

A CHARACTERIZATION OF (γ_t, γ_2) -TREES *

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Abstract

Let $\gamma_t(G)$ and $\gamma_2(G)$ be the total domination number and the 2-domination number of a graph G , respectively. It has been shown that: $\gamma_t(T) \leq \gamma_2(T)$ for any tree T . In this paper, we provide a constructive characterization of those trees with equal total domination number and 2-domination number.

Keywords: domination, total domination, 2-domination, (λ, μ) -tree.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of a vertex $v \in V(G)$ are denoted by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $\deg_G(v) = |N_G(v)|$, respectively. For $u \in V(G)$, u is a leaf of G if $\deg_G(u) = 1$ and a support vertex of G if u has a leaf as its neighbor in G . For a pair of vertices $u, v \in V(G)$, the distance $d_G(u, v)$ of u and v is the length of a shortest uv -path in G . The diameter of G is $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}$.

For any set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$ and we write $G - S$ for $G[V(G) - S]$. For convenience, we write $G - v$ for

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$G - \{v\}$ for $v \in V(G)$. For any edge $xy \in E(G)$, we use $G - xy$ to denote the subgraph induced by $E(G) - \{xy\}$.

Total domination in graphs was introduced by Cockayne *et al.* [3]. A subset $S \subseteq V(G)$ is a total dominating set (denoted by TDS) if every vertex of $V(G)$ has at least one neighbor in S . The total domination number (denoted by $\gamma_t(G)$) is the minimum cardinality among the total dominating sets of G . The total dominating set of G with cardinality $\gamma_t(G)$ will be called a γ_t -set of G . For a survey on total domination in graphs one can refer to Henning [12].

Let p be a positive integer. In [6], Fink and Jacobson introduced the concept of p -domination. A p -dominating set of G is a subset S of $V(G)$ such that every vertex not in S has at least p neighbors in S . The p -domination number $\gamma_p(G)$ is the minimum cardinality of a p -dominating set of G . The p -dominating set of G with cardinality $\gamma_p(G)$ will be called a γ_p -set of G . Note that p -domination is the classic domination when $p = 1$. For any $S, T \subseteq V(G)$, S p -dominates T in G if every vertex of T not in S has at least p neighbors in S .

An area of research in domination of graphs that has received considerable attention is the characterization of classes of graphs with equal domination parameters. For any two graph parameters λ and μ , G is called a (λ, μ) -graph if $\lambda(G) = \mu(G)$. Characterizing the (λ, μ) -graphs has been investigated in many papers (for example [1, 4, 7, 11, 13]).

In [8], Haynes *et al.* showed that for all trees the total domination number is equal or less than the 2-domination number, and they also gave a necessary condition for all trees with equal total domination number and 2-domination number. In this paper, we give a constructive characterization of trees with equal total domination number and 2-domination number.

2. A CHARACTERIZATION

Let $P_n = u_1 \cdots u_n$ ($n \geq 1$) be a path with vertex set $\{u_1, \dots, u_n\}$ and $K(t)$ ($t \geq 2$) be the tree obtained from a star $K_{1,t}$ with support vertex u by adding a path P_2 to every leaf of $K_{1,t}$. Denote u by $cent(K(t))$. For convenience, we denote a path P_4 by $K(1)$ and let $cent(K(1))$ represent one leaf of P_4 .

To state the characterization of (γ_t, γ_2) -trees, we introduce the six types of operations.

Type-1 operation: Attach a path P_1 to each of the two vertices u, w of a tree T , respectively, where u, w locate at a component P_l of $T - xy$ for some edge xy such that either x is in a γ_2 -set of T and $P_l = P_4 = uvwx$ or y is in a γ_2 -set of T and $P_l = P_5 = uvwx'$.

Type-2 operation: Attach a path P_2 to a vertex v of a tree T by joining one leaf of P_2 to v , where v is a vertex such that $T - v$ has a component P_2 .

Type-3 operation: Attach t (≥ 1) paths P_3 to a vertex v of a tree T by joining one leaf of each P_3 to v , where v is a vertex such that either $T - v$ has a component P_2 or $T - v$ has two components P_1 and P_3 that a leaf of P_3 is adjacent to v in T .

