

**STRUCTURE OF THE SET OF ALL MINIMAL TOTAL  
DOMINATING FUNCTIONS OF SOME CLASSES  
OF GRAPHS**

K. REJI KUMAR

*Department of Mathematics*  
*N.S.S College, Pandalam – 689 501, India*

**e-mail:** rkkmaths@yahoo.co.in

AND

GARY MACGILLIVRAY

*Department of Mathematics and Statistics*  
*University of Victoria, BC, Canada*

**e-mail:** gmacgill@math.uvic.ca

**Abstract**

In this paper we study some of the structural properties of the set of all minimal total dominating functions ( $\mathfrak{F}_T$ ) of cycles and paths and introduce the idea of function reducible graphs and function separable graphs. It is proved that a function reducible graph is a function separable graph. We shall also see how the idea of function reducibility is used to study the structure of  $\mathfrak{F}_T(G)$  for some classes of graphs.

**Keywords:** minimal total dominating functions (MTDFs), convex combination of MTDFs, basic minimal total dominating functions (BMTDFs), simplex, polytope, simplicial complex, function separable graphs, function reducible graphs.

**2010 Mathematics Subject Classification:** Primary 05C69;  
Secondary 05C35.

## 1. INTRODUCTION

By a graph  $G = (V, E)$ , we mean a finite undirected graph which does not contain loops and multiple edges. In this paper, unless specified otherwise, we follow the terminology of D.B. West [9]. Total domination, as an analogue to domination, is well studied by many graph theorists. For the terms and definitions related to domination, which are not given in this paper, readers may refer the books *Fundamentals of Domination and Domination in Graphs — Advanced Topics* [5, 6]. A *total dominating set* of  $G = (V, E)$  is a subset  $S$  of  $V$  such that every vertex of  $V$  is adjacent to at least one vertex in  $S$ . Smallest such set is called a *minimal total dominating set*. The characteristic functions of the dominating set is a 0 – 1 valued function such that, the sum of the function values over the open neighborhood of each vertex is at least one.

Fractional analog of the total dominating set is a *total dominating function* (TDF) defined as the real valued function  $f : V \rightarrow [0, 1]$  such that

$$\sum_{x \in N(v)} f(x) \geq 1$$

for all  $v \in V$ , where  $N(v)$  is the open neighborhood of  $v$ . This definition was first given by Hedetniemi and Wimer [3] in 1994. A *minimal total dominating function* (MTDF) is a TDF such that  $f$  is not a TDF if for any  $v \in V$ , the value of  $f(v)$  is decreased.

For an MTDF  $f$  of  $G$ , denote  $\sum_{x \in N(v)} f(x)$  by  $f(N(v))$ . The *boundary* of  $f$  or  $B_f$  is  $\{v \in V : \sum_{x \in N(v)} f(x) = 1\}$  and the *positive set* of  $f$  or  $P_f$  is  $\{v \in V : f(v) > 0\}$ . For two subsets  $A$  and  $B$  of  $V$ , we write  $A \rightarrow_t B$  if every vertex in  $B$  is adjacent to some vertex in  $A$ . Identifying BMTDFs from a collection of MTDFs is not a difficult task, if we use the following theorem.

**Theorem 1.1** [3]. *A total dominating function  $f$  of the graph  $G$  is a minimal total dominating function if and only if  $B_f \rightarrow_t P_f$ .*

An interpolation problem motivated to define the convex combination of minimal dominating functions. This problem can be stated as follows. “Given two minimal dominating functions  $f$  and  $g$  of the graph  $G$  and for any real number  $x$ , such that  $ag(f) < x < ag(g)$ , where  $ag(f) = \sum_{v \in V} f(v)$ , does there exist a minimal dominating function  $h$  of  $G$  such that  $ag(h) = x$ ?”.

A similar question raised about total domination, motivated to define the convex combination of two MTDFs. Let  $f$  and  $g$  be two MTDFs of  $G$ , a convex combination of  $f$  and  $g$  is  $h_\lambda = \lambda f + (1 - \lambda)g$  where  $0 < \lambda < 1$ . This function is clearly a TDF. Hence the set of all TDFs forms a convex set. However it is evident from the following theorem that the convex combination of two MTDFs need not always be an MTDF.

**Theorem 1.2** [3]. *A convex combination of two MTDFs  $f$  and  $g$  is minimal if and only if  $B_f \cap B_g \rightarrow_t P_f \cup P_g$ .*

An MTDF  $f$  of  $G$  is called a *universal minimal total dominating function* if and only if every convex combination of  $f$  and any other MTDF is minimal. Theorem 1.2 is true for any finite number of MTDFs.

**Theorem 1.3** [7]. *A convex combination of  $n$  MTDFs  $f_1, f_2, \dots, f_n$  is minimal if and only if  $B_{f_1} \cap B_{f_2} \cap \dots \cap B_{f_n} \rightarrow_t P_{f_1} \cup P_{f_2} \cup \dots \cup P_{f_n}$ .*

Fractional version of total domination, convexity of two MTDFs and the existence of universal MTDFs have been studied by many authors [2, 3, 4]. Since the set of TDFs is convex, some TDFs cannot be expressed as a convex combination two or more TDFs. Motivated by this, in 2000 K. Reji Kumar introduced basic total dominating functions (BTDFs) and basic minimal total dominating functions (BMTDFs) [7]. An MTDF is called a *basic minimal total dominating function* or BMTDF, if it cannot be expressed as a proper convex combination of two distinct MTDFs. A necessary and sufficient condition for an MTDF to be a basic MTDF is known and based on this we have developed an algorithm to decide whether a given MTDF is basic.

**Theorem 1.4** [7]. *Let  $f$  be an MTDF. Then  $f$  is a BMTDF if and only if there does not exist an MTDF  $g$  such that  $B_f = B_g$  and  $P_f = P_g$ .*

**Theorem 1.5** [7]. *Let  $f$  be an MTDF of a graph  $G = (V, E)$  with  $B_f = \{v_1, v_2, \dots, v_m\}$  and  $P_f = \{u \in V : 0 < f(u) < 1\} = \{u_1, u_2, \dots, u_n\}$ . Let  $A = (a_{ij})$  be an  $m \times n$  matrix defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } u_j, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the system of linear equations given by

$$(1.1) \quad \sum_j a_{ij}x_j = 0, \text{ where } 1 \leq i \leq m.$$

Then  $f$  is a BMTDF if and only if (1.1) does not have a non-trivial solution.

