THE EDGE $C_4$ GRAPH OF SOME GRAPH CLASSES

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Abstract

The edge $C_4$ graph of a graph $G$, $E_4(G)$ is a graph whose vertices are the edges of $G$ and two vertices in $E_4(G)$ are adjacent if the corresponding edges in $G$ are either incident or are opposite edges of some $C_4$. In this paper, we show that there exist infinitely many pairs of non isomorphic graphs whose edge $C_4$ graphs are isomorphic. We study the relationship between the diameter, radius and domination number of $G$ and those of $E_4(G)$. It is shown that for any graph $G$ without isolated vertices, there exists a super graph $H$ such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$. Also we give forbidden subgraph characterizations for $E_4(G)$ being a threshold graph, block graph, geodetic graph and weakly geodetic graph.

Keywords: edge $C_4$ graph, threshold graph, block graph, geodetic graph, weakly geodetic graph.

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1. Introduction

We consider the graph operator $E_4(G)$, whose vertices are the edges of $G$ and two vertices in $E_4(G)$ are adjacent if the corresponding edges in $G$ are either incident or are opposite edges of some $C_4$. This graph class is also known by the name edge graph in [11]. In $E_4(G)$ any two vertices are adjacent if the union of the corresponding edges in $G$ induce any one of the graphs $P_3$, $C_3$, $C_4$, $K_4 - \{e\}$, $K_4$. If $a_1 - a_2$ is an edge in $G$, the corresponding
vertex in \( E_4(G) \) is denoted by \( a_1a_2 \). In [9], we obtained characterizations for \( E_4(G) \) being connected, complete, bipartite etc and also some dynamical behaviour of \( E_4(G) \) are studied. It was also proved that \( E_4(G) \) has no forbidden subgraphs.

For a vertex \( v \in V(G) \), \( N(v) \) denotes the set of all vertices in \( G \) which are adjacent to \( v \) and \( N[v] = N(v) \cup \{v\} \). A vertex \( x \) dominates a vertex \( y \) if \( N(y) \subseteq N[x] \). If \( x \) dominates \( y \) or \( y \) dominates \( x \), then \( x \) and \( y \) are comparable. Otherwise, they are incomparable. The Dilworth number of a graph \( G \), \( \text{dilw}(G) \) is the largest number of pairwise incomparable vertices of \( G \). A vertex \( v \) is a universal vertex if it is adjacent to all the other vertices in \( G \). A subset \( S \) of \( V \) is a dominating set if each vertex of \( G \) that is not in \( S \) is adjacent to at least one vertex of \( S \). If \( S \) is a dominating set then \( N[S] = V \). A dominating set of minimum cardinality is called a minimum dominating set, its cardinality is called the domination number of \( G \) and it is denoted by \( \gamma(G) \). Many types of domination and its characteristics are discussed in [5]. In [4], it is observed that for graphs \( G \) without isolated vertices, \( \gamma(G) \leq \text{dilw}(G) \).

All the graphs considered here are finite, undirected and simple. We denote by \( P_n \) (respectively \( C_n \)), a path (respectively cycle) on \( n \) vertices. The graph obtained by deleting any edge \( e \) of \( K_n \) is denoted by \( K_n - \{e\} \). The join of two graphs \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \) is denoted by \( G \vee H \) and has \( V(G \vee H) = V_1 \cup V_2 \) and \( E(G \vee H) = E_1 \cup E_2 \cup \{ (u, v) : u \in V_1 \text{ and } v \in V_2 \} \). A ‘bow’ is \( K_1 \vee 2K_2 \). The graph obtained by attaching a pendant vertex to any vertex of \( C_n \), is called an ‘\( n \)-pan’ and a ‘paw’ is a 3-pan. The graph in Figure 1 is called a ‘moth’.

