digraph, e.g., in [14] is proved that

**Theorem 1.1.** *If $D$ is a digraph without directed cycles, then $D$ has a kernel.*

In [10], Richardson generalizes this result as follows[1]

**Theorem 1.2** (Richardson [10]). *If $D$ is a digraph such that the length of every directed cycle is congruent to* $0 \pmod 2$*, then $D$ has a kernel.*

This two theorems are examples of results than can be generalized for $k$-kernels, our attention is focused in the generalization of the second theorem.

M. Kwaśnik stated the following generalization for $k$-kernels.

**Definition 1.3.** A digraph $D$ is *strongly connected* if and only if for every pair of vertices $u, v \in V(D)$, there exists a $uv$-directed path in $D$.

A digraph $D$ is *unilaterally connected*, or simply *unilateral*, if and only if for every pair of vertices $u, v \in V(D)$, there exists an $uv$-directed path or a $vu$-directed path in $D$.

**Theorem 1.4** (Kwaśnik [8]). *Let $D$ be a strongly connected digraph. If every directed cycle in $D$ has length congruent to* $0 \pmod k$*, then $D$ has a $k$-kernel.*

It has been noticed that the hypothesis of being strongly connected cannot be dropped, and, altough diverse counterexamples have been considered for the non strongly conected case (e.g. [11]), all of these examples are non unilateral, so the question arises. Can the strong connectedness be substituted for unilaterality? The answer is no, and the next digraph is a counterexample, showing that the hypothesis in Theorem 1.4 is sharp.

If the digraph in Figure 1 had a 3-kernel, since vertex 10 has outdegree 0 (and thus can not be absorbed by any other vertex) it should be in the 3-kernel, hence vertices $2, 7, 4, 8, 6$ and 9 would be 2-absorbed. The only vertices that could 2-absorb vertex 1 are $2, 3$ and 7, but the distance from vertex 7 to vertex 10 is one, and distance from vertex 2 to vertex 10 is two, so they can not be in the 3-kernel and the only remaining possibilities are that vertex 1 is in the 3-kernel or vertex 3 is in the 3-kernel. We will show that vertex 3 can not be in the 3-kernel and by symmetry vertex 1 neither can be in the 3-kernel. Let us assume that 3 is in the 3-kernel. Now, vertex 5 can be 2-absorbed by vertices $1, 6$ or 9 but $d(1, 3) = 2$, $d(6, 10) = 2$ and $d(9, 10) = 1$ and hence none of them can be added to the 3-kernel but neither can vertex 5, since vertex 3 is at distance two from vertex 5. Consequently, digraph in

[1] See [3] for a simpler proof of Theorem 1.2.

Figure 1 does not have a 3-kernel, its only directed cycle, $(1, 2, 3, 4, 5, 6, 1)$ has length $\equiv 0 \pmod 3$, and is unilaterally connected.

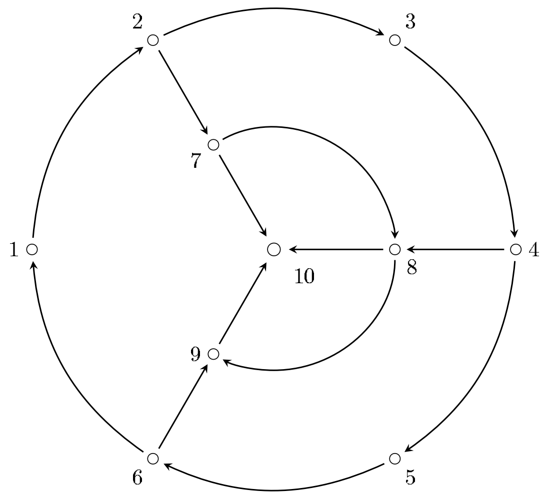


Figure 1. Counterexample to a version of Theorem 1.4 with weaker hypothesis.

## 2. Cyclically $k$-partite Digraphs

We are interested in the different ways the concept of $k$-kernel can be generalized and the possibility to demonstrate equivalent versions of Theorem 1.4 or any of the classical theorems in kernel theory. Also, in the study of the structural properties implicated by the hypotesis of Theorem 1.4, we prove a similar result, relaxing the hypothesis of connectedness to unilaterality but increasing the length restrictions to more than just the directed cycles. In [6], Galeana-Sánchez proves Theorem 1.4 showing that any strongly connected digraph $D$ such that every directed cycle has length $\equiv 0 \pmod k$ is cyclically $k$-partite i.e., there exists a $k$-partition of $V(D)$, $V_0, V_1, \ldots, V_{k-1}$ such that every arc of $D$ is a $V_iV_{i+1}$-arc $(\text{mod } k)$; and, thanks to the strong connectedness, that every $V_i$ is a $k$-absorbent set for every $i \in \{0, 1, \ldots, k-1\}$, i.e., that for every vertex $u \in V(D) \setminus V_i$, there is a vertex $v \in V_i$ such that $d(u, v) \leq k$. We propose new sufficient conditions for a digraph to be cyclically $k$-partite and to find a $k$-absorbing set in this partition.

