

THE GEODETIC NUMBER OF STRONG PRODUCT GRAPHS *

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Abstract

For two vertices u and v of a connected graph G , the set $I_G[u, v]$ consists of all those vertices lying on $u - v$ geodesics in G . Given a set S of vertices of G , the union of all sets $I_G[u, v]$ for $u, v \in S$ is denoted by $I_G[S]$. A set $S \subseteq V(G)$ is a geodetic set if $I_G[S] = V(G)$ and the minimum cardinality of a geodetic set is its geodetic number $g(G)$ of G . Bounds for the geodetic number of strong product graphs are obtained and for several classes improved bounds and exact values are obtained.

Keywords: geodetic number, extreme vertex, extreme geodesic graph, open geodetic number, double domination number.

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1. INTRODUCTION

By a graph $G = (V(G), E(G))$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. The *distance* $d_G(u, v)$ between two vertices u and v in a

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connected graph G is the length of a shortest $u-v$ path in G . An $u-v$ path of length $d_G(u, v)$ is called an $u-v$ *geodesic*. It is known that the distance is a metric on the vertex set $V(G)$. The set $I_G[u, v]$ consists of all vertices lying on $u-v$ geodesics of G , while for $S \subseteq V(G)$, $I_G[S] = \bigcup_{u, v \in S} I_G[u, v]$. A set S of vertices of G is called a *geodetic set* of G if $I_G[S] = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set* of G . The cardinality of a minimum geodetic set of G is the *geodetic number* $g(G)$ of G . A geodetic set of cardinality $g(G)$ is a *g -set* of G . The geodetic number of a graph was introduced in [6] and further studied in [3]. The geodetic number of Cartesian product graphs was discussed in [1]. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. For a vertex v in G , $N(v)$ denotes the set of all neighbors of v , and $N[v] = N(v) \cup \{v\}$. A vertex v in G is an *extreme vertex* if the subgraph induced by $N(v)$ is complete. The set of all extreme vertices is denoted by $Ext(G)$ and $e(G) = |Ext(G)|$. A graph G is an *extreme geodesic graph* if $Ext(G)$ forms a geodetic set of G . A set $S \subseteq V(G)$ is an *open geodetic set* if for each vertex v , either (1) v is an extreme vertex of G and $v \in S$, or (2) v lies as an internal vertex of an $x-y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a *minimum open geodetic set* or *og-set* of G and this cardinality is the *open geodetic number* $og(G)$. The open geodetic number of a graph was studied in [4]. A set $S \subseteq V(G)$ is a *double dominating set* if $|N[v] \cap S| \geq 2$ for all $v \in V(G)$. A double dominating set of minimum cardinality is the *double domination number* $\gamma_{\times 2}(G)$. Any double dominating set of cardinality $\gamma_{\times 2}(G)$ is a *$\gamma_{\times 2}$ -set* of G . The double domination number of a graph was introduced and studied in [7].

The *strong product* of graphs G and H , denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent with respect to the strong product if

- (a) $x_1 = x_2$ and $y_1 y_2 \in E(H)$ or
- (b) $y_1 = y_2$ and $x_1 x_2 \in E(G)$ or
- (c) $x_1 x_2 \in E(G)$ and $y_1 y_2 \in E(H)$.

The mappings $\pi_G : (x, y) \mapsto x$ and $\pi_H : (x, y) \mapsto y$ from $V(G \boxtimes H)$ onto G and H respectively are called *projections*. For a set $S \subseteq V(G \boxtimes H)$, we define the *G -projection* on G as $\pi_G(S) = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$, and the *H -projection* $\pi_H(S) = \{y \in V(H) : (x, y) \in S \text{ for some } x \in V(G)\}$.

$V(G)\}$. For a walk $P : (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in $G \boxtimes H$, we define the G -projection $\pi_G(P)$ of P as a sequence that is obtained from (x_1, x_2, \dots, x_n) by changing each constant subsequence with its unique element. For example, if $P : (x_2, y_3), (x_2, y_4), (x_2, y_5), (x_4, y_5), (x_4, y_2), (x_3, y_2), (x_2, y_2)$, then $\pi_G(P)$ is (x_2, x_4, x_3, x_2) (it is obtained from the sequence $(x_2, x_2, x_2, x_4, x_4, x_3, x_2)$). The H -projection $\pi_H(P)$ is defined similarly. It is clear from the definition of strong product that for any walk P in $G \boxtimes H$, both $\pi_G(P)$ and $\pi_H(P)$ are walks in the factor graphs G and H respectively.