Type-4 operation: Attach a path P_3 to a vertex v of a tree T by joining its support vertex to v , where v is a vertex such that v is not contained in any γ_t -set of T and $T - v$ has a component P_3 that one of its leaves is adjacent to v in T .

Type-5 operation: Attach a tree $K(t)$ ($t \geq 1$) to a vertex v of a tree T by joining $cent(K(t))$ to v , where v is in a γ_2 -set of T if $t = 1$.

Type-6 operation: Attach a path P_5 to a vertex v of a tree T by joining one of its support vertices to v , where v is a vertex such that $T - v$ has a component $H \in \{P_2, P_3, P_5\}$ and v is adjacent to a support vertex of H if $H = P_5$.

From the survey on total domination in graphs [12], it is hard to recognize whether a vertex v is in no γ_t -set or no γ_2 -set.

Let \mathcal{A} be the family of trees with equal total domination number and 2-domination number, that is

$$\mathcal{A} = \{T : T \text{ is a tree satisfying } \gamma_t(T) = \gamma_2(T)\}.$$

We also define the family \mathcal{B} as:

$$\mathcal{B} = \{T : T \text{ is obtained from } P_3 \text{ by a finite sequence of operations of Type-}i, \text{ where } 1 \leq i \leq 6\}.$$

We shall show that

Theorem 1. $\mathcal{A} = \mathcal{B} \cup \{P_2\}$.

3. THE PROOF OF THEOREM 1

We need some known results.

Lemma 2 ([8]). *Let T be a tree without isolated vertices, then $\gamma_t(T) \leq \gamma_2(T)$.*

Lemma 3 ([2]). *Every 2-dominating set of a graph G contains all leaves of G .*

Lemma 4 ([8]). *If T is a tree satisfying $\gamma_t(T) = \gamma_2(T)$, then every support vertex of T is adjacent to at most two leaves.*

Let T be a rooted tree. For every $v \in V(T)$, let $C(v)$ and $D(v)$ denote the set of children and descendants of v , respectively, and $D[v] = D(v) \cup \{v\}$. Define

$$C'(v) = \{u \in C(v) : \text{every vertex of } D[u] \text{ has distance at most two from } v \text{ in } T\}.$$

By Lemma 4, each vertex of $C'(v)$ has degree at most three. Hence we can partition $C'(v)$ into $C'_1(v), C'_2(v), C'_3(v)$ such that every vertex of $C'_i(v)$ has degree i in T , $i = 1, 2$ or 3 .

Lemma 5. *Let T be a rooted tree satisfying $\gamma_t(T) = \gamma_2(T)$ and $w \in V(T)$. We have*

- (1) *If $C'_2(w) \neq \emptyset$, then $C'_1(w) = C'_3(w) = \emptyset$.*
- (2) *If $C'_3(w) \neq \emptyset$, then $C'_1(w) = C'_2(w) = \emptyset$ and $|C'_3(w)| = 1$.*
- (3) *If $C(w) = C'(w) \neq C'_1(w)$, then $C'_1(w) = C'_3(w) = \emptyset$.*

Proof. Let $C'_1(w) = \{x_1, \dots, x_r\}$, $C'_2(w) = \{y_1, \dots, y_s\}$ and $C'_3(w) = \{z_1, \dots, z_t\}$. Then $|C'_1(w)| = r$, $|C'_2(w)| = s$ and $|C'_3(w)| = t$. For each $i = 1, \dots, t$, let u_i be a leaf adjacent with z_i in T . Let $T' = T - \{x_1, \dots, x_r, u_1, \dots, u_t\}$.

(1). We prove that if $s \geq 1$ then $r + t = 0$. Assume $r + t \geq 1$. Since $s \geq 1$, we can choose a γ_2 -set D of T such that $w \in D$, and a γ_t -set S' of T' such that $w \in S'$. It is not difficult to check that $D - \{x_1, \dots, x_r, u_1, \dots, u_t\}$ is a 2-dominating set of T' and S' is a TDS of T . Hence,

$$\begin{aligned} \gamma_t(T') &= |S'| \geq \gamma_t(T) = \gamma_2(T) \\ &= |D| > |D - \{x_1, \dots, x_r, u_1, \dots, u_t\}| \geq \gamma_2(T'), \end{aligned}$$

a contradiction with Lemma 2.