![Moth Graph](image)

A graph \( G \) is \( H \)-free if \( G \) does not contain \( H \) as an induced subgraph. A graph \( H \) is a forbidden subgraph for a property \( P \), if any graph \( G \) which satisfies the property \( P \) cannot have \( H \) as an induced subgraph. The distance between any two vertices \( u \) and \( v \) of a connected graph \( G \), \( d_G(u, v) \) is the
length of a shortest path joining them. The eccentricity of a vertex \( v \in V(G) \) is \( e(v) = \max\{d(u,v) : u \in V(G)\} \). The radius and diameter of \( G \) are respectively \( \text{rad}(G) = \min\{e(v) : v \in V(G)\} \), \( \text{diam}(G) = \max\{e(v) : v \in V(G)\} \). A vertex \( v \) is called a central vertex of \( G \) if \( e(v) = \text{rad}(G) \). The center, \( C(G) \) of a connected graph \( G \) is the subgraph of \( G \) induced by its central vertices. The girth of \( G \), \( g(G) \) is the length of a shortest cycle in \( G \). A clique in \( G \) is a complete subgraph of \( G \). For all basic concepts and notations not mentioned in this paper we refer [13].

The line graph \( L(G) \) of a graph \( G \) is a graph that has a vertex for every edge of \( G \), and two vertices of \( L(G) \) are adjacent if and only if they correspond to two edges of \( G \) with a common end vertex. In [8], it is shown that for any graph \( G \) without isolated vertices, there is a graph \( H \) such that \( C(H) = G \) and \( C(L(H)) = L(G) \). It is further proved that \( \text{diam}(L(G)) \leq \text{diam}(G) + 1 \) and \( \text{rad}(L(G)) \leq \text{rad}(G) + 1 \).

In [1], several graph classes and their forbidden subgraph characterizations for many properties are discussed in detail. We consider the graph classes — threshold graphs, cographs, block graphs, geodetic graphs and weakly geodetic graphs with regard to \( E_4(G) \).

Threshold graphs were introduced by Chvátal and Hammer in [2]. It is known that a graph \( G \) is a threshold graph if and only if \( \text{dilw}(G) = 1 \) and that \( G \) is \( \{2K_2, C_4, P_4\} \)-free graph [2, 5].

In [7], it is proved that a connected graph \( G \) is a block graph if and only if every maximal 2-connected subgraph (block) is complete. A cycle \( C \) of \( G \) is a b-cycle of \( G \) if \( C \) is not contained in a clique of \( G \). The bulge of \( G \), \( b(G) \) is the minimum length of a b-cycle in \( G \) if \( G \) contains a b-cycle and is \( \infty \) otherwise. Also, \( G \) is a block graph if and only if \( b(G) = \infty \) [6, 7].

A graph \( G \) is a geodetic graph [10] if any two vertices of \( G \) are joined by a unique shortest path and \( G \) is weakly geodetic if for every pair of vertices of distance two, there is a unique common neighbour [7]. A graph \( G \) is weakly geodetic if and only if \( b(G) \geq 5 \) [6, 7]. It is known that block graphs \( \subseteq \) geodetic graphs \( \subseteq \) weakly geodetic graphs [1].

\( P_4 \)-free graphs are called cographs [3]. The domination number of cographs is at most two [12].

It is well known that \( K_{1,3} \) and \( K_3 \) are the only non isomorphic graphs with isomorphic line graphs. Even though \( L(G) \subseteq E_4(G) \), it is proved in this paper that there exist infinitely many pairs of non isomorphic graphs with isomorphic edge \( C_4 \) graphs. We study relations between \( \gamma(G) \) and \( \gamma(E_4(G)) \), \( \text{diam}(G) \) and \( \text{diam}(E_4(G)) \), and \( \text{rad}(G) \) and \( \text{rad}(E_4(G)) \). We prove that for
any graph \( G \) without isolated vertices, it is possible to construct a super graph \( H \) such that \( C(H) = G \) and \( C(E_k(H)) = E_k(G) \). We also obtain forbidden subgraph characterizations for \( E_k(G) \) being threshold graph, block graph, geodetic graph and weakly geodetic graph.