**Definition 2.1.** A closed walk $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_{n+1} = x_0)$ is directed with an obstruction at vertex $x_n$ if there exists a directed walk $\mathscr{C}' = (x_0, x_1, \ldots, x_n)$ and an arc $(x_0, x_n) \in A(D) \setminus A(\mathscr{C}')$ such that $\mathscr{C} = \mathscr{C}' \cup (x_0, x_n)$.

Figure 2 shows a cycle $\mathscr{C} = (0, 1, 2, 3, 4, 5, 6, 7, 0)$ with an obstruction at vertex 7, where $\mathscr{C}' = (0, 1, 2, 3, 4, 5, 6, 7)$ and $\mathscr{C} = \mathscr{C}' \cup (0, 7)$. If we reverse the arc $(0, 7)$, the sequence $\mathscr{C}$ will denote a directed cycle.

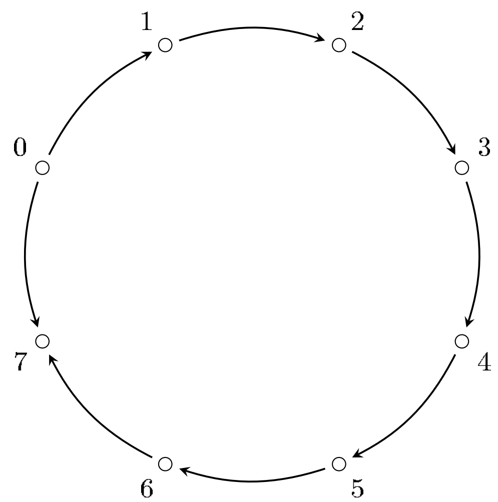


Figure 2. A cycle with an obstruction at vertex 7.

In Definition 2.1 it is important to notice that $(x_0, x_n) \notin A(\mathscr{C}')$ so its reversal turns $\mathscr{C}$ into a closed directed walk. Figure 3 digraph (i) shows a digraph with a closed walk $\mathscr{C} = (0, 1, 2, 0, 3, 4, 2, 3, 0)$ such that there exist a directed walk $\mathscr{C}' = (0, 1, 2, 0, 3, 4, 2, 3)$ and an arc $(0, 3) \in A(\mathscr{C}')$ such that $\mathscr{C} = \mathscr{C}' \cup (0, 3)$, but as it can be observed in digraph (ii), the reversal of $(x_0, x_n)$ does not turn $\mathscr{C}$ into a directed walk.

With this definition we state some lemmas leading to a characterization of unilateral cyclically $k$-partite digraphs.

**Lemma 2.2.** *If $\mathscr{C}$ is a directed closed walk with one obstruction, then $\mathscr{C}$ contains a cycle with at most one obstruction.*

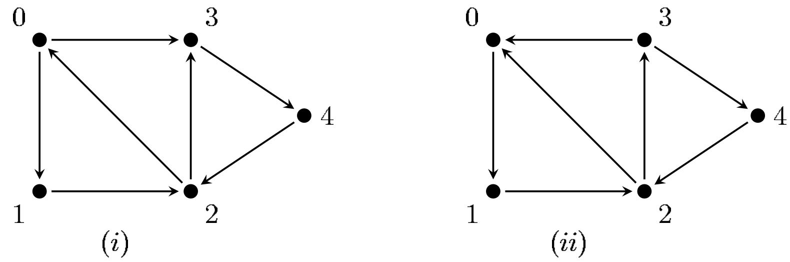


Figure 3. Digraph $(ii)$ is obtained from digraph $(i)$ by the reversal of the arc $(0, 3)$. The closed walk $(0, 1, 2, 0, 3, 4, 2, 3, 0)$ is not directed in digraph $(i)$ nor in digraph $(ii)$.