(2) and (3). Suppose either $C'_3(w) \neq \emptyset$ or $C(w) = C'(w) \neq C'_1(w)$. Then $s + t \geq 1$. Choose a γ_t -set S' of T' such that $w \in S'$. Then S' is also a TDS of T . Hence $\gamma_t(T') = |S'| \geq \gamma_t(T)$. By the definition of γ_2 -set and Lemma 3, there is a γ_2 -set, denoted by D , of T satisfying $D \cap \{y_1, \dots, y_s, z_1, \dots, z_t\} = \emptyset$. Then $(D \cap V(T')) \cup \{w\}$ is a 2-dominating set of T' . Hence

$$\begin{aligned} \gamma_2(T') &\leq |(D \cap V(T')) \cup \{w\}| \\ &\leq |D| - (r + t) + 1 \\ &= \gamma_2(T) - (r + t) + 1 \\ &= \gamma_t(T) - (r + t) + 1. \end{aligned}$$

If $t \geq 1$, then $\gamma_2(T') \leq \gamma_t(T) \leq \gamma_t(T') \leq \gamma_2(T')$, the last inequality is by Lemma 2, which implies that $r + t = 1$ and $w \notin D$. So $r = 0$ and $t = 1$. By (1), we have $s = 0$. Hence (2) is valid.

If $C(w) = C'(w) \neq C'_1(w)$, then $s + t \geq 1$. By (1) and (2), $r = 0$. We show that $t = 0$. If not, similar to the proof of (2), we have $w \notin D$, $t = 1$ and $s = 0$. Since $C(w) = C'(w)$, we know that $\text{deg}_T(w) = 2$. To 2-dominate w , $z_1 \in D$, which contradicts with the choice of D . ■

Lemma 6. *If $T' \in \mathcal{A}$ with order at least three and T is obtained from T' by an operation of Type- i , $1 \leq i \leq 6$, then $T \in \mathcal{A}$.*

Proof. Since $T' \in \mathcal{A}$, we have $\gamma_t(T') = \gamma_2(T')$. By Lemma 2, we only need to prove that $\gamma_t(T) \geq \gamma_2(T)$.

Case 1. $i = 1$. Assume that T is obtained from T' by attaching u' and w' to u and w , respectively, where u and w satisfy the conditions of Type-1 operation. Then there is an edge xy in T' such that either x is in a γ_2 -set of T' and $T' - xy$ has a component $P_4 = uvwx$, or y is in a γ_2 -set of T' and $T' - xy$ has a component $P_5 = uvwx'$. Clearly, $\gamma_t(T') = \gamma_t(T) - 1$.

If $T' - xy$ contains a path $P_4 = uvwx$, then let D' be a γ_2 -set of T' containing x . From Lemma 3 and the definition of γ_2 -set, we have $D' \cap$

$\{u, v, w, x\} = \{u, w\}$ or $\{u, v\}$. Thus $D = (D' - \{u, v, w\}) \cup \{u', v, w'\}$ is a 2-dominating set of T with $|D| = |D'| + 1 = \gamma_2(T') + 1$. So, $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D| \geq \gamma_2(T)$.

If $T' - xy$ contains a path $P_5 = uvwx'$, then let D' be a γ_2 -set of T' containing y . By Lemma 3 and the definition of γ_2 -set, we have $D' \cap \{u, v, w, x, x'\} = \{u, w, x'\}$. Thus $D = (D' \setminus \{u, w\}) \cup \{u', v, w'\}$ is a 2-dominating set of T with $|D| = |D'| + 1 = \gamma_2(T') + 1$. So, $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D| \geq \gamma_2(T)$.

Case 2. $i = 2$. Assume that T is obtained from T' by attaching a path $P_2 = uu'$ to a vertex v of T' such that $uv \in E(T)$, where $T' - v$ has a component $P_2 = wx$ satisfying $vw \in E(T')$. It is easy to show that $\gamma_t(T) = \gamma_t(T') + 1$. By the definition of γ_2 -set, there exists a γ_2 -set D' of T' containing the vertex v . Then $D' \cup \{u'\}$ is a 2-dominating set of T . Hence, $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D' \cup \{u'\}| \geq \gamma_2(T)$.