**Corollary 1.6** [7]. *If  $f(v) \in \{0, 1\}$  for all  $v \in V$ , then the MTDF  $f$  of  $G$  is a BMTDF.*

Let  $G$  be a graph. We define

$$C_0(G) = \{v \in V : f(v) = 0 \text{ for any MTDF } f \text{ of } G\}, \text{ and}$$

$$C_1(G) = \{v \in V : f(v) = 1 \text{ for any MTDF } f \text{ of } G\}.$$

The set of *leaves* of  $G$  is,  $L = \{v \in V : d(v) = 1\}$  and the set of *remote vertices* is defined by  $R = \{v \in V : v \in N(u) \text{ for } u \in L\}$ . Here  $d(v)$  is the number of vertices adjacent to a vertex  $v \in V$ . Nice characterizations of the sets  $C_0(G)$  and  $C_1(G)$  of a graph  $G$  are given by Cockayne *et al.* in [3].

**Proposition 1.7** [3]. *For any graph  $G$  and vertex  $v$ ,*

1.  $v \in C_0(G)$  if and only if  $v$  is in no MTDS of  $G$ ;
2.  $v \in C_1(G)$  if and only if  $v$  is in every MTDS of  $G$ .

**Theorem 1.8** [3]. *A graph  $G$  has either a unique MTDF or infinitely many MTDFs.*

**Theorem 1.9** [3]. *For any graph  $G$ ,  $C_1(G) = R$ .*

**Theorem 1.10** [3]. *The vertex  $v \in C_0(G)$ , if and only if for any  $u \in N(v)$  there exists a vertex  $w$  such that  $N(w) \subseteq N(u) - v$ .*

Let  $K$  be a convex subset of  $\mathbf{R}^n$ . A point  $x \in K$  is an *extreme point* of  $K$  if  $y, z \in K$ ,  $0 < \lambda < 1$ , and  $x = \lambda y + (1 - \lambda)z$  imply  $x = y = z$ . The set of all extreme points of  $K$  is denoted by  $ext(K)$ . A set  $F \subseteq K$  is a *face* of  $K$  if either  $F = \emptyset$  or  $F = K$  or there exists a supporting hyperplane  $H$  of  $K$  such that  $F = K \cap H$ . An  $n$ -*simplex* in the Euclidean space is the convex hull of  $n + 1$  affinely independent points. A *convex polytope* is the convex hull of a finite set. A finite family  $\mathfrak{B}$  of convex polytopes in  $\mathbf{R}^n$  is called a *simplicial complex* if it satisfies the following conditions

1. Every face of a member of  $\mathfrak{B}$  is itself a member of  $\mathfrak{B}$ ;
2. The intersection of any two members of  $\mathfrak{B}$  is a face of each of them.

For further study of simplices, polytopes and complexes, the reader is referred to [1]. We use the notations  $\mathfrak{F}_T(G)$  and  $\mathfrak{F}_{BT}(G)$  to denote the set of all MTDFs and the set of all BMTDFs of a graph  $G$ , respectively.

**Theorem 1.11** [8]. *Let  $A \subseteq \mathfrak{F}_{BT}(G)$  such that the convex combination  $f_{A_i}$  of all BMTDFs in  $A_i \subseteq A$  is an MTDF for any subset  $A_i$  and  $B_{A_1} \neq B_{A_2}$  or  $P_{A_1} \neq P_{A_2}$  for any two nonempty subsets  $A_1$  and  $A_2$ . Then the convex combination  $f_A$  is a simplex with dimension  $|A| - 1$ .*

**Theorem 1.12** [8]. *Let  $G$  be a graph with  $|V| = n$ . Then the Euclidian dimension of  $\mathfrak{F}_T(G)$  is at most  $n$ .*

**Theorem 1.13** [8]. *Let  $G$  be a graph having order  $n$  such that  $|\mathfrak{F}_{BT}(G)| = r$ , and  $\mathfrak{F}_T(G)$  is convex.*

1. *If  $r \leq (n+1)$  and for all different subsets  $A_1$  and  $A_2$  of  $\mathfrak{F}_{BT}(G)$ ,  $B_{f_{A_1}} \neq B_{f_{A_2}}$  or  $P_{f_{A_1}} \neq P_{f_{A_2}}$  then  $\mathfrak{F}_T(G)$  is an  $r-1$  simplex. Otherwise  $\mathfrak{F}_T(G)$  is a convex polytope having dimension at most  $n - 1$ .*
2. *If  $r > (n+1)$ ,  $\mathfrak{F}_T(G)$  is a convex polytope having dimension at most  $n$  and there exists two subsets  $A_1$  and  $A_2$  of  $\mathfrak{F}_{BT}(G)$ , such that  $B_{f_{A_1}} = B_{f_{A_2}}$  and  $P_{f_{A_1}} = P_{f_{A_2}}$ .*

**Theorem 1.14** [8]. *If  $\mathfrak{F}_T(G)$  is not convex, then it is a simplicial complex.*

**Theorem 1.15** [8]. *For the complete bipartite graph  $G = K_{m,n}$ , the set  $\mathfrak{F}_T(K_{m,n})$  is isomorphic to*

1. *the  $n-1$ -simplex if  $m = 1$  and  $n \geq 2$ .*
2. *a convex polytope otherwise.*

## 2. STRUCTURE OF THE SET OF ALL MTDFs OF SOME CLASSES OF GRAPHS

In this section, our focus is on the study of the structure of the set of all MTDFs of cycles and paths. We shall show that,  $\mathfrak{F}_T(C_n)$  is convex only if  $n = 4$  or  $8$  and  $\mathfrak{F}_T(P_n)$  is convex only if  $n \leq 8$ . There exists a bijection

from the set of all functions  $f : V \rightarrow [0, 1]$  of the graph  $G = (V, E)$  to the  $n$  dimensional cube  $(I^n)$  in  $\mathbb{R}^n$ . So the set of all TDFs is isomorphic to a subset of  $I^n$ .