***Proof.*** Since $\mathscr{C}$ is a directed closed walk with an obstruction, then $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_{n+1} = x_0)$ where $\mathscr{C}' = (x_0, \ldots, x_n)$ is a directed walk and $(x_0, x_n)$ is an arc not in $A(\mathscr{C}')$. If $\mathscr{D}$ is the directed closed walk obtained from $\mathscr{C}$ by reversing the arc $(x_0, x_n)$, then it is a well known result[2] that $\mathscr{D}$ contains a directed cycle $\mathscr{D}_1$. If $\mathscr{D}_1$ contains the arc $(x_n, x_0)$, then $\mathscr{C}$ contains the directed cycle with one obstruction at vertex $x_n$ $\mathscr{C}_1$, obtained by the reversing of the arc $(x_n, x_0)$ in $\mathscr{D}_1$; if $\mathscr{D}_1$ does not contain the arc $(x_n, x_0)$, then $\mathscr{C}$ contains the directed cycle $\mathscr{D}_1$. ■

**Lemma 2.3.** *If $D$ is a digraph such that every directed cycle in $D$ has length $\equiv 0 \pmod{k}$, then every directed closed walk has length $\equiv 0 \pmod{k}$.*

***Proof.*** By induction on $\ell(\mathscr{C}) = n$, where $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ is the directed closed walk. If $n \leq k$, since every directed closed walk contains a directed cycle and every directed cycle in $D$ has length $\equiv 0 \pmod{k}$, then $n = k$. If $n > k$, then $\mathscr{C}$ contains a directed cycle $\mathscr{C}_1 = x_i \mathscr{C} (x_j = x_i)$, where $j > i$. It is clear that if $\mathscr{C}_2 = x_0 \mathscr{C} x_i \cup x_j \mathscr{C} x_n$, then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ and $\ell(\mathscr{C}) = \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2)$. By induction hypothesis $\ell(\mathscr{C}_2) \equiv 0 \pmod{k}$ and $\ell(\mathscr{C}_1) \equiv 0 \pmod{k}$ because $\mathscr{C}_1$ is a directed cycle. Hence, $\ell(\mathscr{C}) \equiv 0 \pmod{k}$. ■

**Lemma 2.4.** *Let $D$ be a digraph. If every directed cycle has length $\equiv 0 \pmod{k}$ and every directed cycle with one obstruction has length $\equiv r \pmod{k}$, then every directed closed walk $\mathscr{C}$ with one obstruction fulfills that $\ell(\mathscr{C}) \equiv r \pmod{k}$.*

[2]This result can be found in [2].

***Proof.*** By induction on $\ell(\mathscr{C})$. If $\ell(\mathscr{C}) \leq k$ then $\mathscr{C}$ cannot repeat interior vertices, because it wolud contain a directed closed walk and then a directed cycle, but every directed cycle has length $\equiv 0 \pmod{k}$, thus, $\mathscr{C}$ is a directed cycle with one obstruction and by hypothesis has length $\equiv r \pmod{k}$. If $\ell(\mathscr{C}) > k$ and $\mathscr{C}$ does not repeat interior vertices, then again $\mathscr{C}$ is a directed cycle with one obstruction. Otherwise, there exist an interior vertex $x_i$ such that $x_i\mathscr{C}(x_j = x_i) = \mathscr{C}_1$ is a directed closed walk and as in the proof of Lemma 2.3, $\mathscr{C}_2 = x_0\mathscr{C}x_i \cup x_j\mathscr{C}x_n$ is such that $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ and $\ell(\mathscr{C}) = \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2)$. But, in virtue of Lemma 2.3, $\ell(\mathscr{C}_1) \equiv 0 \pmod{k}$ and by induction hypothesis $\ell(\mathscr{C}_2) \equiv r \pmod{k}$. Thence, $\ell(\mathscr{C}) \equiv r \pmod{k}$. ∎

The next theorem was proved while we were looking for new sufficient conditions for digraphs to have $k$-kernel.

**Lemma 2.5.** *If $D$ is an unilateral digraph such that every directed cycle has length $\equiv 0 \pmod{k}$ and every directed cycle with one obstruction has length $\equiv 2 \pmod{k}$, then $D$ is cyclically $k$-partite.*