Case 3. $i = 3$. Assume that T is obtained from T' by attaching t (≥ 1) paths P_3 , denoted by $\{x_i y_i z_i : 1 \leq i \leq t\}$, to a vertex v of T' such that $x_i v \in E(T)$ for $1 \leq i \leq t$, where either $T' - v$ has a component $P_2 = uu'$ satisfying $uv \in E(T')$, or $T' - v$ has a component $P_1 = u_0$ and a component $P_3 = uu'u''$ satisfying $uv \in E(T')$. By the definitions of γ_t -set and γ_2 -set, we can easily prove that $\gamma_t(T) \geq \gamma_t(T') + 2t$ and $\gamma_2(T') + 2t \geq \gamma_2(T)$. Since $\gamma_t(T') = \gamma_2(T')$, we have $\gamma_t(T) \geq \gamma_t(T') + 2t = \gamma_2(T') + 2t \geq \gamma_2(T)$.

Case 4. $i = 4$. Assume that T is obtained from T' by attaching a path $P_2 = xyz$ to a vertex v of T' such that $yv \in E(T)$, where v is not in any γ_t -set of T' and $T' - v$ has a component $P_3 = uu'u''$ satisfying $uv \in E(T')$. For any γ_2 -set D' of T' , $D' \cup \{x, z\}$ is a 2-dominating set of T . So $\gamma_2(T') + 2 \geq \gamma_2(T)$. Let S be a γ_t -set of T containing the vertex u , then $y \in S$ and $|S \cap \{v, x, z\}| = 1$.

If $v \notin S$, then $|S \cap V(T')| = |S| - 2 = \gamma_t(T) - 2 \geq \gamma_t(T')$ since $S \cap V(T')$ is a TDS of T' . By $\gamma_t(T') = \gamma_2(T')$, $\gamma_t(T) \geq \gamma_t(T') + 2 = \gamma_2(T') + 2 \geq \gamma_2(T)$.

If $v \in S$, then $S \cap \{v, x, z\} = \{v\}$ and $|S \cap V(T')| = |S| - 1 = \gamma_t(T) - 1 \geq \gamma_t(T')$ since $S \cap V(T')$ is a TDS of T' . Suppose that $\gamma_t(T) \leq \gamma_2(T) - 1$, then, by $\gamma_t(T') = \gamma_2(T')$, $\gamma_2(T) \geq \gamma_t(T) + 1 \geq \gamma_t(T') + 2 = \gamma_2(T') + 2 \geq \gamma_2(T)$. So $|S \cap V(T')| = \gamma_t(T) - 1 = \gamma_t(T')$, and $S \cap V(T')$ is a γ_t -set of T' containing v , which contradicts with v is not in any γ_t -set of T' . Hence $\gamma_t(T) \geq \gamma_2(T)$.

Case 5. $i = 5$. Assume that T is obtained from T' by attaching a $K(t)$ ($t \geq 1$) to a vertex v of T' by joining $u = \text{cent}(K(t))$ to v , where v satisfies the condition of Type-5 operation. Clearly, $\gamma_t(T) \geq \gamma_t(T') + 2t$.

If $t \geq 2$, then, by $\gamma_t(T') = \gamma_2(T')$, it is obvious that $\gamma_t(T) \geq \gamma_t(T') + 2t = \gamma_2(T') + 2t \geq \gamma_2(T)$.

If $t = 1$, then let $K(1) = uxyz$ and D' be a γ_2 -set of T' containing v . Thus $D' \cup \{z, x\}$ is a 2-dominating set of T . Hence $\gamma_t(T) \geq \gamma_t(T') + 2 = \gamma_2(T') + 2 = |D' \cup \{z, x\}| \geq \gamma_2(T)$.