2. Some Properties of \( E_k(G) \)

**Theorem 1.** There exist infinitely many pairs of non isomorphic graphs whose edge \( C_k \) graphs are isomorphic.

**Proof.** Let \( G = K_{1,n} \). If \( n = 2k - 1 \), then take \( H = K_2 \lor (k - 1)K_1 \) and if \( n = 2k \), then take \( H = 2K_1 \lor kK_1 \). Clearly \( G \) and \( H \) are non isomorphic graphs. But \( E_k(G) = E_k(H) = K_n \).

**Theorem 2.** For a connected graph \( G \), \( \text{diam}(G) - 1 \leq \text{diam}(E_k(G)) \leq \text{diam}(G) + 1 \) and \( \text{rad}(G) - 1 \leq \text{rad}(E_k(G)) \leq \text{rad}(G) + 1 \).

**Proof.** By the definition of \( E_k(G) \) and \( L(G) \), \( \text{diam}(E_k(G)) \leq \text{diam}(L(G)) \) and \( \text{rad}(E_k(G)) \leq \text{rad}(L(G)) \). But, \( \text{diam}(L(G)) \leq \text{diam}(G) + 1 \) and \( \text{rad}(L(G)) \leq \text{rad}(G) + 1 \). Thus \( \text{diam}(E_k(G)) \leq \text{diam}(G) + 1 \) and \( \text{rad}(E_k(G)) \leq \text{rad}(G) + 1 \).

Next let \( \text{diam}(G) = k \). We want to prove that \( \text{diam}(E_k(G)) \geq k - 1 \). On the contrary, assume that \( \text{diam}(E_k(G)) < k - 1 \). Let \( u \) and \( v \) be any two vertices in \( G \) and let \( u - u', v - v' \) be any two edges incident with \( u \) and \( v \) respectively. But \( d_{E_k(G)}(uu', vv') < k - 1 \). So \( d_G(u, v) \leq d_{E_k(G)}(uu', vv') + 1 < k \), which is a contradiction to the fact that \( \text{diam}(G) = k \).

Finally, let \( \text{rad}(G) = k \). It is required to prove that \( \text{rad}(E_k(G)) \geq k - 1 \). On the contrary, suppose that \( \text{rad}(E_k(G)) < k - 1 \). Then there exists a vertex \( uu' \) in \( E_k(G) \) such that \( e(uu') < k - 1 \). Consider the vertex \( u \) in \( G \). Let \( v \) be any vertex in \( G \) and \( vv' \) be any edge incident with \( v \). Then \( d_G(u, v) \leq d_{E_k(G)}(uu', vv') + 1 < k \), and hence \( e(u) < k \), which is a contradiction to the fact that \( \text{rad}(G) = k \).

**Note 1.** The bounds in Theorem 2 are strict.

If \( G \) is a bow, then \( \text{diam}(G) = 2 \), \( \text{diam}(E_k(G)) = 3 \), \( \text{rad}(G) = 1 \) and \( \text{rad}(E_k(G)) = 2 \).

If \( G \) is \( C_4 \), then \( \text{diam}(G) = 2 \), \( \text{diam}(E_k(G)) = 1 \), \( \text{rad}(G) = 2 \) and \( \text{rad}(E_k(G)) = 1 \).
Theorem 3. For any graph G without isolated vertices, there exists a super graph H such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$.

Proof. Consider $G \setminus 2K_2$. Let the $K_2$’s be $a - a'$ and $b - b'$. Attach $a'' - a'''$ to $a - a'$ such that $a$ is adjacent to $a'''$ and $a'$ is adjacent to $a''$. Similarly attach $b'' - b'''$ to $b - b'$ such that $b$ is adjacent to $b'''$ and $b'$ is adjacent to $b''$. The graph so obtained is H.