***Proof.*** First observe that to have a $k$-partition of $D$ we need at least $k$ vertices, so we will suppose that $|V(D)| \geq k$. Since $D$ is unilateral, there exists a spanning directed walk $\mathscr{C} = (v_0, v_1, \ldots, v_n)$ and we can consider the subsets $V_i = \{v_r | r \equiv i \ (\mathrm{mod} k)\}$, $0 \leq i \leq k-1$ of $V(D)$. The set $\{V_i\}_{i=0}^{k-1}$ is a partition of $V(D)$. To prove that $\bigcup_{i=0}^{k-1} V_i = V(D)$ and $V_i \neq \emptyset$ for $0 \leq i \leq k-1$ it suffices to observe that $\mathscr{C}$ is a spanning directed walk and thus has length greater than or equal to $k$, it follows that $v_i \in V_i$ for $0 \leq i \leq k-1$, then $V_i \neq \varnothing$. Also, if $v \in V(D)$ then $v = v_r$ for some $0 \leq r \leq n$, but $\{0, 1, \ldots, k-1\}$ is a complete system of distinct representatives (mod $k$) hence $r \equiv i \pmod{k}$ for some $0 \leq i \leq k-1$ and $v \in V_i$. Finally, to prove that $V_j \cap V_k = \emptyset$, let $v_r$ be a vertex in $V(D)$, if $v_r$ appears only once in $\mathscr{C}$ then $r \equiv i \pmod{k}$ for a unique $i \in \{0, 1, \ldots, k-1\}$ and consequently $v_r$ belongs to $V_i$ for a unique $i \in \{0, 1, \ldots, k-1\}$; if $v_r$ appears more than once in $\mathscr{C}$ we can suppose without loss of generality that $v_r = v_s$ with $r < s$ and then $v_r\mathscr{C}v_s$ is a directed closed walk which, in virtue of Lemma 2.3, has length $\equiv 0 \pmod{k}$ so $r \equiv s \pmod{k}$ and $v_r \in V_i$ for a unique $i$.

Let $(x, y) \in A(D)$, then $x = v_r$, $y = v_s$ for some $r, s \in \{0, 1, \ldots, n\}$. If $s < r$, then $y\mathscr{C}x \cup (x, y)$ is a directed closed walk and it follows from Lemma 2.3 that $\ell(y\mathscr{C}x \cup (x, y)) \equiv 0 \pmod{k}$ and since $\ell(y\mathscr{C}x) = r - s$, then $r - s + 1 \equiv 0$ and hence $s \equiv r + 1 \pmod{k}$ therefore $(x, y)$ is a $V_iV_{i+1}$-arc for some $i \in \{0, 1, \ldots, k-1\}$. If $r < s$, then $s = r+1$ when $(x, y) \in A(D)$

or $x\mathscr{C}y \cup (x,y)$ is a directed closed walk with one obstruction in $y$ and in virtue of Lemma 2.4 $\ell\,(x\mathscr{C}y \cup (x,y)) \equiv 2 \pmod k$, but $\ell(x\mathscr{C}y) = s - r$, thus $s - r + 1 \equiv 2 \pmod k$ and $s \equiv r + 1 \pmod k$; in either case $(x,y)$ is a $V_iV_{i+1}$-arc for some $i \in \{0,1,\dots,k-1\}$ and we can conclude that $D$ is a cyclically $k$-partite digraph. ■

As it can be noticed, we just found sufficient conditions for an unilateral digraph to be cyclically $k$-partite, so the natural question arose. Are these sufficient condition also necessary? The answer to this question is yes, not only for unilateral digraphs, but for every cyclically $k$-partite digraph as well.

**Lemma 2.6.** *If $D$ is a cyclically k-partite digraph, then every directed cycle has length $\equiv 0 \pmod k$ and every directed cycle with one obstruction has length $\equiv 2 \pmod k$.*

***Proof.*** Let $D$ be a cyclically $k$-partite digraph. Is clear that every directed cycle has length $\equiv 0 \pmod k$, so let $\mathscr{C} = (x_0, x_1, \dots, x_n, x_{n+1} = x_0)$ be a directed cycle with one obstruction at vertex $x_n$ (and hence $(x_0, x_n) \in A(D)$). Without loss of generality let us assume that $\{V_i\}_{i=0}^{k-1}$ is the cyclical $k$-partition and that $x_0 \in V_0$, then $x_n \in V_1$. Since $x_1 \in V_1$, $\ell\,(x_1 \dots x_n) \equiv 0 \pmod k$, but $\mathscr{C} = (x_0, x_1) \cup x_1\mathscr{C}x_n \cup (x_0, x_n)$, so $\ell(\mathscr{C}) \equiv 2 \pmod k$. ■

The characterization is then obtained.

**Theorem 2.7.** *If $D$ is an unilateral digraph then $D$ is cyclically $k$-partite if and only if every directed cycle has length $\equiv 0 \pmod k$ and every directed cycle with one obstruction has length $\equiv 2 \pmod k$.*

***Proof.*** It follows from Lemma 2.5 and Lemma 2.6. ■

And as an immediate consequence of these theorem, we have the next corollary that generalizes a classical characterization of bipartite digraphs. It is known that a strongly connected digraph is bipartite if and only if every directed cycle has even length. We have the following characterization for unilateral digraphs.