Case 6. $i = 6$. Assume that T is obtained from T' by attaching a path $P_5 = x_1x_2x_3x_4x_5$ to a vertex v of a tree T such that $x_2v \in E(T)$, where T' and v satisfy the condition of Type-6 operation. Then we can choose a subset S of $V(T)$ as a γ_t -set of T such that $S \cap N_{T'}(v) \neq \emptyset$. Thus $S \cap V(T')$ is a TDS of T' and then $|S \cap V(T')| \geq \gamma_t(T')$. By the definition of γ_2 -set, we have $\gamma_2(T') + 3 \geq \gamma_2(T)$. Hence $\gamma_t(T) = |S| = |S \cap V(P_5)| + |S \cap V(T')| \geq 3 + \gamma_t(T') = 3 + \gamma_2(T') \geq \gamma_2(T)$. ■

Lemma 7. *If $T \in \mathcal{A}$ with order at least three, then $T \in \mathcal{B}$.*

Proof. Let $n = |V(T)|$. Since $T \in \mathcal{A}$, we have $\gamma_t(T) = \gamma_2(T)$. If $d(T) = 2$, then T is a star $K_{1, n-1}$. Since $2 = \gamma_t(T) = \gamma_2(T) = n - 1$, $n = 3$. So $T = P_3 \in \mathcal{B}$. If $d(T) = 3$, then T contains exactly $n - 2$ leaves. Since $2 = \gamma_t(T) = \gamma_2(T) \geq n - 2$, $n = 4$. So $T = P_4$. However, $\gamma_2(P_4) = 3 \neq \gamma_t(P_4)$, a contradiction. If $d(T) = 4$, then there is a vertex w of T with distance at most two from the other vertices in T . Hence $C(w) = C'(w) \neq C'_1(w)$ if we root T at w . By (3) of Lemma 5, T is a tree obtained from a star $K_{1, t}$ by attaching a vertex to every leaf of $K_{1, t}$, where $2t + 1 = n$. Clearly, T can be obtained from P_3 by $t - 1$ operations of Type-2. By Lemma 6, $T \in \mathcal{B}$. In the following, we will assume that $d(T) \geq 5$ and prove $T \in \mathcal{B}$ by induction on the order of $n = |V(T)|$.

If $n < 6$, then $d(T) \leq 4$. The result is true from the above proof. If $n = 6$, then $T = P_6 \in \mathcal{B}$. This establishes the base cases. Assume that $n > 6$ and the result is true for all the trees T' with order $|V(T')| < n$, that is, if $T' \in \mathcal{A}$ with order $|V(T')| < n$ then $T' \in \mathcal{B}$.

Claim 1. *If there is a vertex $a \in V(T)$ such that $T - a$ contains at least two components P_2 , then $T \in \mathcal{B}$.*

Proof. Assume that $P_2 = bb'$ and $P_2 = cc'$ are two components of $T - a$ such that $ab, ac \in E(T)$. Let $T' = T - \{b, b'\}$, then we use S' and D to

denote a γ_t -set of T' containing a and a γ_2 -set of T , respectively. Since $a \in S'$, $S' \cup \{b\}$ is a TDS of T , and so $\gamma_t(T') \geq \gamma_t(T) - 1$. Since D is a γ_2 -set of T , $D \cap \{a, b, b'\} = \{a, b'\}$ by the definition of γ_2 -set. So $D \cap V(T')$ is a 2-dominating set of T' . Hence $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| \geq \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$, and so $T' \in \mathcal{A}$. By the induction on T' , $T' \in \mathcal{B}$. Since T can be obtained from T' by Type-2 operation. So $T \in \mathcal{B}$. The claim holds.

By Claim 1, we only need consider the case that, for every vertex a , $T - a$ has at most one component P_2 . Let $P = uvwxyz \cdots r$ be a longest path in T and we root T at r .

Clearly, $C(w) = C'(w) \neq C'_1(w)$. By (3) of Lemma 5, $C'_1(w) = C'_3(w) = \emptyset$. Hence $P_3 = uvw$ is a component of $T - x$. Let t be the number of components P_3 of $T[D(x)]$ such that a leaf of every P_3 is adjacent to x . Note that $T[D(x)]$ possible has other components. We suppose $T[D(x)]$ has s components P_3 with its support vertex is adjacent to x , k components P_2 and h components P_1 . By Lemmas 4 and 5, $s, k \in \{0, 1\}$ and $h \in \{0, 1, 2\}$. Denote the t components P_3 of $T[D(x)]$ with one of its leaves is adjacent to x in T by $P_3 = u_i v_i w_i$ ($1 \leq i \leq t$), where $xw_i \in E(T)$ for $1 \leq i \leq t$. We prove the result according to the values of $\{s, k, h\}$.

Case 1. $s = k = h = 0$.