In this paper, we characterize graphs G and H for which $g(G \boxtimes H) = 2$. We obtain bounds for the geodetic number of $G \boxtimes H$ in terms of the geodetic number of the factor graphs. Improved bounds for the same are obtained for several classes of strong product graphs and exact values of $g(G \boxtimes H)$ are also obtained for some classes of graphs. Further, we characterize graphs G and H for which $g(G \boxtimes H) = e(G)e(H)$. We also obtain upper bounds for the geodetic number for some classes of strong product graphs in terms of the open geodetic number and double domination number of the factor graphs and improve the upper bounds for special classes of graphs. For basic graph theoretic terminology, we refer to [5]. We also refer to [2] for results on distance in graphs and to [8] for metric structures in strong product graphs. Throughout the following G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1 [8]. *Let G and H be connected graphs with (u, v) and (x, y) arbitrary vertices of the strong product $G \boxtimes H$ of G and H . Then $d_{G \boxtimes H}((u, v), (x, y)) = \max\{d_G(u, x), d_H(v, y)\}$.*

Theorem 1.2 [2]. *Each extreme vertex of a connected graph G belongs to every geodetic set of G .*

Theorem 1.3 [9]. *Let G and H be connected graphs. Then $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$.*

2. BOUNDS FOR THE GEODETIC NUMBER

Proposition 2.1. *Let G and H be connected graphs and P a $(u, v) - (u', v')$ geodesic in $G \boxtimes H$ of length n . If $d_G(u, u') \geq d_H(v, v')$, then $\pi_G(P)$ is a $u - u'$ geodesic in G of length n , and if $d_G(u, u') \leq d_H(v, v')$, then $\pi_H(P)$ is a $v - v'$ geodesic in H of length n .*

Proof. Let $P : (u, v) = (u_0, v_0), (u_1, v_1), \dots, (u_n, v_n) = (u', v')$ be a $(u, v) - (u', v')$ geodesic of length n in $G \boxtimes H$. If $d_G(u, u') \geq d_H(v, v')$, then it follows from Theorem 1.1 that $d_G(u, u') = n$ and so $\pi_G(P)$ must be a $u - u'$ geodesic in G . The other case follows similarly. ■

Remark 2.2. If P is a geodesic in $G \boxtimes H$, then both $\pi_G(P)$ and $\pi_H(P)$ need not be geodesics in the factor graphs G and H respectively. For the graph $G = K_{2,2}$ with partite sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $H = P_4$ with $V(H) = \{v_1, v_2, v_3, v_4\}$, it is clear from Theorem 1.1 that $P : (x_1, v_1), (y_1, v_2), (x_2, v_3), (y_2, v_4)$ is a $(x_1, v_1) - (y_2, v_4)$ geodesic in $G \boxtimes H$. However, $\pi_G(P) : x_1, y_1, x_2, y_2$ is a $x_1 - y_2$ path in G , which is not a geodesic and $\pi_H(P) : v_1, v_2, v_3, v_4$ is a geodesic in H .

Theorem 2.3. *Let G and H be nontrivial connected graphs. Then $g(G \boxtimes H) \geq 4$.*

Proof. Suppose that there is a geodetic set of $G \boxtimes H$ of cardinality 3, say $W = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$. We consider three cases.

Case 1. $x_1 = x_2 = x_3 = x$ (say). Then y_1, y_2 and y_3 are distinct. Let $x' \in V(G)$ be such that $x' \neq x$. Then it follows from Proposition 2.1 that $(x', y_1) \in I_{G \boxtimes H}[(x, y_2), (x, y_3)]$ and so $y_1 \in I_H[y_2, y_3]$. Similarly, we have $y_2 \in I_H[y_1, y_3]$ and $y_3 \in I_H[y_1, y_2]$. Thus we get a contradiction.

Case 2. $x_1 = x_2 \neq x_3$. Then $y_1 \neq y_2$. Hence $y_3 \neq y_1$ or $y_3 \neq y_2$. Assume that $y_3 \neq y_1$. Hence it follows from Proposition 2.1 that $(x_3, y_1) \in I_{G \boxtimes H}[(x_1, y_2), (x_3, y_3)]$ and so $y_1 \in I_H[y_2, y_3]$. Thus $y_2 \neq y_3$. Hence it follows similarly from Proposition 2.1 that $(x_1, y_3) \in I_{G \boxtimes H}[(x_1, y_1), (x_1, y_2)]$ and $(x_3, y_2) \in I_{G \boxtimes H}[(x_1, y_1), (x_3, y_3)]$. Again Proposition 2.1 shows that $y_3 \in I_H[y_1, y_2]$ and $y_2 \in I_H[y_1, y_3]$, which is a contradiction.

Case 3. $x_1 \neq x_2 \neq x_3$. We consider only the case $y_1 \neq y_2 \neq y_3$, since the other cases are similar to the above cases. As in the previous case, we have $(x_2, y_1) \in I_{G \boxtimes H}[(x_1, y_1), (x_3, y_3)]$ or $(x_2, y_1) \in I_{G \boxtimes H}[(x_2, y_2), (x_3, y_3)]$.

Subcase 3.1. Assume that $(x_2, y_1) \in I_{G \boxtimes H}[(x_1, y_1), (x_3, y_3)]$. Then, by Proposition 2.1, $d_G(x_1, x_3) > d_H(y_1, y_3)$ and $x_2 \in I_G[x_1, x_3]$. Again, it follows from Proposition 2.1 that $(x_1, y_2) \in I_{G \boxtimes H}[(x_2, y_2), (x_3, y_3)]$. Hence

$d_G(x_2, x_3) > d_H(y_2, y_3)$ and $x_1 \in I_G[x_2, x_3]$. Now, it is clear from Proposition 2.1 that $(x_3, y_1) \in I_{G \boxtimes H}[(x_1, y_1), (x_2, y_2)]$ and $x_3 \in I_G[x_1, x_2]$, which is a contradiction.