**Corollary 2.8.** *Let $D$ be an unilateral digraph, then $D$ is bipartite if and only if every directed cycle with at most one obstruction has even length.*

***Proof.*** The sufficiency is trivial as every cycle (directed or not) of the digraph is of even length. For the necessity set $k = 2$, then $2 \equiv 0 \pmod{k}$ and the hypothesis of Theorem 2.7 are fullfiled, so $D$ is cyclically 2-partite and then bipartite. ■

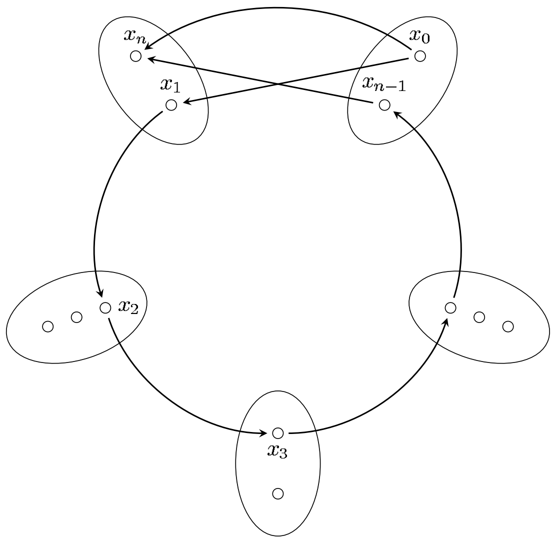


Figure 4. An illustration for the proof of Lemma 2.6, a directed cycle with one obstruction in a cyclically $k$-partite digraph.

Thus, we have characterizations for strongly connected and unilateral cyclically $k$-partite digraphs, in terms of connectedness the next step would be connected digraphs, unfortunately the method used in the proof of the existing characterizations use strongly the existence of a directed spanning walk, which we do not have in merely connected digraphs. The next theorem gives a sufficient condition for a graph to have a cyclically $k$-partite orientation. This theorem is of great interest on its own because it generalizes a classic result in Graph Theory, and also, its contrapositive form gives some information on the structural properties of non cyclically $k$-partite digraphs (and graphs).

Besides, we introduce the bridge graph of a given graph, a new tool that we found very useful in the proof of the theorem.

**Definition 2.9.** If $G$ is a graph, the *bridge graph of* $G$ is the graph $Br(G)$ with vertex set $\{H \subseteq G | H \text{ is a maximal bridgeless subgraph of } G\}$ and such that $H_1H_2 \in E\left(Br(G)\right)$ if and only if there is a bridge between $H_1$ and $H_2$ in $G$.

It is clear from the definition that every edge of $Br(G)$ is a bridge, and thus, $Br(G)$ is a tree. Moreover, there is a bijection between edges in $Br(G)$ and bridges in $G$.

**Theorem 2.10.** *Let $G$ be a graph such that every cycle has length $\equiv 0 \pmod k$, then $G$ admits a cyclically $k$-partite orientation.*

***Proof.*** By induction on $n = |V\left(Br(G)\right)|$. If $n = 1$, then $G$ is a bridgeless graph, so it admits a strongly connected orientation $\mathcal{O}(G)$. Since every cycle of $G$ has length $\equiv 0 \pmod k$, then $\mathcal{O}(G)$ is strongly connected and every directed cycle has length $\equiv 0 \pmod k$, thus $\mathcal{O}(G)$ is cyclically $k$-partite. Assume the result valid for every graph $G$ with $|V\left(Br(G)\right)| < n$ and let $G$ be a graph such that $|V\left(Br(G)\right)| = n$. If $H$ is a leaf in $Br(G)$, then $G - H$ is a connected graph with $|V\left(Br(G)\right)| = n - 1$, and by induction hypothesis it is cyclically $k$-partite with $k$-partition $\mathsf{P} = \{V_0, V_1, \ldots, V_{k-1}\}$. Since $H$ is bridgless, it is also cyclically $k$-partite with $k$-partition $\mathsf{Q} = \{W_0, W_1, \ldots, W_{k-1}\}$ and there is only one edge $e \in E(G)$ between $H$ and $G - H$. If we orient $e$ so it has tail in $H$ and head in $G - H$, and we rename the elements of $\mathsf{Q}$ to obtain $\mathsf{Q}'$ such that the arc obtained by the orientation of $e$ has tail in $W_i$ and head in $V_{i+1} \pmod k$, as this is the only arc between the orientations of $H$ and $G - H$, $\mathsf{R} = \{V_0 \cup W_0, V_1 \cup W_1, \ldots, V_{k-1} \cup W_{k-1}\}$ is a cyclical $k$-partition of $G$. ∎

This condition is suficient, but not necessary as the example in Figure 5 shows.