**Theorem 2.1.** *Let  $G$  be a vertex transitive graph. The set  $\mathfrak{F}_T(G)$  is convex if and only if  $B_f = V$  for all MTDF  $f$  of  $G$ .*

**Proof.** Suppose that  $\mathfrak{F}_T(G)$  is convex. Assume that  $B_f \neq V$  for some MTDF  $f$  of  $G$ . Let  $f_v$  be an MTDF of  $G$  such that,  $v \notin B_{f_v}$ . Since the graph is vertex transitive, there exists a function  $f_u$  such that,  $u \notin B_{f_u}$  for each  $u \in V$ . Now, by the convexity of  $\mathfrak{F}_T(G)$ , there must exist an MTDF  $g$  such that,  $B_g = \emptyset$ . This is a contradiction. Conversely, if  $B_f = V$  for all MTDF  $f$  of  $G$ , then it directly follows that,  $\mathfrak{F}_T(G)$  is a convex set. ■

**Lemma 2.2.** *If  $f$  is a BMTDF of an even cycle or a path, then  $f$  is a 0 – 1 BMTDF. The odd cycle has exactly one BMTDF which is not a 0 – 1 BMTDF.*

**Proof.** Suppose that  $f$  is not a 0-1 BMTDF. Then there exist vertices  $u_1, u_2, \dots, u_r \in P'_f$ . Now consider the corresponding system of equations (1.1), (Theorem 1.5). In this system, each equation should contain exactly two  $u_i$ 's and each should appear in at most two equations. Rank of all possible system of equations obeying this rule, is less than the number of equations used in it, except when the graph is a cycle and  $P'_f = V$ . Using row echelon form, we can prove that, the system has trivial solution only if the graph is an odd cycle and  $P'_f = V$ . If the graph is a path or an even cycle, the system has a non-trivial solution. Then  $f$  is not basic, which is a contradiction. Hence the function  $f$  of the odd cycle defined by,  $f(v) = \frac{1}{n}$  for all  $v \in V$  is a BMTDF. ■

**Theorem 2.3.** *The set  $\mathfrak{F}_T(C_n)$  is convex only if  $n = 4$  or  $8$ .*

**Proof.** First we shall show that,  $C_n$  except when  $n = 4$  or  $8$  have one MTDF  $f$  with  $B_f \neq V$ . Since cycles are vertex transitive, by Theorem 2.1 it follows that  $\mathfrak{F}_T(C_n)$  is not convex when  $n \neq 4$  or  $8$ . Let  $V(C_n) = \{v_1, v_2, \dots, v_n\}$ .

*Case 1.* When  $n = 3$ , the required function has the following values  $f(v_1) = 1$ ,  $f(v_2) = 1$  and  $f(v_3) = 0$ .

*Case 2.* When  $n = 5$ ,  $f$  is defined as  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 1$ ,  $f(v_4) = 0$  and  $f(v_5) = 0$ .

*Case 3.* When  $n = 6$ ,  $f$  has the values  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 1$ ,  $f(v_5) = 1$  and  $f(v_6) = 0$ .

*Case 4.* When  $n = 7$ , the function is defined as  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 1$ ,  $f(v_5) = 1$ ,  $f(v_6) = 0$  and  $f(v_7) = 0$ .

*Case 5.* When  $n = 9$ ,  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 0$ ,  $f(v_5) = 1$ ,  $f(v_6) = 1$ ,  $f(v_7) = 1$ ,  $f(v_8) = 0$  and  $f(v_9) = 0$  are the function values.

*Case 6.* When  $n = 10$ ,  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 1$ ,  $f(v_5) = 1$ ,  $f(v_6) = 0$ ,  $f(v_7) = 1$ ,  $f(v_8) = 1$ ,  $f(v_9) = 0$  and  $f(v_{10}) = 0$  are the function values.

*Case 7.* When  $n = 11$ ,  $f$  takes the values,  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 1$ ,  $f(v_5) = 1$ ,  $f(v_6) = 0$ ,  $f(v_7) = 0$ ,  $f(v_8) = 1$ ,  $f(v_9) = 1$ ,  $f(v_{10}) = 0$ , and  $f(v_{11}) = 0$ .

*Case 8.* When  $n = 12$ ,  $f(v_1) = 1$ ,  $f(v_2) = 1$ ,  $f(v_3) = 0$ ,  $f(v_4) = 1$ ,  $f(v_5) = 1$ ,  $f(v_6) = 0$ ,  $f(v_7) = 1$ ,  $f(v_8) = 1$ ,  $f(v_9) = 0$ ,  $f(v_{10}) = 1$ ,  $f(v_{11}) = 1$  and  $f(v_{12}) = 0$  are the values at different vertices.

Next we shall show that, if  $C_n$  has an MTDF  $f$  such that,  $f(v_i) = 0$ ,  $f(v_{i+1}) = 1$ ,  $f(v_{i+2}) = 1$ ,  $f(v_{i+3}) = 0$  and  $B_f \neq V(C_n)$ , then  $f$  can be extended to an MTDF  $f'$  of  $C_{n+4}$  satisfying  $B_{f'} \neq V(C_{n+4})$ . We can make  $C_{n+4}$  from  $C_n$  by joining the path  $u_1, u_2, u_3$  and  $u_4$  between the vertices  $v_{i+1}$  and  $v_{i+2}$ . Consider the function  $f' : V(C_{n+4}) \rightarrow [0, 1]$ , defined by  $f'(v_i) = f(v_i)$  for all  $i$ ,  $f'(u_1) = 1$ ,  $f'(u_2) = 0$ ,  $f'(u_3) = 0$  and  $f'(u_4) = 1$ . Now the function  $f'$  has the property,  $B_{f'} \neq V(C_{n+4})$ .

Since  $C_n$ , when  $n = 9, 10, 11$ , and  $12$  has an MTDF  $f$  such that  $B_f \neq V(C_n)$ , we can apply the above procedure repeatedly over these cycles and get similar kind of MTDFs for higher order cycles. Hence for  $C_n$  where  $n \geq 9$  has at least one MTDF  $f$  with  $B_f \neq V(C_n)$ .

