

RAINBOW NUMBERS FOR SMALL STARS WITH ONE EDGE ADDED

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Abstract

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. For a graph H and a positive integer n , the *anti-Ramsey number* $f(n, H)$ is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of H . The *rainbow number* $rb(n, H)$ is the minimum number of colors such that any edge-coloring of K_n with $rb(n, H)$ number of colors contains a rainbow copy of H . Certainly $rb(n, H) = f(n, H) + 1$. Anti-Ramsey numbers were introduced by Erdős *et al.* [5] and studied in numerous papers.

We show that $rb(n, K_{1,4} + e) = n + 2$ in all nontrivial cases.

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1. INTRODUCTION

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. For a graph H and a positive integer n , the *anti-Ramsey number* $f(n, H)$ is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of H . The *rainbow number* $rb(n, H)$ is the minimum number of colors such that any edge-coloring of K_n with $rb(n, H)$ number of colors contains a rainbow copy of H . Certainly $rb(n, H) = f(n, H) + 1$. Anti-Ramsey numbers were introduced by Erdős *et al.* [5]. They showed that these are closely related to Turán numbers. The *Turán number* $ex(n, \mathcal{H})$ of

\mathcal{H} is the maximum number of edges of an n -vertex simple graph having no member of \mathcal{H} as a subgraph.

For a given graph H let $\mathcal{H} = \{H - e : e \in E(H)\}$. Erdős *et al.* [5] showed $f(n, H) - ex(n, \mathcal{H}) = o(n^2)$, as $n \rightarrow \infty$. If $d = \min\{\chi(G) : G \in \mathcal{H}\} \geq 3$, then by an earlier result of Erdős and Simonovits [4], we have $ex(n, \mathcal{H}) = \frac{d-2}{d-1} \binom{n}{2} + o(n^2)$. So the above theorem determines rainbow numbers asymptotically in that case. If $d \leq 2$, however, we have $ex(n, \mathcal{H}) = o(n^2)$ and the above theorem says little about rainbow numbers. Therefore it was suggested by Erdős *et al.* [5] to study Ramsey numbers for graphs that contain an edge whose deletion leaves a bipartite subgraph and put forward two conjectures about paths and cycles.

Simonovits and Sós proved the conjecture for paths determining $f(n, P_k)$ for n large enough [15]. As for the conjecture for cycles, they proved it for C_3 by themselves. For C_4 it was proved by Alon [1] and for C_5 and C_6 independently by Jiang and West [9] and by Schiermeyer [13] and completely solved by Montellano-Ballesteros and Neuman-Lara [12].

Moreover rainbow numbers were studied for complete bipartite graphs by Axenovich and Jiang [2], for trees by Jiang and West [10], for subdivided graphs by Jiang [7] and for complete graphs and matchings by Schiermeyer [14]. Recently cycles with an edge added were studied by Montellano-Ballesteros [11] and Gorgol [6].

The aim of the paper is to prove Theorem 4 which says that we need $n + 2$ colors to be sure that in any coloring of the edges of K_n with this number of colors we always obtain a rainbow $K_{1,4} + e$.

2. PRELIMINARIES

Graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation (see, e.g., [3]). In particular, $G \cup H$, K_n and $K_{1,r}$ stand, respectively, for disjoint sum of graphs G and H , the complete graph on n vertices and a star with r rays. A graph K_3 we call a triangle. For a graph G and its subgraph H by $G - H$ we mean a graph obtained from G by deleting all vertices of H . For a set S by $|S|$ we denote the cardinality of S .

We will need the following theorems.

Theorem 1 [5]. $rb(n, K_3) = n$ for $n \geq 3$.