Subcase 3.2. This is similar to Subcase 3.1. Thus the proof is complete. ■

Theorem 2.4. *Let G and H be connected graphs and S a geodetic set of $G \boxtimes H$. Then, $\pi_G(S)$ is a geodetic set of G or $\pi_H(S)$ is a geodetic set of H .*

Proof. Suppose that both $\pi_G(S)$ and $\pi_H(S)$ are not geodetic sets of G and H respectively. Then there exist vertices x in G and y in H such that $x \notin I_G[\pi_G(S)]$ and $y \notin I_H[\pi_H(S)]$. Since S is a geodetic set of $G \boxtimes H$, there exist $(g, h), (g', h') \in S$ such that (x, y) lies on a $(g, h) - (g', h')$ geodesic P in $G \boxtimes H$. Now, it follows from Proposition 2.1 that $x \in I_G[\pi_G(S)]$ or $y \in I_H[\pi_H(S)]$, which is a contradiction. Hence $\pi_G(S)$ is a geodetic set of G or $\pi_H(S)$ is a geodetic set of H . ■

Corollary 2.5. *Let G and H be connected graphs. Then $\min\{g(G), g(H)\} \leq g(G \boxtimes H)$.*

The following theorem is useful in giving an improved lower bound of $g(G \boxtimes H)$ for a class of graphs.

Theorem 2.6. *Let G and H be connected graphs and S a geodetic set of $G \boxtimes H$. If $Ext(G) \neq \emptyset$, then $\pi_H(S)$ is a geodetic set of H .*

Proof. Let $S_1 = \pi_H(S)$. We show that S_1 is a geodetic set of H . Let $x \in Ext(G)$ and $y \in V(H)$. Since S is a g -set of $G \boxtimes H$, the vertex (x, y) lies on a geodesic $P : (g_0, h_0), (g_1, h_1), \dots, (g_i, h_i) = (x, y), \dots, (g_n, h_n)$ of length n with $(g_0, h_0), (g_n, h_n) \in S$. First, suppose that $d_G(g_0, g_n) \leq d_H(h_0, h_n)$. Then it follows from Proposition 2.1 that $\pi_H(P)$ is a $h_0 - h_n$ geodesic in H containing the vertex y , with $h_0, h_n \in S_1$. Next, suppose that $d_G(g_0, g_n) > d_H(h_0, h_n)$. Then, as above, by Proposition 2.1, $\pi_G(P)$ is a $g_0 - g_n$ geodesic in G containing the vertex x . Now, since the vertex x is extreme, either $x = g_0$ or $x = g_n$ and it follows that either $y = h_0$ or $y = h_n$. Hence S_1 is a geodetic set of H . ■

Corollary 2.7. *Let G and H be connected graphs such that $Ext(G) \neq \emptyset$. Then $g(H) \leq g(G \boxtimes H)$.*

Corollary 2.8. *Let G and H be connected graphs such that $\text{Ext}(G) \neq \emptyset$ and $\text{Ext}(H) \neq \emptyset$. Then $\max\{g(G), g(H)\} \leq g(G \boxtimes H)$.*

Corollary 2.9. *Let G be a connected graph and $m \geq 2$ an integer. Then*

- (i) $g(G) \leq g(G \boxtimes K_m)$.
- (ii) $g(G) \leq g(G \boxtimes K_{1,m})$.

The following lemma is useful in proving an upper bound for the geodetic number of $G \boxtimes H$.

Lemma 2.10. *Let G and H be connected graphs. If $g \in I_G[g', g'']$ and $h \in I_H[h', h'']$, then $(g, h) \in I_{G \boxtimes H}[S]$, where $S = \{g', g''\} \times \{h', h''\}$.*

Proof. Let g be a vertex of the geodesic $P : g' = g_0, g_1, \dots, g_i = g, \dots, g_n = g''$ in G and h a vertex of the geodesic $Q : h' = h_0, h_1, \dots, h_j = h, \dots, h_m = h''$ in H . Then $d_G(g_0, g_i) = i$ and $d_G(g_i, g_n) = n - i$ for all $0 \leq i \leq n$. Similarly, $d_H(h_0, h_j) = j$ and $d_H(h_j, h_m) = m - j$ for all $0 \leq j \leq m$. Without loss of generality, we may assume that $m \leq n$. Suppose that $(g, h) \notin I_{G \boxtimes H}[S]$. We consider two cases.

Case 1. $j \leq i$. First we show that $m - j > n - i$. Assume the contrary. Let P_1 be a $(g_0, h_0) - (g_i, h_j)$ geodesic and P_2 a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Then it follows from Theorem 1.1 that $l(P_1) = i$ and $l(P_2) = n - i$. Now, $P_3 = P_1 \cup P_2$ is a $(g_0, h_0) - (g_n, h_m)$ walk in $G \boxtimes H$, which contains (g, h) . Since $l(P_3) = n$, it follows from Theorem 1.1 that P_3 is a $(g_0, h_0) - (g_n, h_m)$ geodesic in $G \boxtimes H$ containing the vertex (g, h) , which is a contradiction to our assumption that $(g, h) \notin I_{G \boxtimes H}[S]$. Hence $m - j > n - i$. Similarly, we can show that $j > n - i$.