When  $n = 4$ , let us denote the four 0-1 MTDFs of  $C_4$  by  $f_1 = (1, 1, 0, 0)$ ,  $f_2 = (0, 1, 1, 0)$ ,  $f_3 = (0, 0, 1, 1)$  and  $f_4 = (1, 0, 0, 1)$ . Let  $g$  be any MTDF of  $C_4$ . We shall show that,  $g$  can be expressed as a convex combination of these four MTDFs. Clearly  $B_g(C_4) = V$ . If  $g(v_1) = \delta$  then  $g(v_3) = (1 - \delta)$

and if  $g(v_2) = \Delta$  then  $g(v_4) = (1 - \Delta)$ . By equating the function values at each vertex, we get the following system of equations.

$$\begin{aligned}\lambda_1 + \lambda_4 &= \delta, \\ \lambda_1 + \lambda_2 &= \Delta, \\ \lambda_2 + \lambda_3 &= (1 - \delta) \text{ and} \\ \lambda_3 + \lambda_4 &= (1 - \Delta).\end{aligned}$$

This system is consistent. To get the solution, assign an arbitrary value to one of the  $\lambda_i$ s. By Theorem 1.5,  $g$  is a non-basic MTDF. Convex combinations of the BMTDFs taken two, three or four at a time have same pairs of boundary and positive sets except for the convex combination of the pairs of functions  $(f_1, f_2)$ ,  $(f_2, f_3)$ ,  $(f_3, f_4)$  and  $(f_4, f_1)$ . So the set  $\mathfrak{F}_T(C_4)$  is isomorphic to  $I^2$ .

When  $n = 8$ , we claim that  $P_8$  has no MTDF  $f$  such that  $B_f \neq V$ . Otherwise, it must have a BMTDF  $g$  such that  $B_g \neq V$ . But by Lemma 2.2, it must be a 0-1 BMTDF. Without loss of generality let us assume that,  $v_1 \notin B_g$ . Then the function values at the vertices  $v_2, v_4, v_8$  and  $v_6$  are 1, 0, 1 and 0 respectively. Consequently we get  $g(N(v_5)) = 0$ , which is a contradiction. Thus it is clear that  $\mathfrak{F}_T(C_8)$  is a convex set. To know more about the structure of this set, we have to consider all 0-1 MTDFs of the graph. The 0-1 MTDFs are  $f_1 = (1, 1, 0, 0, 1, 1, 0, 0)$ ,  $f_2 = (0, 1, 1, 0, 0, 1, 1, 0)$ ,  $f_3 = (0, 0, 1, 1, 0, 0, 1, 1)$  and  $f_4 = (1, 0, 0, 1, 1, 0, 0, 1)$ . Their convex combinations taken two, three or four at a time, have same pairs of boundary and positive sets except for the pairs of functions  $(f_1, f_2)$ ,  $(f_2, f_3)$ ,  $(f_3, f_4)$  and  $(f_4, f_1)$ . So the set  $\mathfrak{F}_T(C_8)$  is isomorphic to  $I^2$ . ■

**Theorem 2.4.** *For a path  $P_n$ ,*

1. *if  $n = 2$  or  $4$ , then  $\mathfrak{F}_T(P_n)$  is a 0-simplex,*
2. *if  $n = 3$  or  $5$ , then  $\mathfrak{F}_T(P_n)$  is a 1-simplex,*
3.  *$\mathfrak{F}_T(P_n) \cong I^2$  if  $n = 6$  or  $8$ ,*
4. *if  $n = 7$ , then  $\mathfrak{F}_T(P_n)$  is a 2-simplex and*
5. *the set  $\mathfrak{F}_T(P_n)$  is not a convex set if  $n = 9$  or  $n \geq 11$ .*

**Proof.** Let the vertices of  $P_n$  be labeled as  $v_1, v_2, \dots, v_n$ . One can easily verify that  $P_n$ , when  $n = 2$  and  $4$ , has unique MTDF. The case  $n = 3$  follows from Theorem 1.15. When  $n = 5$ , let  $g$  be an arbitrary MTDF



of  $P_5$ . Then  $g(v_2) = g(v_4) = 1$ . Let  $g(v_1) = \Delta$ . Then  $v_2 \in B_g$  and  $g(v_3) = (1 - \Delta)$ . Consequently,  $g(v_5) > 0$  and hence  $v_4 \in B_g$ . So  $g(v_5) = \Delta$ . Clearly  $g = \lambda f_1 + (1 - \lambda)f_2$ , where  $f_1$  and  $f_2$  are defined by  $f_1(v_1) = f_1(v_5) = 1$ ,  $f_1(v_3) = 0$ ,  $f_2(v_1) = f_2(v_5) = 0$ ,  $f_2(v_3) = 1$  and  $f_i(v_2) = f_i(v_4) = 1$  for  $i = 1$  and  $2$ .

When  $n = 7$ , let  $g$  be an arbitrary MTDF of the graph. Then  $v_1, v_2, v_6, v_7 \in B_g$ . The vertex  $v_3 \notin B_g$ . Otherwise,  $g(v_4) > 0$  and there is no vertex in  $B_g$  to dominate  $v_4$  and this contradicts the assumption that  $g$  is an MTDF. Similarly,  $v_5 \in B_g$ . The set  $\bigcap_g B_g$  can dominate any vertex in  $P_7$ . Hence the set  $\mathfrak{F}_T(P_7)$  is convex. To find all BMTDFs of  $P_7$ , we have to consider two cases.

First case: when  $v_4 \in B_g$ . We have two 0-1 MTDFs, having this property. They are  $f_1 = (0, 1, 1, 0, 0, 1, 1)$  and  $f_2 = (1, 1, 0, 0, 1, 1, 0)$ . Let  $g$  be an arbitrary MTDF such that  $v_4 \in B_g$  and  $g(v_3) = \Delta$  and  $g(v_5) = (1 - \Delta)$ . We get  $f_1$  and  $f_2$  when  $\Delta = 1$  and  $\Delta = 0$  respectively. If  $0 < \Delta < 1$ , then  $g(v_1) = (1 - \Delta)$  and  $g(v_7) = \Delta$ . Hence,  $g = \Delta f_1 + (1 - \Delta)f_2$ . Second case: when  $v_4 \notin B_g$ . There exists only one 0 - 1 MTDF having this property. Let that function be  $f_3 = (0, 1, 1, 0, 1, 1, 0)$ . If  $g$  is not a 0-1 MTDF, we take  $g(v_3) = \delta$  and  $g(v_5) = \Delta$ . Consequently,  $g(v_1) = (1 - \delta)$  and  $g(v_7) = (1 - \Delta)$ . Next, assume that  $g = \sum_i \lambda_i f_i$ . Then by equating the function values at different vertices, we get the system of equations.

$$\begin{aligned} \lambda_1 + \lambda_3 &= \delta, \\ \lambda_2 + \lambda_3 &= \Delta, \\ \lambda_2 &= (1 - \delta) \text{ and} \\ \lambda_1 &= (1 - \Delta). \end{aligned}$$

Solving them, we get  $\delta = \Delta$  and subsequently the values of  $\lambda_i$ s. So  $\mathfrak{F}_T(P_7)$  is isomorphic to a two simplex.