Theorem 2 [8]. *Given positive integers n and r , where $r \leq n - 2$, $rb(n, K_{1,r+1}) = \lfloor \frac{1}{2}n(r - 1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor + \varepsilon$, where $\varepsilon = 1$ or 2 if n is odd, r is even and $\lfloor \frac{2n}{n-r+1} \rfloor$ is odd; $\varepsilon = 1$ otherwise.*

We will also need some lemmas to conduct the inductive proof of Theorem 4. Throughout all proofs we use the following notation. $C(G)$ is a set of colors used on the edges of a graph G ; $C(G, H)$ is a set of colors used on the edges with one endvertex in the vertex-set of a graph G and the other in the vertex-set of a graph H ; $C(v)$ is a set of colors used on the edges incident to a vertex v and $c(e)$ denotes the color of the edge e .

In a graph $K_{1,4} + e$ a vertex of degree 4 we call a center.

A vertex v is called *monochromatic* if $|C(v)| = 1$.

We call a color c *private* for a vertex v if all edges of color c are incident to v .

Claim 1. If a color c is private for two vertices v and w then an edge vw is the only edge of color c .

Proof. It follows immediately from the definition of a private vertex. □

Claim 2. An arbitrary color can be private for at most two vertices.

Proof. It is a straightforward consequence of Claim 1. □

For a fixed coloring of the edges of $K_n = K$ we construct a bipartite graph B with bipartition sets V and C as follows. Let $V = V(K)$ and $C = C(K)$. We put an edge between $v \in V$ and $c \in C$ if and only if c is private for v . Note that by Claim 2 each vertex from C has degree at most 2.

Note that by Claim 1, B cannot contain C_4 .

Lemma 1. $rb(n, K_{1,4} + e) = n + 2$ for $n \in \{5, 6\}$.

Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. We color the edges of $K_n = K$ with $n + 2$ colors arbitrarily and show that there exists a rainbow $K_{1,4} + e$. By Theorem 2 there exists a rainbow $K_{1,4} = S$. The existence of a rainbow $K_{1,4} + e$ is obvious for $n \in \{5, 6\}$. ■

3. MAIN RESULTS

Although the next theorem is proved in a more general case in [6], we state it here with a proof to make the paper self-contained.

Theorem 3. $rb(n, K_{1,3} + e) = n$ for $n \geq 4$.

Proof. By Theorem 1 we have $rb(n, K_{1,3} + e) \geq n$ for $n \geq 4$. So we have to prove the opposite inequality. It is easy to check it for $n \in \{4, 5, 6\}$. Therefore let $n \geq 7$. We color the edges of $K_n = K$ with n colors. By Theorem 1 there exists a rainbow triangle T with the set of colors $C(T)$. If the condition $C(T, K_n - T) \cap (C(K) - C(T)) \neq \emptyset$ holds then there exists a rainbow $K_{1,3} + e$. Otherwise $|V(K_n - T)| = n - 3$ and $|C(K_n - T)| \geq n - 3$ and we have a rainbow copy of $K_{1,3} + e$ in $K_n - T$ by induction. ■

Theorem 4. $rb(n, K_{1,4} + e) = n + 2$ for $n \geq 5$.

Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. The proof will be conducted by induction with respect to n . For $n \in \{5, 6\}$ it is Lemma 1.

Therefore let $n \geq 7$. We color the edges of $K_n = K$ with $n + 2$ colors arbitrarily and construct an appropriate bipartite graph B .

If there exists a vertex v such that $|C(K_n) - C(v)| \geq n + 1$ then $K_n - v$ is a K_{n-1} colored with at least $n + 1$ colors, so a rainbow $K_{1,4} + e$ exists by induction. A contradiction. Therefore for each vertex $v \in V(K_n)$ there exist at least two private colors. So each vertex from V has degree at least 2.

Before the next part of the proof we will show the following two facts.

Fact 1. If there is an isolated vertex $c_0 \in C$ in B then K contains a rainbow $K_{1,4} + e$.

Proof of Fact 1. Assume there is an isolated vertex $c_0 \in C$ in B , but there is not any rainbow $K_{1,4} + e$ in K .