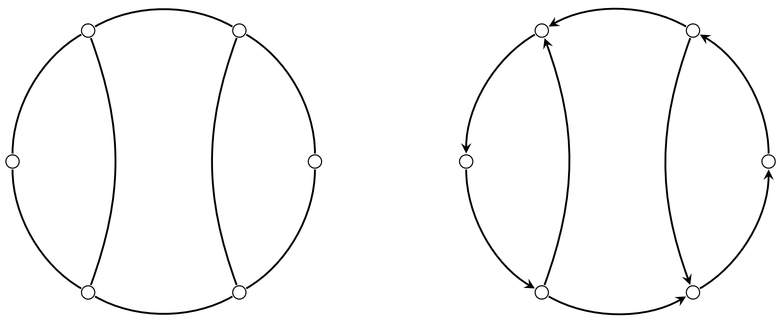

Figure 5. A graph with cycles of length 3, 4 and 5 and a cyclically 3-partite orientation of the same graph.

However Theorem 2.10 has interesting consequences.

**Theorem 2.11.** *If $G$ is a graph such that every cycle has length $\equiv 0 \pmod{k}$, then $G$ is cyclically $k$-partite.*

***Proof.*** It suffices to consider a cyclically $k$-partite orientation of $G$, it result obvious that $G$ is itself cyclically $k$-partite. ■

For $k = 2$ this is a classical Graph Theory theorem, asserting that if every cycle of $G$ is even, then $G$ is bipartite. For the $k = 2$ (bipartite) case the necessity is also true, but Figure 5 demonstrates that it is not true for every $k$, as a matter of fact, for every other $k$ we can find a cyclically $k$-partite graph with a 4-cycle as Figure 6 shows. The idea of this construction can be extended to find cyclically $k$-partite graphs with cycles of every even length.

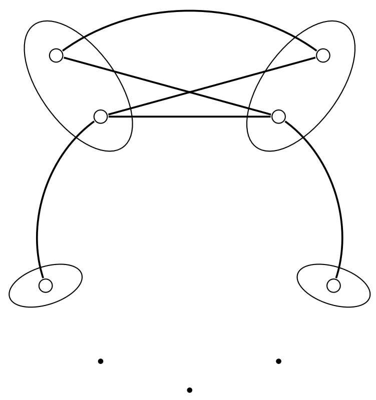

Figure 6. Example of a cyclically $k$-partite digraph with a 4-cycle.

As a final consequence of Theorem 2.10 in this section, we give the following corollary.

**Corollary 2.12.** *Let $G$ be graph such that every cycle has length $\equiv 0 \pmod{k}$ with $k = 2n - 1$, $n \in \mathbb{N}$, then $\chi(G) \leq 3$.*

***Proof.*** Let us recall that for any graph $G$, $\chi(G) < 3$ if and only if $G$ has no cycles of odd length, so if we assume that $G$ has at least one cycle, since $k$ is odd the equality $\chi(G) = 3$ must hold. It follows from Theorem 2.11 that $G$ is cyclically $k$-partite with partition $\{V_1, V_2, \ldots, V_k\}$ and the elements of the partition form an odd cycle. As every element of the partition is an independent set, it suffices to give a 3-colouring for the $k$-cycle $(V_1, V_2, \ldots, V_k, V_0)$ and assign the same color as $V_i$ to each vertex in $V_i$ for every $i \in \{1, 2, \ldots, k\}$. ■

## 3. Cyclically $k$-partite Digraphs and $k$-kernels

From the proof of definition of a cyclically $k$-partite digraph we can observe the following.

**Proposition 3.1.** *If $D$ is a cyclically $k$-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$, then $V_i$ is $k$-independent in $D$ for every $i \in \{0, 1, \ldots, k-1\}$.*

***Proof.*** Since every arc of $D$ is a $V_iV_{i+1}$-arc (mod $k$) for some $i \in \{0, 1, \ldots, k-1\}$ then for each $i \in \{0, 1, \ldots, k-1\}$, every $V_iV_i$-walk must pass throug each $V_j$, $j \neq i$ before getting back to $V_i$. ■

Before proving the main theorem of this section, we need to state a simple result that generalizes the first theorem in Kernel Theory due to Von Neumann and Morgenstern.