When  $n = 6$ , take an arbitrary MTDF, say  $g$ . If  $g(v_1) = \Delta$ , then  $g(v_3) = (1 - \Delta)$ . Similarly, if  $g(v_4) = \delta$ , then  $g(v_6) = (1 - \delta)$ . The function  $g$  is a convex combination of the BMTDFs  $f_1 = (1, 1, 0, 0, 1, 1)$ ,  $f_2 = (1, 1, 0, 1, 1, 0)$ ,  $f_3 = (0, 1, 1, 0, 1, 1)$  and  $f_4 = (0, 1, 1, 1, 1, 0)$ . The boundary and positive sets of all possible convex combinations of these functions are same, except for the pairs of functions  $(f_1, f_2)$ ,  $(f_2, f_4)$ ,  $(f_4, f_3)$  and  $(f_3, f_1)$ . So  $\mathfrak{F}_T(P_6)$  is isomorphic to  $I^2$ .

When  $n = 8$ , the set of vertices  $\{v_1, v_2, v_7, v_8\} \subseteq B_f$  for all MTDF  $f$  of  $P_8$ . Next let  $g$  be an arbitrary MTDF of  $P_8$  and  $g(v_4) = \Delta$ . If  $\Delta = 0$

then  $g(v_6) = 1$  and  $g(v_8) = 0$ . If  $\Delta = 1$  then  $g(v_6) = 0$  and  $g(v_8) = 1$ . If  $0 < \Delta < 1$  then  $v_5 \in B_g$ . So  $g(v_6) = (1 - \Delta)$  and  $g(v_8) = \Delta$ . Let us define  $f_1$  and  $f_2$  such that,  $f_1(v_4) = 1$ ,  $f_1(v_6) = 0$ ,  $f_1(v_8) = 1$ ,  $f_2(v_4) = 0$ ,  $f_2(v_6) = 1$ ,  $f_2(v_8) = 0$  and  $f_i(v) = g(v)$  for all other vertices. The functions  $f_1$  and  $f_2$  are MTDFs. Also  $g = \Delta f_1 + (1 - \Delta)f_2$ . Next by considering the function  $f_1$  at the place of  $g$  and starting with the vertex  $v_5$  and applying the same procedure, we can show that  $f_1 = \delta f_{11} + (1 - \delta)f_{12}$ . The functions  $f_{11}$  and  $f_{12}$  are defined as,  $f_{11}(v_5) = 1$ ,  $f_{11}(v_3) = 0$ ,  $f_{11}(v_1) = 1$ ,  $f_{12}(v_5) = 0$ ,  $f_{12}(v_3) = 1$  and  $f_{12}(v_1) = 0$ . Also when  $i = 1$  or  $2$ ,  $f_{1i}(v) = f_1(v)$  for all remaining vertices.

Similarly we can express the function  $f_2$  as a convex combination of two MTDFs  $f_{21}$  and  $f_{22}$  having functions values,  $f_{21} = (1, 1, 0, 0, 1, 1, 1, 0)$  and  $f_{22} = (0, 1, 1, 1, 0, 0, 1, 1)$ . Now, the MTDF  $g = \Delta(\delta f_{11} + (1 - \delta)f_{12}) + (1 - \Delta)(\delta f_{21} + (1 - \delta)f_{22})$  and hence  $g$  is the convex combination of the MTDFs  $f_{ij}$  where  $i, j = 1, 2$ . Exactly as in the case of  $P_6$ , we can verify that the boundary and positive sets of all possible convex combinations of these functions are same, except for the pairs of functions  $(f_{11}, f_{12})$ ,  $(f_{11}, f_{21})$ ,  $(f_{21}, f_{22})$  and  $(f_{12}, f_{22})$ . Hence the result.

When  $n = 10$ , the vertices  $v_1, v_2, v_9$  and  $v_{10}$  are in  $B_f$  for every MTDF  $f$  of  $P_{10}$ . We shall show that, the vertices  $v_5$  and  $v_6$  are also in  $B_f$ . Suppose that  $v_5 \notin B_f$ . Then  $f(v_4) > 0$  and  $f(v_6) > 0$  and the vertex  $v_3$  must be in the boundary of  $f$ . Otherwise  $B_f$  cannot dominate  $P_f$ . But this is impossible as  $f(v_2) = 0$  for all MTDF  $f$  of  $P_{10}$ . Similarly we can show that  $v_6 \in B_f$  for all MTDFs  $f$  of  $P_{10}$ . Consequently, the set  $\{v_1, v_2, v_5, v_6, v_9, v_{10}\} \subseteq \bigcap_f B_f$ , where the intersection is taken over all MTDFs of  $P_{10}$ . Since the set  $\{v_1, v_2, v_5, v_6, v_9, v_{10}\}$  dominates  $V(P_{10})$ , the convex combination of any two MTDFs is an MTDF and hence  $\mathfrak{F}_T(P_{10})$  is convex.

Next we proceed to prove that  $P_9$  and  $P_{12}$  have two MTDFs, whose convex combinations are not MTDFs. For any MTDF of a path, the function values of odd labeled vertices are independent of the function values of even labeled vertices. In other words, if  $f$  and  $g$  are any two MTDFs of  $P_n$ , the new function  $h$  — defined by  $h(x) = f(x)$  if  $x$  is an odd vertex and  $h(x) = g(x)$  if  $x$  is an even vertex — is an MTDF. So, if necessary we can concentrate on either odd vertices or even vertices, without mentioning the other set of vertices. In  $P_9$ , let  $f$  and  $g$  be any two MTDFs such that  $f(v_1) = 0$ ,  $f(v_3) = 1$ ,  $f(v_5) = 1$ ,  $f(v_7) = 0$ ,  $f(v_9) = 1$ ,  $g(v_1) = 1$ ,  $g(v_3) = 0$ ,  $g(v_5) = 1$ ,  $g(v_7) = 1$  and  $g(v_9) = 0$ . Convex combinations of  $f$  and  $g$  are not MTDFs. So  $\mathfrak{F}_T(P_9)$  is not a convex set. Similarly in  $P_{12}$ , consider any

two MTDFs  $f$  and  $g$  such that,  $f(v_1) = 1, f(v_3) = 0, f(v_5) = 1, f(v_7) = 1, f(v_9) = 0, f(v_{11}) = 1, g(v_1) = 0, g(v_3) = 1, g(v_5) = 1, g(v_7) = 0, g(v_9) = 1$  and  $g(v_{11}) = 1$ . Again the convex combination of  $f$  and  $g$  is not an MTDF, implying that  $\mathfrak{F}_T(P_{12})$  is not a convex set.