By Claim 2 each vertex from C has degree at most 2. By Claim 1 it means that in coloring of K the respective colors appear exactly once. Hence at least $n - 1$ vertices from C have degree exactly 2.

For each $v \in V$ choose exactly two private colors at v , and consider a subgraph B' of B with $V(B') = V(B) - \{c_0\}$, but with an edge between $v \in V$ and $c \in C$ if c is one of the two chosen private colors at v . Thus

$E(B') \subset E(B)$, $|E(B')| = 2n \leq |E(B)| \leq 2n + 2$ and the maximal degree in B' is 2.

The above degree conditions determine the structure of B' . Namely B' consists of a path with an odd number ≥ 1 of vertices, starting and ending in C , and of zero or more cycle components. Without loss of generality this path could be assumed to be $c_r v_r c_{r+1} \dots v_n c_{n+1}$, where $1 \leq r \leq n + 1$ ($r = n + 1$ if and only if c_{n+1} is an isolated vertex in B').

Note that the graph on $V(K)$ consisting only of the edges colored in the three or two colors c_0, c_r, c_{n+1} is connected. Thus there is some vertex v_i with at least two or three colors occurring among its edges: it is easy to see that we may choose $v_i \notin \{v_r, v_n\}$, if $r \leq n$. Hence, in fact, either v_i belongs to a 6-cycle $v_{i-1}c_i v_i c_{i+1} v_{i+1} c'$ or to a path $v_{i-1}c_{i-1} v_i c_i v_{i+1} c_{i+1} v_{i+2}$ in B' . In either case, $|C(v_i)| \geq 4$. In the case of C_6 , v_i would be a center of a rainbow $K_{1,4} + e$, where e would be colored with c' . Similarly in the case of the path, v_i would be a center of a rainbow $K_{1,4} + e$, where $e = v_i v_{i+2}$. Indeed, let $c(v_i v_{i+2}) = c''$ be the color of the edge $v_i v_{i+2}$. It can be c_0 or c_{n+1} (if $i + 2 = n$) and by the choice of v_i there is another edge coming out of it of color from $\{c_0, c_r, c_{n+1}\} \setminus \{c''\}$. \square

Fact 2. If K contains two disjoint rainbow triangles, then K contains a rainbow $K_{1,4} + e$.

Proof of Fact 2. Let T_1 and T_2 be these triangles. Note that if there is $c \in C(T_1) \cap C(T_2)$ then indeed c is not private at any vertex in K and thus is isolated in B , whence then Fact 1 applies. Therefore we have to consider the case when $T_1 \cup T_2$ is a rainbow $2K_3$.

Let $V(T_1) = \{x, y, z\}$, $V(T_2) = \{a, b, c\}$, $C(T_1) = \{c_1, c_2, c_3\}$ and $C(T_2) = \{c_4, c_5, c_6\}$.

If $|C(T_1) \cup C(T_2) \cup C(T_1, T_2)| \geq 8$ then Fact 2 follows from Lemma 1 so we can assume that $|C(T_1) \cup C(T_2) \cup C(T_1, T_2)| \leq 7$ which means that there can be at most one color in $C(T_1, T_2)$ not belonging to $C(T_1) \cup C(T_2)$. Let $K' = K - (T_1 \cup T_2)$.

Suppose that there exists an edge e between the triangles T_1 and T_2 of the color not belonging to $C(T_1) \cup C(T_2)$. Without loss of generality we can assume that $e = xa$ and $c(e) = c$. Let $C_R = C(K) - (C(T_1) \cup C(T_2) \cup \{c\})$. Note that either we have a rainbow $K_{1,4} + e$ or $c(xv) \in C(T_1) \cup \{c\}$ for all vertices $v \in V(K - T_1)$ and $c(av) \in C(T_2) \cup \{c\}$ for all vertices $v \in V(K - T_2)$. If there is at least one edge between $T_1 \cup T_2$ and K' of color from C_R , say yw , where $w \in V(K')$, then we obtain a rainbow $K_{1,4} + e$. It is the triangle