Then $T[D[x]] = K(t)$, $t \geq 1$. Let $T' = T - D[x]$. Then $3 \leq |V(T')| < n$. Clearly, $\gamma_t(T') \geq \gamma_t(T) - 2t$. Let D be a γ_2 -set of T such that D contains as few vertices of $T[D[x]]$ as possible. Then, $x \notin D$ and $|D \cap D[x]| = 2t$ by the definition of γ_2 -set. So $D \cap V(T')$ is a 2-dominating set of T' . Thus $\gamma_t(T') \geq \gamma_t(T) - 2t = \gamma_t(T) - 2t = |D \cap V(T')| \geq \gamma_2(T)$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$ and $D \cap V(T')$ is a γ_2 -set of T' . So $T' \in \mathcal{A}$. Applying the inductive hypothesis on T' , $T' \in \mathcal{B}$.

If $t \geq 2$, then it is obvious that T is obtained from T' by Type-5 operation, and so $T \in \mathcal{B}$.

If $t = 1$, then $T[D[x]] = K(1) = P_4 = uvwx$, and so $D \cap \{u, v, w, x\} = \{u, w\}$. To 2-dominate x , $y \in D$, and so $y \in D \cap V(T')$, which implies that y is in some γ_2 -set of T' . Hence T can be obtained from T' by Type-5 operation, and $T \in \mathcal{B}$, too.

Case 2. $s \neq 0$. By the proof procedure of Lemma 5, $s = 1$ and $k = h = 0$. Denote the component P_3 of $T[D[x]]$ whose support vertex is adjacent to x in T by $P_3 = abc$ and let $T' = T - \{a, b, c\}$. Clearly, $3 \leq |V(T')| < n$. Let D be a γ_2 -set of T which does not contain b .

We claim that x is not in any γ_t -set of T' . Suppose that T' has a γ_t -set containing x , denoted by S' , then $S' \cup \{b\}$ is a TDS of T . So $\gamma_t(T') \geq \gamma_t(T) - 1$. Since $b \notin D$, then $D \cap V(T')$ is a 2-dominating set of T' . Hence $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| + 1 \geq \gamma_2(T') + 1$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$. The claim holds. Therefore, T can be obtained from T' by Type-4 operation.

Now we prove that $T' \in \mathcal{B}$. Let S' be a γ_t -set of T' . By the above claim, $x \notin S'$. Since $S' \cup \{x, b\}$ is a TDS of T , $\gamma_t(T') \geq \gamma_t(T) - 2$. Since $b \notin D$, $D \cap V(T')$ is a 2-dominating set of T' . Hence $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2(T) - 2 = |D \cap V(T')| \geq \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$, which implies $T' \in \mathcal{A}$. Applying the inductive hypothesis on T' , $T' \in \mathcal{B}$, and so $T \in \mathcal{B}$.

Case 3. $k \neq 0$. By the proof procedure of Lemma 5, $s = h = 0$.

Let $T' = T - \cup_{i=1}^t \{u_i, v_i, w_i\}$. It is clearly that $3 \leq |V(T')| < n$ and T is obtained from T' by Type-3 operation.

We only need to prove that $T' \in \mathcal{B}$. Let $S' \subseteq V(T')$ be a γ_t -set of T' , then $S' \cup (\cup_{i=1}^t \{v_i, w_i\})$ is a TDS of T . So $\gamma_t(T') \geq \gamma_t(T) - 2t$. Since $T - x$ has a component $P_2 = ab$, we can choose $D \subseteq V(T)$ as a γ_2 -set of T containing x . Then $D \cap V(T')$ is a 2-dominating set of T' , and so $\gamma_2(T) = |D| = 2t + |D \cap V(T')| \geq 2t + \gamma_2(T')$. By $\gamma_t(T) = \gamma_2(T)$, we have $\gamma_t(T') = \gamma_2(T')$, and so $T' \in \mathcal{A}$. Applying the inductive hypothesis on T' , $T' \in \mathcal{B}$.

Case 4. $h \neq 0$. By Lemmas 4 and 5, $h \in \{1, 2\}$ and $s = k = 0$.