Finally, we shall show that if  $f$  is a 0-1 MTDF of  $P_n$ , then it can be extend to an MTDF of  $P_{n+2}$ . The values of  $f(v_{(n+1)})$  and  $f(v_{(n+2)})$  are decided depending upon  $f(v_{(n-1)})$  and  $f(v_n)$ . As an example, let  $f(v_{(n-1)}) = 0$  and  $f(v_n) = 1$ . Then  $f(v_{(n+1)}) = 1$  and  $f(v_{(n+2)}) = 0$ . Similarly we can always find two values for the vertices  $v_{n+1}$  and  $v_{n+2}$ . So the set of all MTDFs of the paths  $P_{(9+i)}$  and  $P_{(12+i)}$  for  $i = 2, 4, 6, \dots$  are not convex. ■

### 3. GRAPHS HAVING $\mathfrak{F}$ ISOMORPHIC TO A PRODUCT OF SIMPLICIAL COMPLEXES

Let  $A \subseteq V(G)$ . The subgraph of  $G$  induced by  $A$  is denoted by  $\langle A \rangle$ . Let  $G$  be a graph and  $V_1, V_2 \subset V(G)$  such that  $V_1 \cup V_2 = V(G)$ . Note that, the possibility of  $V_1 \cap V_2 \neq \emptyset$  is not eliminated. Let  $W \subset V$  and  $f$  is an MTDF of  $G$ . We denote the *restriction* of  $f$  to  $W$  by  $f/W$ . A graph  $G$  is a *function reducible graph* with respect to a partition  $V_1$  and  $V_2$ , if  $\mathfrak{F}(\langle V_i \rangle) = \{f/V_i : f$  is an MTDF of  $G\}$  for  $i = 1$  and  $2$  and if for any  $f_1 \in \mathfrak{F}(V_1)$  and  $f_2 \in \mathfrak{F}(V_2)$  the new function defined by

$$f(v) = \begin{cases} f_1(v), & \text{if } v \in V_1, \\ f_2(v), & \text{if } v \in V_2 \end{cases}$$

is an MTDF of the whole graph.

Clearly for all  $v \in (V_1 \cap V_2)$  and any MTDF  $f$  of  $G$ ,  $f(v)$  must be a constant. Disconnected graphs are examples. But the following example shows that some connected graphs also possess this property.

**Lemma 3.1.** *If  $f(v)$  is a constant for all MTDF  $f$  of a graph  $G$ , then  $f(v) = 0$  or  $1$ .*

**Proof.** Suppose that,  $0 < f(v) = \Delta < 1$ . We know that every graph has at least one MTDS. If we take the characteristic function of an MTDS, then the function value at  $v$  is either 0 or 1. This is a contradiction. ■

**Lemma 3.2.** *If the graph  $G$  is function reducible with respect to the pair  $V_1, V_2$  and  $V_1 \cap V_2 \neq \emptyset$ , then  $V_1 \cap V_2 \subseteq C_0 \cup C_1$ .*

**Proof.** By the definition of function reducible graphs, if  $f$  is an arbitrary MTDF of  $G$  then  $f(v) = 0$  or  $1$  for all  $v \in V_1 \cap V_2$ . ■

**Example 3.3.** Take two star graphs  $K_{(1,n)}$  and  $K_{(1,m)}$  where  $n, m \geq 2$ . The graph  $G$  is made by joining one pendant vertex from each star, by an edge.

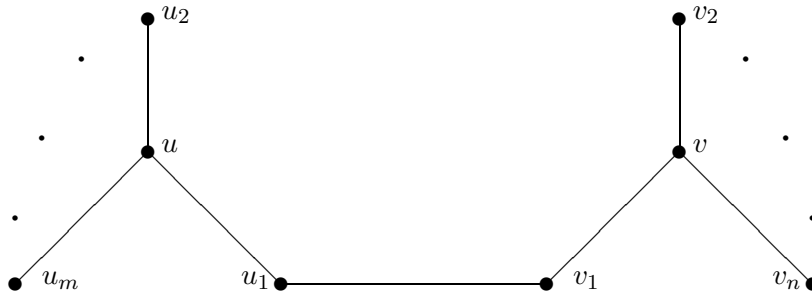


Figure 3.1

Consider the partition  $V_1 = \{u, u_1, u_2, \dots, u_m\}$  and  $V_2 = \{v, v_1, v_2, \dots, v_n\}$  of the graph. The vertices  $u, v \in C_1$ . We can change the function values at the vertices  $u_i$ 's without affecting the function values at  $v_i$ 's, such that  $u \in B_g$  for any MTDF  $g$  of the graph.

Next we consider the graphs having at least two MTDFs. We call the graph  $G$  a *function separable graph*, if  $V(G)$  has at least one partition, say  $\{V_1, V_2\}$  such that for any two MTDFs  $f$  and  $g$  of  $G$ , the functions defined by

$$(f, g)(v) = \begin{cases} f(v), & \text{if } v \in V_1, \\ g(v), & \text{if } v \in V_2 \end{cases}$$

and

$$(g, f)(v) = \begin{cases} g(v), & \text{if } v \in V_1, \\ f(v), & \text{if } v \in V_2 \end{cases}$$

are MTDFs of  $G$ . Let the graph  $G$  be function separable with respect to a partition  $\{V_1, V_2\}$ , we shall call the function  $f_{V_1} : V_1 \rightarrow [0, 1]$ , a *basic function* of  $V_1$  if and only if there exists a BMTDF  $f$  of  $G$  such that,  $f/V_1 = f_{V_1}$ .

We avoid the words “minimal dominating” because, the basic functions may not always be a minimal dominating function. Next we proceed to prove some properties of basic functions. We define a convex combination of the functions  $f/V_1$  and  $g/V_1$  if and only if the convex combination of the MTDFs  $f$  and  $g$  of  $G$  is minimal.

