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Note

A NOTE ON HAMILTONIAN CYCLES IN GENERALIZED HALIN GRAPHS

Magdalena Bojarska

Makowskiego 5, 02–784 Warsaw, Poland

e-mail: m.bojarska@gmail.com

Abstract

We show that every 2-connected (2)-Halin graph is Hamiltonian.

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1. Introduction

We generalize the well-known notion of a Halin graph in the following way. An (n)-*Halin graph* is a planar simple graph having the property that its edge set E can be partitioned as $E = T \cup C_1 \cup C_2 \cup \cdots \cup C_n$ where T is a tree with no vertices of degree two and $C_1, C_2, \ldots, C_n$ are pairwise disjoint cycles such that $V(C_1) \cup \cdots \cup V(C_n)$ is the set of all leaves of T (see Figure 1). Thus, (1)-Halin graphs are the usual Halin graphs. It is well known that Halin graphs are Hamiltonian, even Hamiltonian connected (see, for instance, Barefoot [1]). In this note we show that each 2-connected (2)-Halin graph is Hamiltonian.

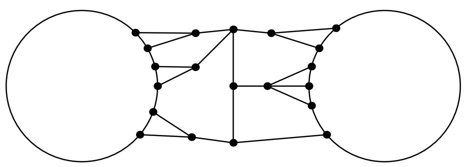

Figure 1. An example of a (2)-Halin graph.

Theorem 1. *A* (2)*-Halin graph is Hamiltonian if and only if it is* 2*-connected.*

Our proof relies on Lemma 2 below. By a rooted Halin graph we mean a planar graph F which is the union of a rooted tree T, where the root of T is a vertex of degree at least two and all other vertices, except the leaves, are of degree at least three, and a path $P = \ell_1\ell_2\cdots\ell_m$ whose vertices are precisely the leaves of T. The endvertices ℓ_1 and ℓ_m of P are called left and rights corners of H respectively.

Lemma 2 [2]**.** *Let F be a rooted Halin graph and let x, y be two different vertices from the set which consists of the root of F and its two corners. Then F contains a Hamiltonian path joining x and y.*

***Proof of Theorem* 1.** Clearly, since each Hamiltonian graph is 2-connected, we need only to prove that 2-connectivity is a sufficient condition for a (2)-Halin graph to be Hamiltonian. Let G be a (2)-Halin graph which decomposes into a tree T and two cycles C_1 and C_2, and let $\hat{G}$ be an embedding of G into the plane. Without loss of generality we may assume that in the embedding $\hat{G}$ the faces corresponding to C_1 and C_2 are both bounded. Let x_1y_1 $[x_2y_2]$ denote an edge of C_1 $[C_2]$ which belongs to the unbounded face, and let P_x and P_y denote the disjoint paths contained in T which join vertices x_1, x_2 and y_1, y_2 respectively. Note that because H is 2-connected P_x and P_y have to exist. Finally, let $P = v_1v_2\cdots v_n$, $n \geq 2$, be the unique path which joins the paths P_x and P_y in T.

Observe that if we remove P_m from the tree T, it decomposes into a number of 'rooted Halin trees', attached to vertices of the cycles C_1 and C_2. Moreover, since v_{n-1} has degree at least three, it must have a neighbor which does not lie on P; thus, without loss of generality, we may assume that it has a neighbor which is the root of a Halin tree attached to C_1. Now, using Lemma 2, we can define a Hamiltonian cycle H in G in the following way (see Figure 2). Start at the vertex y_1 and move to v_n, going through all vertices of the rooted Halin tree which contains y_1. Then go through y_2 and collect the vertices of all Halin rooted trees attached to C_2 up to v_1. Next, pass through the first $n-1$ vertices of P and then visit all vertices of the remaining Halin rooted trees attached to C_1 up to x_1 and finally, go back to y_1. ■

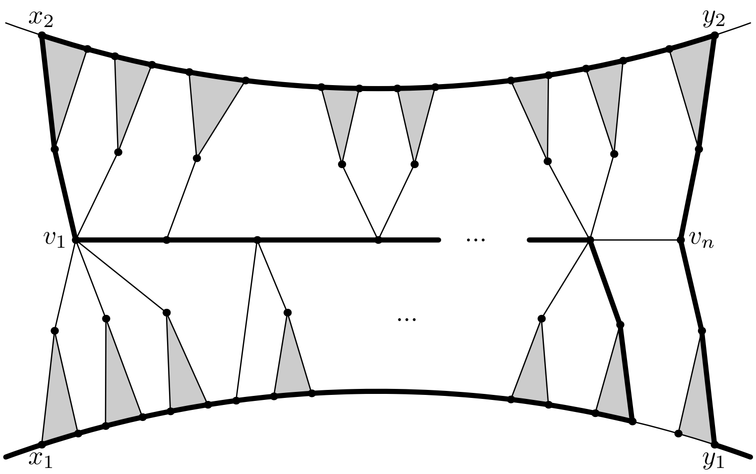

Figure 2. A construction of a Hamiltonian cycle in (2)-Halin graph.

Let us conclude the note with a few remarks. It is tempting to generalize the above result to (n)-Halin graphs and conjecture that, say, a (3)-Halin graph is Hamiltonian whenever it is 1-tough. Unfortunately, it is not the case; Figure 3 shows a 1-tough (3)-Halin graph which, as one can easily check, contains no Hamiltonian cycle. Moreover, unlike (1)-Halin graphs (which are always 3-connected), 3-connected (2)-Halin graphs are not always Hamiltonian connected (see Figure 4). Finally, we remark that from the proof of Theorem 1 it follows that a Hamiltonian cycle in (2)-Halin graph, if exists, can be found in polynomial time. It is not clear whether the same holds for (3)-Halin graph, and more generally, if there exists k such that the problem of deciding hamiltonicity of (k)-Halin graph is NP-complete.

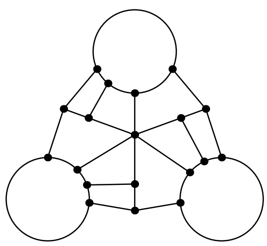

Figure 3. An example of a non-Hamiltonian 1-tough (3)-Halin graph.

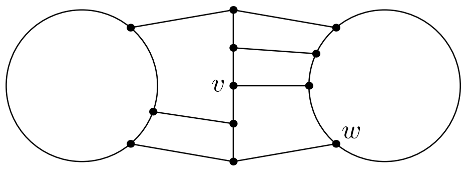

Figure 4. An example of a 3-connected (2)-Halin graph, which is not Hamiltonian connected (there are no Hamiltonian path between v and w).

References

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