Now, let P' be a $(g_n, h_0) - (g_i, h_j)$ geodesic and P'' a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Since $m - j > n - i$ and $j > n - i$, it follows from Theorem 1.1 that $l(P') = j$ and $l(P'') = m - j$. Now, $P' \cup P''$ is a $(g_n, h_0) - (g_n, h_m)$ walk in $G \boxtimes H$, which contains (g, h) . Since $l(P' \cup P'') = m$, it follows from Theorem 1.1 that $P' \cup P''$ is a $(g_n, h_0) - (g_n, h_m)$ geodesic, which contains (g, h) . Thus $(g, h) \in I_{G \boxtimes H}[S]$, which is a contradiction.

Case 2. $i < j$. As in Case 1, we can prove that $n - i > m - j$ and $i > m - j$. Let Q' be a $(g_0, h_m) - (g_i, h_j)$ geodesic and Q'' a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Then, as in Case 1, we can show that $Q' \cup Q''$ is a $(g_0, h_m) - (g_n, h_m)$ geodesic, which contains (g, h) . Thus $(g, h) \in I_{G \boxtimes H}[S]$, which is a contradiction. Hence the result follows. ■

Theorem 2.11. *Let G and H be connected graphs. If S and T are geodetic sets of G and H respectively, then $S \times T$ is a geodetic set of $G \boxtimes H$.*

Proof. Let $U = S \times T$. Let $(g, h) \in V(G \boxtimes H)$. Since S and T are geodetic sets of G and H respectively, there exist $g', g'' \in S$ and $h', h'' \in T$ such that $g \in I_G[g', g'']$ and $h \in I_H[h', h'']$. Then, by Lemma 2.10, $(g, h) \in I_{G \boxtimes H}[W] \subseteq I_{G \boxtimes H}[U]$, where $W = \{g', g''\} \times \{h', h''\}$. Hence U is a geodetic set of $G \boxtimes H$. ■

Corollary 2.12. *Let G and H be connected graphs. Then $g(G \boxtimes H) \leq g(G)g(H)$.*

Theorem 2.13. *Let G and H be connected graphs. Then $\min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$.*

Proof. This follows from Corollaries 2.5 and 2.12. ■

Now, we proceed to characterize graphs G and H for which $g(G \boxtimes H) = e(G)e(H)$.

Theorem 2.14. *Let G and H be connected graphs. Then G and H are extreme geodesic graphs if and only if $G \boxtimes H$ is an extreme geodesic graph.*

Proof. Let G and H be extreme geodesic graphs. Then $Ext(G)$ and $Ext(H)$ are geodetic sets of G and H respectively. Then it follows from Theorems 1.3 and 2.11 that $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$ is a geodetic set of $G \boxtimes H$. Hence $G \boxtimes H$ is an extreme geodesic graph.

Conversely, let $G \boxtimes H$ be an extreme geodesic graph. Then $Ext(G \boxtimes H)$ is a geodetic set of $G \boxtimes H$. Then it follows from Theorems 1.3 and 2.6 that $Ext(G)$ and $Ext(H)$ are geodetic sets of G and H respectively. Thus G and H are extreme geodesic graphs. ■

Corollary 2.15. *Let G and H be connected graphs. Then G and H are extreme geodesic graphs if and only if $g(G \boxtimes H) = e(G)e(H)$.*

Proof. This follows from Theorems 1.3 and 2.14. ■

A vertex x in a set S of vertices of G is a *geodetic interior vertex* of S if $x \in I_G[S - \{x\}]$. The set of all geodetic interior vertices of S is denoted by S^o . For a geodetic set S , we have (i) $S^o \subseteq S - Ext(G)$ and (ii) $S^o = S - Ext(G)$ if and only if S is an open geodetic set of G .

Theorem 2.16. *Let G and H be connected graphs such that H has a full degree vertex v_0 . Then*

$g(G \boxtimes H) \leq \min\{|S||T| - (|T| - 1)|S^o| : S \text{ and } T \text{ are geodetic sets of } G \text{ and } H \text{ respectively}\}$. Moreover, if H is an extreme geodesic graph, then

$g(G \boxtimes H) = \min\{e(H)|S| - (e(H) - 1)|S^o| : S \text{ is a geodetic set of } G\}$.

Proof. Let S and T be geodetic sets of G and H respectively and let $W = ((S - S^o) \times T) \cup (S^o \times \{v_0\})$. Then $|W| = |S||T| - (|T| - 1)|S^o|$. We show that W is a geodetic set of $G \boxtimes H$. Let $(x, y) \in V(G \boxtimes H)$. Since T is a geodetic set of H , y lies on a $h - h'$ geodesic $P : h = h_0, h_1, \dots, h_j = y, \dots, h_m = h'$ in H with $h, h' \in T$. Now, we consider the following two cases.

Case 1. $x \in S - S^o$. Then, it follows from Theorem 1.1 that $P' : (x, h) = (x, h_0), (x, h_1), \dots, (x, h_j) = (x, y), \dots, (x, h_m) = (x, h')$ is a geodesic in $G \boxtimes H$ with $(x, h), (x, h') \in (S - S^o) \times T$. Hence $(x, y) \in I_{G \boxtimes H}[(S - S^o) \times T] \subseteq I_{G \boxtimes H}[W]$.