**Theorem 3.4.** *The function  $f_{V_1}$  is a basic function if and only if there does not exist functions  $f_{V_1}^i$  where  $i = 1, 2, \dots, r$  such that,  $f_{V_1} = \sum_i \lambda_i f_{V_1}^i$ ,  $\sum_i \lambda_i = 1$  and  $0 < \lambda_i < 1$ .*

**Proof.** Let the function  $f_{V_1}$  be a basic function on the set  $V_1$ . Suppose that, there exist functions  $f_{V_1}^i$  where  $i = 1, 2, \dots, r$  such that,  $f_{V_1} = \sum_i \lambda_i f_{V_1}^i$ , where  $\sum_i \lambda_i = 1$  and  $0 < \lambda_i < 1$ . By the definition of basic function, there exists a BMTDF  $f$  of  $G$  such that  $f/V_1 = f_{V_1}$ . The functions  $f_i : V \rightarrow [0, 1]$  are defined such that

$$f_i(v) = \begin{cases} f_{V_1}^i(v), & \text{if } v \in V_1, \\ f(v), & \text{if } v \in V_2. \end{cases}$$

It is an easy exercise to show that  $f = \sum_i (\lambda_i f_i)$ , where  $\sum_i \lambda_i = 1$  and  $0 < \lambda_i < 1$ . This contradicts the fact that  $f$  is a BMTDF.

To prove the converse, we have to show that if  $f_{V_1}$  cannot be expressed as a convex combination of a set of functions over  $V_1$ , then  $f$  is a BMTDF. Suppose on the contrary that  $f$  is not basic. Then there exist MTDFS  $f_1, f_2, \dots, f_r$  such that  $f = \sum_i \lambda_i f_i$ , where  $\sum_i \lambda_i = 1$  and  $0 < \lambda_i < 1$ . This implies that  $f_{V_1}$  is a convex combination of  $f_i/V_1$ , a contradiction. ■

**Theorem 3.5.** *Let  $G$  be a function separable graph with respect to a partition  $\{V_1, V_2\}$ . If the sets of all basic functions over  $V_1$  and  $V_2$  are  $A = \{f_1, f_2, \dots, f_r\}$  and  $B = \{g_1, g_2, \dots, g_s\}$  respectively, then the set of all BMTDFs of  $G$  is  $A \times B = \{(f_i, g_j) : i = 1, 2, \dots, r \text{ and } j = 1, 2, \dots, s\}$ .*

**Proof.** If  $f \in \mathfrak{F}_{BT}(G)$ , then  $f/V_1$  and  $f/V_2$  are basic functions and  $f = (f/V_1, f/V_2)$ . So  $f \in \mathfrak{F}_{BT}(G) \subseteq A \times B$ . To prove the converse, let  $(f, g) \in A \times B$ . We have to show that  $(f, g)$  is a BMTDF of  $G$ . Suppose not. Since  $(f, g)$  is an MTDF, there exists MTDFS  $f_1, f_2, \dots, f_r$  such that  $f$  is a convex combination of these functions. Then  $f/V_1$  is a convex combination of the restrictions,  $f_1/V_1, f_2/V_1, \dots, f_r/V_1$  and by Theorem 3.4 we get a contradiction. ■

**Theorem 3.6.** *Let  $f$  and  $g$  be two MTDFs of a function separable graph  $G$ . These functions are basic if and only if both MTDFs  $(f, g)$  and  $(g, f)$  are BMTDFs.*

**Proof.** Let the function  $f$  and  $g$  be BMTDFs of  $G$  and let  $G$  be function separable with respect to the partition  $\{V_1, V_2\}$ . Then  $f/V_1$  is a basic function over  $V_1$ . Suppose that  $(f, g)$  is not basic. Then there exists MTDFs  $f_1, f_2, \dots, f_r$  of  $G$  such that  $(f, g) = \sum_i \lambda_i f_i$ . Using the Theorem 3.4, we get that  $f/V_1$  is not a basic function. This is a contradiction. Proof of the function  $(g, f)$  and that of the converse are similar. ■

**Theorem 3.7.** *All bipartite graphs are function separable.*

**Proof.** Let  $V_1$  and  $V_2$  be the partition of  $V(G)$ . Consider any two MTDFs say,  $f$  and  $g$  of  $G$  and the new function  $(f, g)$ . We claim that this function is an MTDF of  $G$ . First note that,  $(f, g)(N(v)) = g(N(v))$  if  $v \in V_1$  and  $(f, g)(N(v)) = f(N(v))$  if  $v \in V_2$ . Also  $V_2 \cap B_f \rightarrow V_1 \cap P_f$  and  $V_1 \cap B_g \rightarrow V_2 \cap P_g$ . Thus  $B_{(f,g)} \rightarrow P_{(f,g)}$  and hence the result. ■

It is interesting to note that if  $G$  is a function reducible graph, then it is a function separable graph. The previous theorem provides an example, which shows the converse is not always true.

**Theorem 3.8.** *Function reducible graphs are function separable graphs.*

**Proof.** Let  $G$  be a function reducible graph with respect to the vertex subsets  $V_1$  and  $V_2$ . Then consider the new vertex subsets  $V'_1 = V_1 - (V_1 \cap V_2)$  and  $V'_2 = V_2$ . With respect to the partition  $\{V'_1, V'_2\}$ , the graph is function separable. ■

**Theorem 3.9.** *Cycles are not function reducible.*

**Proof.** Suppose that the cycle  $G$  is function reducible with respect to the vertex subsets  $V_1$  and  $V_2$ . Since we consider only connected induced subgraphs,  $\langle V_1 \rangle$  and  $\langle V_2 \rangle$  must be paths. Then  $V(\langle V_i \rangle) \cap C_1(\langle V_i \rangle) \neq \emptyset$ . But  $C_1(G) = \emptyset$ . These are not possible simultaneously. ■

This result shows that, bipartite graphs are not function reducible graphs in general, because even cycles are bipartite graphs. Next we discuss some necessary conditions for a graph to be function reducible. In the result we use the following sets.  $C'_1 = \{x \in C_1 : x \text{ is adjacent to at least one vertex in } C_1\}$ ,  $L' = \{x \in L : x \in N(y) \text{ and } y \in C'_1\}$  and  $C = C'_1 \cup L'$ .