**Theorem 3.2.** *Every acyclic digraph has a unique $k$-kernel for every $k \geq 2$.*

***Proof.*** Let us proceed by induction on $|V(D)|$ with fixed $k \geq 2$. If $|V(D)| = 1$, the only vertex of $D$ is the desired $k$-kernel. Supposing the result valid for every acyclic digraph $D$ such that $|V(D)| < n$, let $D$ be an acyclic digraph with $|V(D)| = n$. Since $D$ is an acyclic digraph, there exists $v \in V(D)$ such that $d^-(v) = 0$. Now, $D - v$ is an acyclic digraph on $n-1$ vertices and by induction hypothesis has a unique $k$-kernel $N'$. There are two cases:

*Case* 1. If $v$ is $k$-absorbed by $N'$ in $D$, then $N'$ is the $k$-kernel we have been looking for.

*Case* 2. If $v$ is not $k$-absorbed by $N'$ in $D$, then there are not $vN'$-directed paths of length less or equal than $k-1$ and, as $v$ has indegree 0 there are not $N'v$-directed paths in $D$, in particular there are not $N'v$-directed paths of length less or equal than $k-1$ and hence $N = N' \cup \{v\}$ is $k$-independent and $k-1$-absorbent in $D$. We have found in $N$ the desired kernel.

Finally, observe that in either case $N'$ is unique by induction hypothesis. If $M$ is a $k$-kernel for $D$, $M \setminus \{v\}$ is $k$-independent in $D - v$ and, as $d_D^-(v) = 0$, $v$ can't absorb any other vertex, therefore $M \setminus \{v\}$ is $(k-1)$-absorbent in $D - v$ and a $k$-kernel of $D - v$. It follows than $M \setminus \{v\} = N \setminus \{v\}$ and hence $M = N$, the unique $k$-kernel of $D$. ■

**Theorem 3.3.** *If $D$ is a unilateral digraph such that every directed cycle has length $\equiv 0 \pmod{k}$ and every directed cycle with one obstruction has length $\equiv 2 \pmod{k}$, then $D$ has a $k$-kernel.*

***Proof.*** If $D$ has less than $k$ vertices, then $D$ cannot contain directed cycles (since every directed cycle has at least $k$ vertices), using Theorem 3.2 we can conclude that $D$ has a $k$-kernel. So, we can suppose without loss of generality that $|V(D)| \geq k$.

In virtue of Theorem 2.5, $D$ is a cyclically $k$-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$ and as a consequence of the unilaterality, there exists a directed spanning walk $\mathscr{C} = (v_0, v_1, \ldots, v_n)$ in $D$. Let be $V_j$ such that $v_n \in V_j$. It is a direct observation that $V_j$ is $(k-1)$-absorbent; for every $u \in V(D) \setminus V_j$, $u = x_r$, $r \equiv i \pmod{k}$ for some $0 \leq i \leq k-1$, $i \neq j$ and $r \neq n$, as $u \notin V_j$. We have two cases:

*Case* 1. If $j < i$ it suffices to consider the directed walk $(x_r, x_{r+1}, \ldots, x_{r+(k-i+j)})$. In the virtue that $x_r \in V_i$, it is the case that $x_{r+(k-i+j)} \in V_{i+(k-i+j)} \pmod{k}$, but $i + (k - i + j) \equiv k + j \equiv j \pmod{k}$, and as $j - i < 0$ it follows that $k - i + j \leq k - 1$ therefore $\ell(x_r, x_{r+1}, \ldots, x_{r+(k-i+j)}) \leq k - 1$ and $x_r$ results to be $(k-1)$-absorbed by $V_j$.

*Case* 2. If $i < j$, then $0 \leq i < j \leq k - 1$ and thence $j - i \leq k - 1$. Considering the directed walk $(x_r, x_{r+1}, \ldots, x_{r+(j-i)})$, analogously to Case 1, $x_{r+(j-i)} \in V_{i+(j-i)} \pmod{k}$, but $i + (j - i) = j$ so $x_{r+(j-i)} \in V_j$ and $\ell\left(x_r, x_{r+1}, \ldots, x_{r+(j-i)}\right) = j - i \leq k - 1$, finally $x_r$ results $(k-1)$-absorbed by $V_j$.

Besides, it follows from Proposition 3.1 that $V_j$ is a $k$-independent set. $V_j$ is then $k$-independent and $(k-1)$-absorbent and is therefore the desired $k$-kernel. ■

Also from the observation of the proof of Theorem 3.3, we have good prospects for $k$-kernels in cyclically $k$-partite digraphs, we just have to find an absorbing element of the $k$-partition. It is also clear that unilateral cyclically $k$-partite "like" structures have $k$-kernel.