Case 2. $x \notin S - S^o$. Then x lies on a $g - g'$ geodesic $Q : g = g_0, g_1, \dots, g_i = x, g_{i+1}, \dots, g_n = g'$, where $1 \leq i \leq n - 1$ and $g, g' \in S$. We consider the following three subcases.

Subcase 2.1. Both $g, g' \in S - S^o$. Let $X = \{g, g'\} \times \{h, h'\}$. Then, by Lemma 2.10, $(x, y) \in I_{G \boxtimes H}[X] \subseteq I_{G \boxtimes H}[(S - S^o) \times T] \subseteq I_{G \boxtimes H}[W]$.

Subcase 2.2. Both $g, g' \notin S - S^o$. Then $g, g' \in S^o$. Since v_0 is a full degree vertex of H , it follows from Theorem 1.1 that $Q_1 : (g, v_0) = (g_0, v_0), (g_1, v_0), \dots, (g_{i-1}, v_0), (g_i, y) = (x, y), (g_{i+1}, v_0), \dots, (g_n, v_0) = (g', v_0)$ is a $(g, v_0) - (g', v_0)$ geodesic that contains the vertex (x, y) , where $(g, v_0), (g', v_0) \in S^o \times \{v_0\} \subseteq W$.

Subcase 2.3. $g \in S - S^o$ and $g' \notin S - S^o$. Then $(g, h), (g, h'), (g', v_0) \in W$. Let $y \neq h, h'$. Since $\text{diam}(H) \leq 2$ and y lies on the $h - h'$ geodesic P , it follows that y is adjacent to both h, h' . Now, it is clear from Theorem 1.1 that $Q_2 : (g, h) = (g_0, h), (g_1, y), \dots, (g_i, y) = (x, y), \dots, (g_{n-1}, y), (g_n, v_0) = (g', v_0)$ is a $(g, h) - (g', v_0)$ geodesic in $G \boxtimes H$ containing the vertex (x, y) . If $y = h$ or h' , say $y = h$, then as above (x, y) lies on a $(g, h) - (g', v_0)$ geodesic $Q_3 : (g, h) = (g_0, h), (g_1, h), \dots, (g_i, h) = (x, y), \dots, (g_{n-1}, h), (g_n, v_0) = (g', v_0)$. Thus W is a geodetic set of $G \boxtimes H$ and the first part of the theorem follows.

Now, assume that H is an extreme geodesic graph. Then $T = Ext(H)$ is a geodesic set of H . Let W_1 be a g -set of $G \boxtimes H$. Then $g(G \boxtimes H) = |W_1|$. By Theorem 2.6, $S_1 = \pi_G(W_1)$ is a geodesic set of G . We first claim that $(S_1 - S_1^o) \times T \subseteq W_1$. Let $(x, y) \in (S_1 - S_1^o) \times T$. Then $x \notin S_1^o$. If $(x, y) \notin W_1$, then there exists $(u, v), (u', v') \in W_1$ such that (x, y) lies on a $(u, v) - (u', v')$ geodesic $P : (u, v) = (u_0, v_0), (u_1, v_1), \dots, (u_i, v_i) = (x, y), \dots, (u_m, v_m) = (u', v')$ with $1 \leq i \leq m - 1$. Since y is an extreme vertex of H , it follows from Proposition 2.1 that $\pi_G(P) : u = u_0, u_1, \dots, u_i = x, \dots, u_m = u'$ is a $u - u'$ geodesic in G with $x \neq u, u'$. Thus $x \in I_G(u, u')$ with $u, u' \in S_1$ and so $x \in S_1^o$, which is a contradiction. Hence $(x, y) \in W_1$ and so $((S_1 - S_1^o) \times T) \subseteq W_1$. Let $X = W_1 - ((S_1 - S_1^o) \times T)$. Now, we claim that $S_1^o \subseteq \pi_G(X)$. Let $x \in S_1^o$. Then $x \in S_1$. Since $S_1 = \pi_G(W_1)$, there exists y such that $(x, y) \in W_1$. Since $x \notin S_1 - S_1^o$, we have $(x, y) \in X$ and so $x \in \pi_G(X)$. Thus $S_1^o \subseteq \pi_G(X)$ and so $|S_1^o| \leq |\pi_G(X)| \leq |X|$. If $|S_1^o| < |X|$, let $W_2 = ((S_1 - S_1^o) \times T) \cup (S_1^o \times \{v_0\})$. Then, as in the first part of the proof of this theorem, W_2 is a geodesic set of $G \boxtimes H$. Now, $|W_2| = |(S_1 - S_1^o) \times T| + |S_1^o| < |(S_1 - S_1^o) \times T| + |X| = |W_1|$, which is a contradiction to the fact that W is a minimum geodesic set of $G \boxtimes H$. Hence we have $|X| = |S_1^o|$ and so $|W_1| = |(S_1 - S_1^o) \times T| + |X| = |(S_1 - S_1^o) \times T| + |S_1^o| = |S_1||T| - (|T| - 1)|S_1^o|$. This completes the second part of the theorem. ■

Corollary 2.17. *Let G be a connected graph. Then*

- (i) $g(G \boxtimes K_n) = \min\{n|S| - (n - 1)|S^o| : S \text{ is a geodesic set of } G\}$,
- (ii) $g(G \boxtimes K_{1,n}) = \min\{n|S| - (n - 1)|S^o| : S \text{ is a geodesic set of } G\}$.