**Theorem 3.10.** *Let  $G$  be a connected graph containing at least three vertices. It has two disjoint vertex subsets  $V_1$  and  $V_2$ , where  $V_1 \cup V_2 \cup C = V$  such that, for any path between the vertices  $u \in V_1$  and  $v \in V_2$ , there exist at least two adjacent vertices which are common to the path and the set  $C'_1$ . Then the graph  $G$  is function reducible.*

**Proof.** Let  $G$  be a connected graph containing at least three vertices. Also let  $V_1$  and  $V_2$  be two vertex subsets, satisfying the condition given above. There exists  $C' \subseteq C$  such that  $C'$  is a cut set of  $G$ . We claim that,  $V'_1 = V_1 \cup C$  and  $V'_2 = V_2 \cup C$  are two partitions of  $G$ , with respect to which the graph is function reducible.

**Claim 1.** If  $f$  is an MTDF of  $G$ , then  $f_{V'_1}$  is an MTDF of  $\langle V'_1 \rangle$ .  
Take a vertex  $x \in V'_1$ .

*Case 1.*  $x \in V_1$ . Then  $N(x) \subset V'_1$ . So  $f_{V'_1}(N(x)) = f(N(x)) \geq 1$ . Suppose on the contrary that  $y \in N(x) \cap V_2$ . We get a contradiction because there exists a path connecting  $x$  and  $y$  which does not contain any elements of  $C'_1$ . Also for any  $y \in N(x)$ ,  $N(y) \subset V'_1$ . Suppose not. Let  $z \in N(y) \cap V_2$ . Then the path  $xyz$  can contain at most one vertex from  $C'_1$ . We get contradiction again. Thus for all  $x \in V_1$ ,  $f_{V'_1}(N(x)) \geq 1$  and if  $f_{V'_1}(x) \geq 0$  then  $B_{f_{V'_1}} \rightarrow_t \{x\}$ . Hence  $f_{V'_1}$  is an MTDF of  $\langle V'_1 \rangle$ . Similarly we can prove that  $f_{V'_2}$  is an MTDF of  $\langle V'_2 \rangle$ .

*Case 2.*  $x \in C'_1$ . Since  $N(x) \cap C'_1 \neq \emptyset$ ,  $f_{V'_1}(N(x)) \geq 1$  and  $N(x) \cap L' \neq \emptyset$ . As in case one we get  $B_{f_{V'_1}} \rightarrow_t \{x\}$ . Hence  $f_{V'_1}$  is an MTDF of  $\langle V'_1 \rangle$ . Similarly we can prove that  $f_{V'_2}$  is an MTDF of  $\langle V'_2 \rangle$ .

Next let  $f$  and  $g$  be any MTDFs of  $\langle V'_1 \rangle$  and  $\langle V'_2 \rangle$  respectively. First we shall prove that  $f(x) = g(x)$  for all  $x \in C$ . Take an arbitrary MTDF  $f$  of  $\langle V'_1 \rangle$ . If  $x \in C'_1$ , there exists  $y \in C'_1$  such that  $x$  and  $y$  are adjacent. Since  $N(x) \cap L' \neq \emptyset$  and  $N(y) \cap L' \neq \emptyset$ ,  $x \in C_1(\langle V_1 \rangle)$ . If  $x \in L'$ , then  $f(x) = 0$  because, the only vertex adjacent to  $x$ , in  $\langle V'_1 \rangle$  is not an element of  $B_f$ . So  $f(x) = 1$  for all  $x \in C'_1$  and  $f(x) = 0$  for all  $x \in L'$ . The same is true for any MTDF  $g$  of  $\langle V'_2 \rangle$ . So  $f(x) = g(x)$  for all  $x \in C$ . Define a function  $h : V \rightarrow [0, 1]$  as follows.

$$h(v) = \begin{cases} f(v), & \text{if } v \in V'_1, \\ g(v), & \text{if } v \in V_2. \end{cases}$$

**Claim 2.**  $h$  is an MTDf of  $G$ . Since  $f$  and  $g$  are MTDf's of the induced subgraphs,  $h$  is an MDF of  $G$ . By suitably reproducing the steps in the above paragraph, we can show that  $B_h \rightarrow_t P_h$ . ■

**Theorem 3.11.** *Let  $G$  be a function reducible graph with respect to the vertex subsets  $V_1$  and  $V_2$ . Then the set  $\mathfrak{F}(G)$  is the product of the sets  $\mathfrak{F}(\langle V_1 \rangle)$  and  $\mathfrak{F}(\langle V_2 \rangle)$ .*

**Proof.** Obvious. ■

#### 4. PROBLEMS FOR FURTHER RESEARCH

Structure of the set of all minimal total dominating functions of many families of graphs are still unknown. A characterization is known for only those graphs  $G$ , for which  $\mathfrak{F}_T(G)$  is isomorphic to one simplex. Characterization of graphs, such that  $\mathfrak{F}(G)$  is isomorphic to other higher dimensional simplexes is quite open. The first step to study the structure of  $\mathfrak{F}(G)$  is to find all BMTDFs of a graph. Only a little research is done in this area.

#### Acknowledgement

A part of this research work is completed when the first author visited the Department of Mathematics and Statistics, University of Victoria, BC, Canada with the financial support of the Department of Science and Technology, India in BOYSCAST fellowship scheme. He expresses his gratitude for all facilities provided and support given by the Department of Mathematics and Statistics, University of Victoria, to complete this research.

#### REFERENCES

- [1] B. Grünbaum, *Convex Polytopes* (Interscience Publishers, 1967).
- [2] E.J. Cockayne and C.M. Mynhardt, *A characterization of universal minimal total dominating functions in trees*, *Discrete Math.* **141** (1995) 75–84.
- [3] E.J. Cockayne, C.M. Mynhardt and B. Yu, *Universal minimal total dominating functions in graphs*, *Networks* **24** (1994) 83–90.
- [4] E.J. Cockayne, C.M. Mynhardt and B. Yu, *Total dominating functions in trees: Minimality and convexity*, *J. Graph Theory* **19** (1995) 83–92.
- [5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, Inc., New York, 1998).



- [6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs — Advanced Topics* (Marcel Dekker, Inc., New York, 1998).
- [7] K. Reji Kumar, *Studies in Graph Theory — Dominating functions*, Ph.D Thesis (Manonmaniam Sundaranar University, Tirunelveli, India, 2004).
- [8] K. Reji Kumar, G. MacGillivray and R.B. Bapat, *Topological properties of the set of all minimal total dominating functions of a graph*, manuscript.
- [9] D.B. West, *Graph Theory : An introductory course* (Prentice Hall, New York, 2002).

Received 19 February 2009

Revised 1 September 2009

Accepted 1 September 2009