We claim that $h = 1$. If not, then $h = 2$. We denote the two components P_1 of $T[D(x)]$ by x' and x'' . Let $T' = T - x''$. Clearly, $\gamma_t(T') = \gamma_t(T)$. Let D be a γ_2 -set of T containing $\{w_1, \dots, w_t\}$. By Lemma 3, $\{x', x''\} \subseteq D$. Since $D \cap V(T')$ is 2-dominating set of T' with $|D \cap V(T')| = \gamma_2(T) - 1$, we have $\gamma_t(T') = \gamma_t(T) = \gamma_2(T) > \gamma_2(T) - 1 \geq \gamma_2(T')$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$.

Case 4.1. $t \geq 2$.

Let $T' = T - \cup_{i=2}^t \{u_i, v_i, w_i\}$, then T is obtained from T' by Type-3 operation. By the definitions of γ_t -set and γ_2 -set, it is easy to see that $\gamma_t(T') + 2(t-1) = \gamma_t(T)$ and $\gamma_2(T') + 2(t-1) = \gamma_2(T)$. Hence $\gamma_t(T') = \gamma_2(T')$ and $T' \in \mathcal{A}$. Applying the inductive hypothesis on T' , $T' \in \mathcal{B}$, and so $T \in \mathcal{B}$.

Case 4.2. $t = 1$. Denote the component P_1 of $T[D(x)]$ by $P_1 = x'$.

Case 4.2.1. If $T[D(y) \setminus D[x]]$ has a component $H \in \{P_2, P_3, P_5\}$, then let $T' = T - D[x]$. We can easily check that T is obtained from T' by Type-6 operation. By the definition of γ_2 -set, $\gamma_2(T') + 3 = \gamma_2(T)$. For any γ_t -set S' of T' , $S' \cup \{v, w, x\}$ is a TDS of T . So $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2(T) - 3 = \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$ and $T' \in \mathcal{A}$. Applying the inductive hypothesis on T' , $T' \in \mathcal{B}$, and so $T \in \mathcal{B}$.

Case 4.2.2. If $T[D(y) \setminus D[x]]$ has no component P_2, P_3 or P_5 , we consider the structure of $T[D(y)]$. By the above discussion, $T[D(y)]$ consists of a component $P_5 = uvwx'$ and ℓ components P_1 , denoted by $\{y_1, \dots, y_\ell\}$. By Lemma 4, $\ell \leq 2$. However, if $\ell = 2$, then let $T' = T - D[y]$. It can be easily checked that $\gamma_t(T') + 4 \geq \gamma_t(T) = \gamma_2(T) = \gamma_2(T') + 5$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$. Hence $\ell \leq 1$.

Let $T' = T - \{u, x'\}$. Then we can easily check that $\gamma_t(T') + 1 = \gamma_t(T)$. Let D be a γ_2 -set of T such that D contains as few vertices of $D[y]$ as possible and $D \cap D[x] = \{u, w, x'\}$. Then $D' = (D \setminus \{u, w, x'\}) \cup \{v, x\}$ is a 2-dominating set of T' . So $\gamma_t(T') = \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D'| \geq \gamma_2(T')$, which implies that $\gamma_t(T') = \gamma_2(T')$ and D' is a γ_2 -set of T' . By $\gamma_t(T') = \gamma_2(T')$, $T' \in \mathcal{A}$. Applying the inductive hypothesis to T' , $T' \in \mathcal{B}$.

If $\ell = 0$, then $\deg_T(y) = 2$. Since $x \notin D$, to 2-dominate y , $y \in D$. Thus y is in the γ_2 -set D' of T' . Hence T is obtained from T' by Type-1 operation. Thus $T \in \mathcal{B}$.

If $\ell = 1$, then $\deg_T(y) = 3$. Since $x \notin D$, to 2-dominate y , we have $y \notin D$ and $z \in D$ by the choice of D . Thus z is in the γ_2 -set D' of T' . Hence T is obtained from T' by Type-1 operation. Thus $T \in \mathcal{B}$.

This completes the proof of Lemma 7. ■

Note that $\{P_2, P_3\} \subseteq \mathcal{A}$. Lemma 6 implies that $\mathcal{B} \subseteq \mathcal{A}$ and Lemma 7 implies that $\mathcal{A} \subseteq \mathcal{B} \cup \{P_2\}$. Hence Theorem 1 is true.

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