Corollary 2.18. *Let G and H be connected graphs such that H is an extreme geodesic graph with a full degree vertex. Then*

$$e(G)(g(H) - 1) + g(G) \leq g(G \boxtimes H) \leq e(G)(g(H) - 1) + og(G).$$

Proof. Suppose that $g(G \boxtimes H) < e(G)(g(H) - 1) + g(G)$. Then, by Theorem 2.16, there exists a geodesic set S of G such that $e(H)|S| - (e(H) - 1)|S^o| < e(G)(g(H) - 1) + g(G)$. Thus, $e(H)|S| < e(G)(g(H) - 1) + g(G) + (e(H) - 1)|S^o|$. Since $S^o \subseteq S - Ext(G)$ and $e(H) = g(H)$, we have $g(H)|S| = e(H)|S| < e(G)(g(H) - 1) + g(G) + (e(H) - 1)(|S| - e(G)) = g(G) + (g(H) - 1)|S|$. Hence $|S| < g(G)$, which is a contradiction. Thus $e(G)(g(H) - 1) + g(G) \leq g(G \boxtimes H)$. For the other inequality, let S be a minimum open geodesic set of G . Then $og(G) = |S|$ and $S^o = S - Ext(G)$. By Theorem 2.16, we have $g(G \boxtimes H) \leq e(H)|S| - (e(H) - 1)|S^o| = e(H)|S| - (e(H) - 1)(|S| - e(G)) = e(G)(e(H) - 1) + og(G)$. ■

Theorem 2.19. *Let G be a connected graph and H an extreme geodesic graph with a full degree vertex. Then $g(G \boxtimes H) = e(G)(g(H) - 1) + g(G)$ if and only if $g(G) = og(G)$.*

Proof. Suppose that $og(G) = g(G)$. Then the result follows from Corollary 2.18. Conversely, assume that $g(G \boxtimes H) = e(G)(g(H) - 1) + g(G)$. Let W be a g -set of $G \boxtimes H$. Then $|W| = e(G)(g(H) - 1) + g(G) = e(G)(e(H) - 1) + g(G)$. By Theorem 1.2 and 1.3, $W = (Ext(G) \times Ext(H)) \cup D$, where $D \subseteq V(G \boxtimes H)$ with $(Ext(G) \times Ext(H)) \cap D = \emptyset$. Hence $|D| = g(G) - e(G)$ and so $|\pi_G(W)| \leq e(G) + |\pi_G(D)| \leq e(G) + |D| = g(G)$. By Theorem 2.6, $\pi_G(W)$ is a geodetic set of G and so it follows that $|\pi_G(W)| = g(G)$. Now, we show that $\pi_G(W)$ is an open geodetic set of G . Let $x \in V(G)$ be such that $x \notin Ext(G)$. If $x \notin \pi_G(W)$, then, since $\pi_G(W)$ is a geodetic set of G , x lies as an internal vertex of a $g - g'$ geodesic in G with $g, g' \in \pi_G(W)$. Now, assume that $x \in \pi_G(W)$. First we prove that $\{x\} \times Ext(H) \not\subseteq W$. Otherwise, we have $\{x\} \times Ext(H) \subseteq W$. Then, since $Ext(G) \times Ext(H) \subseteq W$ and $\pi_G(W)$ contains $g(G) - e(G) - 1$ non-extreme vertices other than x , it follows that $|W| \geq e(H) + e(G)e(H) + (g(G) - e(G) - 1) = e(G)(e(H) - 1) + g(G) + (e(H) - 1) > e(G)(e(H) - 1) + g(G)$, which is a contradiction. Thus $\{x\} \times Ext(H) \not\subseteq W$. Hence there exists a $y \in Ext(H)$ such that $(x, y) \notin W$. Since W is a geodetic set of $G \boxtimes H$, it is clear that (x, y) lies on a $(g, h) - (g', h')$ geodesic P in $G \boxtimes H$ with $(g, h), (g', h') \in W$ and $(x, y) \neq (g, h), (g', h')$. Now, if $d_H(h, h') \geq d_G(g, g')$, then it follows from Proposition 2.1 that $\pi_H(P)$ is a $h - h'$ geodesic in H of length that of P so that y lies as an internal vertex of $\pi_H(P)$, which is a contradiction to y an extreme vertex of H . Hence, by Proposition 2.1, $\pi_G(P)$ is a geodesic in G that contains the vertex x with $x \neq g, g'$. Thus $\pi_G(W)$ is an open geodetic set of G and $|\pi_G(W)| = g(G)$. Hence $og(G) = g(G)$. ■

Theorem 2.20. *For integers $2 \leq r \leq s$ and $n \geq 2$, $g(K_{r,s} \boxtimes K_n) = 4$.*