Let us make further observations of the proof of Theorem 3.3. The absorption in the proposed $k$-kernel, $V_0$ (without loss of generality) is granted due to the existence of the spanning directed walk, for any vertex it suffices to "follow" this walk to get eventually $k$-absorbed. The independence follows from the disposition of the arcs between the elements of the $k$-partition, but this disposition guarantee independence for every element of the partition, not only the one we did choose as our $k$-kernel, therefore we can reverse any number of arcs as long as we do not reverse arcs in the directed spanning walk (conserving the absorption) and as long as we do not create any $V_0V_0$-paths of length $< k$. We can also add any number of $V_iV_j$-arcs as long as $j < i \neq 1$, since these arcs will not affect independence.

**Corollary 3.4.** *Let $D$ be an unilateral cyclically $k$-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$, $k$-kernel $V_0$ and spanning directed walk $\mathscr{C}$. If $D'$ is obtained from $D$ by reversing any number of arcs not in $A(\mathscr{C})$ nor of the form $V_0V_1$ or $V_{k-1}V_0$, or adding any number of $V_iV_j$-arcs with $j < i \neq 1$, then $V_0$ is a $k$-kernel for $D'$.*

***Proof.*** The absorption is a consequence of the existence of $\mathscr{C}$ in $D'$. For the independence, observe that every arc with tail in $V_0$ has head in $V_1$, and every arc with head in $V_0$ has tail in $V_{k-1}$, thus, every $V_0V_0$-walk must pass through every element of the $k$-partition of $D'$, and consequently has length greater or equal than $k$. All the added arcs go "backwards" in the $k$-partition, so the $V_0V_0$ distance cannot be shortened. ■

But unilateral digraphs are not the only cyclically $k$-partite digraphs with kernel, directed trees are also cyclically $k$-partite and have $k$-kernel since they are acyclical. Our next corollary continues analyzing the relation between $k$-kernels and cyclically $k$-partite digraphs. Let us recall a definition before the corollary.

**Definition 3.5.** If $D$ is a digraph, $N \subseteq V(D)$ will be called *independent by directed paths* if for every $u, v \in N$ there are not $uv$-paths in $D$. Analogously $N$ will be called *absorbent by directed paths* if for every $u \in V(D) \setminus N$ there exists $v \in N$ such that $d(u, v) \in \mathbb{N}$. If a set is independent by directed paths and absorbent by directed paths it will be called a *kernel by directed paths.*

**Corollary 3.6.** *Let $D = (V_0, V_1, \ldots, V_{k-1})$ be a cyclically $k$-partite digraph. If there exists $N \subset V_i$ for some $i \in \{0, 1, \ldots, k-1\}$ such that $N$ is absorbent by directed paths, then $D$ has a $k$-kernel.*

***Proof.*** Let $N \subseteq V_i$ be the set absorbent by directed paths in $D$. We affirm that $V_i$ is the desired $k$-kernel. Clearly, $V_i$ is independent. For the absortion we have that for every vertex $u \in V(D) \setminus V_i$ there exists a $uV_i$-directed path $\mathscr{C}$ because $N \subseteq V_i$. The digraph $D\,[V(\mathscr{C})]$ induced by the set of vertices of $\mathscr{C}$ is a unilateral cyclically $k$-partite digraph with spanning walk $\mathscr{C}$, so, by Theorem 3.3, $u$ is $(k-1)$-absorbed by $V_i$ in $D\,[V(\mathscr{C})]$ and thence is $(k-1)$-absorbed by $V_i$ in $D$. ■

**Corollary 3.7.** *Let $\mathcal{D} = \{D_i\}_{i=1}^{n}$ be a family of disjoint unilateral cyclically $k$-partite digraphs, $\mathcal{W} = \{\mathscr{W}_i\}_{i=0}^{n}$ a family of directed walks such that $\mathscr{W}_i$ is a directed spanning walk for $D_i$ and $v_i$ is the end vertex of $\mathscr{W}_i$ for every $i \in \{0, 1, \ldots, k-1\}$. If $D_0$ is a cyclically $k$-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$ and $k$-kernel $V_0$ such that $v_i \in V_0$ for every $i \in \{1, 2, \ldots, n\}$, then $\bigcup_0^n \{D_i\}$ has a kernel.*

***Proof.*** This is a direct aplication of Corollary 3.6. Just observe that $V_0$ is a kernel by directed paths for $\bigcup_0^n \{D_i\}$. ■

This last corolary was one of the first generalizations we found for non-unilateral cyclically $k$-partite digraphs, it is a star shaped digraph where each point of the star is a unilateral cyclically $k$-partite digraph, and all these digraphs converge at the $k$-kernel of another cyclically $k$-partite digraph.

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