T_1 with edges ya and yw . Note that surely it is the case for $n = 9$ since $|C_R| = 4$ and $|E(K')| = 3$. If such an edge does not exist it means that $n \geq 10$, $C(T_1 \cup T_2, K') \subset (C(T_1) \cup C(T_2) \cup \{c\})$ and all colors from C_R are used on edges of K' . If there is a rainbow $K_{1,3} + e$ in K' then it gives a rainbow $K_{1,4} + e$ together with one edge coming to $T_1 \cup T_2$. Note that obviously it is the case for $n = 10$ and for $n \geq 11$ and $C(K') = C_R$ it follows from Theorem 3. If $|C(K')| > |C_R|$ for $n \geq 11$ we obtain a rainbow $K_{1,4} + e$ by induction.

Therefore we assume that $C(T_1, T_2) \subset C(T_1) \cup C(T_2)$. Let $C_R = C - (C(T_1) \cup C(T_2))$. If there is at least one edge between $T_1 \cup T_2$ and K' of color from C_R , say xw , where $w \in V(K')$, then all edges coming from x to T_2 are of colors from $C(T_1)$ otherwise we get a rainbow $K_{1,4} + e$. As a further consequence we get that either there is a rainbow $K_{1,4} + e$ in K or all edges coming out from T_2 are of colors from $C(T_1) \cup C(T_2)$. In the latter case the graph $K - T_2$ is colored with at least $n - 1$ colors so the induction completes the proof. Note that surely it is the case for $n \in \{7, 8, 9\}$. So we can assume that $n \geq 10$, $C(T_1 \cup T_2, K') \subset (C(T_1) \cup C(T_2))$ and all colors from C_R are used on edges of K' . Repeating the arguments from the previous part of the proof we prove the fact. \square

Now we are ready to finish the proof of Theorem 4.

By Theorem 1 there exists a rainbow triangle T_1 with the vertex-set $\{x, y, z\}$ and the set of colors $C(T_1)$.

Let $K' = K - T_1$. Note that if K' contains a rainbow triangle then K contains a rainbow $K_{1,4} + e$ by Fact 2. Assume then it is not the case. Then $|C(K')| \leq n - 4$ by Theorem 1.

Let $C_R = C(K) - C(T_1)$. Note that if there is a vertex v in T_1 with $|C(v) \cap C_R| \geq 2$, then there is a rainbow $K_{1,4} + e$ with center v and containing T_1 . Hence the converse can be assumed.

So we are to consider only the case $|C(K')| = n - 4$, $|C(v) \cap C_R| = 1$ and the colors $C(v) \cap C_R$ are distinct for each $v \in \{x, y, z\}$.

Then certainly $|C(T_1, K') \cap C_R| = 3$ and $C(K') \cap (C(T_1) \cup C(T_1, K')) = \emptyset$. Now either we have a rainbow $K_{1,4} + e$ or each edge between T_1 and K' of the color from C_R comes out from a different vertex of T_1 . If such an edge of color c comes out, say from x , to a vertex a which is not monochromatic in K' then $c(ay) = c(xy)$ and $c(za) = c(xz)$ or we have a rainbow $K_{1,4} + e$. But in this case we also get a rainbow $K_{1,4} + e$. It is a rainbow triangle ayz with two edges coming out from the vertex a .

It is easy to note that there can be at most one monochromatic vertex in K' . If there would be at least two such vertices a and b then $C(a) \cap C(K') = C(b) \cap C(K') = \{c(ab)\}$ and so $K' - \{a, b\}$ would be K_{n-5} colored with $n-5$ colors, against the assumption for K' .

Hence the vertex a is monochromatic in K' and all edges between T_1 and K' of the three colors in $C(T_1, K') \cap C_R$ have a as an endpoint. Thus $|C(a)| = 4$ and there is a rainbow $K_{1,4} + e$ with the center a and containing an edge from T_1 . ■

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