Proof. If $r \geq 4$, then it is easily seen that $g(K_{r,s}) = og(K_{r,s}) = 4$ and so by Theorem 2.19, $g(K_{r,s} \boxtimes K_n) = 4$. If $r = 3$, then $g(K_{r,s}) = 3$ and $og(K_{r,s}) = 4$. Hence it follows from Corollary 2.18 and Theorem 2.19 that $g(K_{r,s} \boxtimes K_n) = 4$. Now, let $r = 2$. Let (X, Y) be the partite sets of $K_{2,s}$ with $|X| = 2$. Now, X and Y are geodetic sets of $K_{2,s}$. Let S be any geodetic set of $K_{2,s}$. If $S = X$ or Y , then $S^o = \emptyset$ and so $n|S| - (n - 1)|S^o| = n|S| \geq 4$. Assume that $S \neq X, Y$. Then $|S| \geq 3$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n - 1)|S^o| = 2n + 1 \geq 5$. If $|S| \geq 4$, then $S^o = S$ or $|S^o| = 1$.

If $|S^o| = 1$, then $n|S| - (n - 1)|S^o| \geq 3n + 1 \geq 7$. If $S^o = S$, then $n|S| - (n - 1)|S^o| = |S|$. Now, let $S = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then S is a geodetic set of $K_{2,s}$ with $S^o = S$. Hence it follows from Corollary 2.17 that $g(K_{2,s} \boxtimes K_n) = 4$. ■

3. GEODETIC NUMBER AND DOUBLE DOMINATION

In this section, we obtain an upper bound for the geodetic number of some strong product graphs in terms of the open geodetic number and double domination number of the factor graphs. This upper bound is also improved for certain classes of graphs.

Theorem 3.1. *Let G and H be connected graphs such that G has no extreme vertices. Then $g(G \boxtimes H) \leq og(G)\gamma_{\times 2}(H) - \min\{og(G), \gamma_{\times 2}(H)\}$.*

Proof. Let $S = \{g_1, g_2, \dots, g_p\}$ be an og -set of G and $T = \{h_1, h_2, \dots, h_q\}$ a $\gamma_{\times 2}$ -set of H . Let $r = \min\{p, q\}$ and $U = S \times T - \bigcup_{i=1}^r \{(g_i, h_i)\}$. Then $|U| = pq - r$. We show that U is a geodetic set of $G \boxtimes H$. Let $(g, h) \in V(G \boxtimes H)$. Since S is an og -set of G and G has no extreme vertices, g lies on a $g_i - g_j$ geodesic $P : g_i = u_0, u_1, \dots, u_s = g, u_{s+1} \dots, u_t = g_j$ for some $1 \leq s \leq t - 1$ with $g_i, g_j \in S$. Also, since T is a $\gamma_{\times 2}$ -set of H , it follows that h lies on a $h_k - h_l$ path $Q : h_k, h, h_l$ of length at most 2 with $1 \leq k \neq l \leq m$. Note that if $l(Q) = 1$, then either $h = h_k$ or $h = h_l$.

Case 1. $i = k$. Then $i \neq l$ and $j \neq k$. Hence $(g_i, h_l), (g_j, h_k) \in U$. It follows from Theorem 1.1 that $P' : (g_i, h_l) = (u_0, h_l), (u_1, h_l), \dots, (u_{s-1}, h_l), (u_s, h) = (g, h), (u_{s+1}, h_k), \dots, (u_t, h_k) = (g_j, h_k)$ is a geodesic in $G \boxtimes H$ that contains the vertex (g, h) . Hence U is a geodetic set of $G \boxtimes H$.

Case 2. $i \neq k$. We consider the following two subcases.

Subcase 2.1. $j = l$. Then $i \neq l$ and $j \neq k$. Then as in Case 1, U is a geodetic set of $G \boxtimes H$.

Subcase 2.2. $j \neq l$. Then $(g_i, h_k), (g_j, h_l) \in U$ and it follows from Theorem 1.1 that $P'' : (g_i, h_k) = (u_0, h_k), (u_1, h_k), \dots, (u_{s-1}, h_k), (u_s, h) = (g, h), (u_{s+1}, h_l), \dots, (u_t, h_l) = (g_j, h_l)$ is a geodesic in $G \boxtimes H$ that contains the vertex (g, h) . Hence U is a geodetic set of $G \boxtimes H$. ■

Definition 3.2. Let G be a connected graph. A double dominating set $S = \{g_1, g_2, \dots, g_p\}$ of G is *linear* if for each $g \in V(G)$, there exists an index i with $1 \leq i < n$ such that $g_i, g_{i+1} \in N[g]$.

For the graph G in Figure 3.1, the set $S = \{v_1, v_2, v_3\}$ is a linear minimum double dominating set.

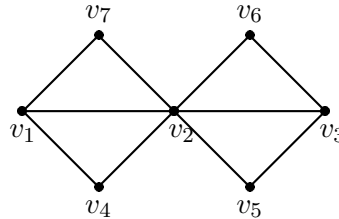


Figure 3.1. G

Any double dominating set consisting of exactly two elements is always linear. For the graph $G = K_{r,s}$ ($r = 1$ and $s \geq 3$), the set of all vertices of G is the unique double dominating set, which is not linear. For the graph $G = K_{r,s}$ ($r, s \geq 3$), let S be a set of four vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is a linear minimum double dominating set of G . The graph $K_{r,s}$ ($r = 2, s \geq 2$) does not admit a linear minimum double dominating set.

Definition 3.3. Let G be a connected graph. An open geodetic set $S = \{g_1, g_2, \dots, g_p\}$ of G is linear if for each $g \notin Ext(G)$, there exists an index i with $1 \leq i < n$ such that g lies as an internal vertex of a $g_i - g_{i+1}$ geodesic in G .

For the graph G in Figure 3.2, the set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a linear minimum open geodetic set of G .

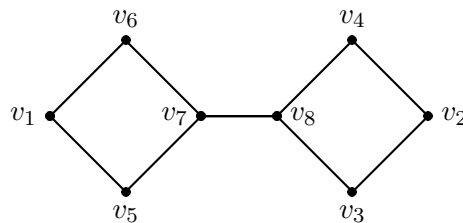


Figure 3.2. G

For the graph $G = K_{r,s}$ ($r, s \geq 2$), let S be a set of four vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is a linear minimum open geodetic set of G .

The following theorem gives an improved upper bound of Theorem 3.1.

Theorem 3.4. *Let G and H be connected graphs such that G has no extreme vertices. If G has a linear og -set and H has a linear $\gamma_{\times 2}$ -set, then*

$$g(G \boxtimes H) \leq \left\lfloor \frac{og(G) \cdot \gamma_{\times 2}(H)}{2} \right\rfloor.$$

Proof. Let $S = \{g_1, g_2, \dots, g_p\}$ be a linear og -set of G and $T = \{h_1, h_2, \dots, h_q\}$ a linear $\gamma_{\times 2}$ of H . Let $U = S \times T - \bigcup_{i+j \text{ even}} \{(g_i, h_j)\}$. Then $|U| = \lfloor \frac{pq}{2} \rfloor$. We prove that U is a geodetic set of $G \boxtimes H$. Let $(g, h) \in V(G \boxtimes H)$. Since G has no extreme vertices and S is a linear og -set of G , it follows that g lies on a g_i - g_{i+1} geodesic $P : u_0, u_1, \dots, u_s = g, u_{s+1}, \dots, u_t = g_{i+1}$ with $1 \leq s \leq t - 1$ for some $1 \leq i < p$. Also, since T is a linear $\gamma_{\times 2}$ -set of H , h lies on a h_j - h_{j+1} path $Q : h_j, h, h_{j+1}$ of length at most 2 with $1 \leq j < q$.

Suppose that $i + j$ is odd. Then $(i + 1) + (j + 1)$ is odd and so $(g_i, h_j), (g_{i+1}, h_{j+1}) \in U$. Now, it follows from Theorem 1.1 that $P' : (g_i, h_j) = (u_0, h_j), (u_1, h_j), \dots, (u_{s-1}, h_j), (u_s, h) = (g, h), (u_{s+1}, h_{j+1}), \dots, (u_t, h_{j+1}) = (g_{i+1}, h_{j+1})$ is a geodesic in $G \boxtimes H$ that contains (g, h) . Hence U is a geodetic set of $G \boxtimes H$.

Next, suppose that $i + j$ is even. Then $i + (j + 1)$ and $(i + 1) + j$ are odd and so $(g_i, h_{j+1}), (g_{i+1}, h_j) \in U$. Now, it follows from Theorem 1.1 that $P'' : (g_i, h_{j+1}) = (u_0, h_{j+1}), (u_1, h_{j+1}), \dots, (u_{s-1}, h_{j+1}), (u_s, h) = (g, h), (u_{s+1}, h_j), \dots, (u_t, h_j) = (g_{i+1}, h_j)$ is a geodesic in $G \boxtimes H$ that contains (g, h) . Hence U is a geodetic set of $G \boxtimes H$. ■

Corollary 3.5. *Let G be a connected graph such that G has no extreme vertices and G has a linear og -set. Then, for integers $r, s \geq 3$, $g(G \boxtimes K_{r,s}) \leq 2 \text{ og}(G)$. Moreover, $g(K_{r_1, s_1} \boxtimes K_{r_2, s_2}) \leq 8$ for $r_i, s_i \geq 3, i = 1, 2$.*

Proof. For the graph $K_{r,s}$ ($r, s \geq 3$), let S be a set of four vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is both a linear og -set as well as a linear $\gamma_{\times 2}$ -set of $K_{r,s}$. Hence the corollary follows from Theorem 3.4. ■

Remark 3.6. Let $r_i, s_i \geq 3$ for $i = 1, 2$. It follows from Corollary 2.12 that $g(K_{r_1, s_1} \boxtimes K_{r_2, s_2}) \leq 9$ if one of r_i or s_i is equal to 3 for $i = 1, 2$ and

$g(K_{r_1, s_1} \boxtimes K_{r_2, s_2}) \leq 16$ for all $r_i, s_i \geq 4$ for $i = 1, 2$. However, Corollary 3.5 gives a better bound for $g(K_{r_1, s_1} \boxtimes K_{r_2, s_2})$.

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