Claim 1. $C(H) = G$.

We prove that among the vertices in $H$, those vertices which are in $G$ also have minimum eccentricity.

$e(u) = 2$, if $u \in V(G)$.
$e(u) = 3$, if $u \in \{a, a', b, b'\}$.
$e(u) = 4$, if $u \in \{a'', a''', b'', b'''\}$.

Hence Claim 1 is proved.

Claim 2. $C(E_4(H)) = E_4(G)$.

$e(x) = 2$, if $x \in \{u_iu_j/u_i$ is adjacent to $u_j \in G, i, j = 1, 2, \ldots, m, i \neq j$. 
$e(x) = 3$, if $x \in \{aa', bb', au_i, a'u_i, bu_i, b'u_i, i = 1, 2, \ldots, m$. 
$e(x) = 4$, if $x \in \{a''a'', aa''', b''b'', bb''', a''a'''', b''b'''\}$. 

Illustration: Let $G = P_3$. Then H:

\[ 
\begin{align*}
 & 
\end{align*}
\]

3. A Bound on the Domination Number of $E_4(G)$

Theorem 4. For a connected graph $G$, $\gamma(G) \leq 2\gamma(E_4(G))$. Given any two integers $a$ and $b$ such that $a \leq 2b$, there exists a graph $G$ such that $\gamma(G) = a$ and $\gamma(E_4(G)) = b$. 

Proof. Let $\gamma(E_4(G)) = b$ and let $\{e_1 = v_1v'_1, e_2 = v_2v'_2, \ldots, e_b = v bv'_b\}$ dominate $E_4(G)$. Consider $S = \{v_1, v'_1, v_2, v'_2, \ldots, v_b, v'_b\}$. Then $S \subseteq V(G)$. Let $w$ be any vertex in $V(G)$. Since $G$ is a connected graph, $w$ must be the end vertex of an edge $w - w'$. But the vertex $ww'$ in $E_4(G)$ is dominated and hence is adjacent to at least one of the $b$ vertices. Let $e_i$ be adjacent to $ww'$ in $E_4(G)$. Then in $G$, either $e_i$ is incident with $w - w'$ or $e_i$ and $w - w'$ are the opposite edges of some $C_4$. In both the cases, $w$ is dominated by $v_i$ or $v'_i$. Thus $S$ is a dominating set of $G$ and hence $\gamma(G) \leq 2\gamma(E_4(G))$.

Construction

| Case 1 | Consider $P_{2b} = \{v_1, v_2, \ldots, v_{2b}\}$. Attach a pendant vertex to each of $v_{2i-1}$, $i = 1, 2, \ldots, b$. Then to each of the $v_{2i}$'s, $i = 1, 2, \ldots, a - b$, attach a pendant vertex. | $a = 4; b = 3$ |
| Case 2 | Consider $K_{1,a}$. Replace a pendant vertex of $K_{1,a}$ by $K_1 \cup (b - a + 1)K_2$. To all the other pendant vertices of $K_{1,a}$, attach a pendant vertex. | $a = 5; b = 6$ |
4. Some Theorems on Graph Classes

Theorem 5 [9]. For a connected graph $G$, $E_4(G)$ is complete if and only if $G$ is a complete multipartite graph.

Theorem 6. Let $G$ be a connected graph such that $E_4(G)$ is a threshold graph. Then $\gamma(G) \leq 2$.

Proof. We know that $E_4(G)$ is a threshold graph if and only if $dilw(E_4(G)) = 1$. Also $dilw(E_4(G)) \geq \gamma(E_4(G))$. Then the theorem follows from Theorem 4. \hfill \blacksquare

The graph obtained from $K_4$ by attaching two pendant vertices to the same vertex of $K_4$ is denoted by $H$.

Theorem 7. If $G$ is a threshold graph then $E_4(G)$ is a threshold graph if and only if $G$ is $\{\text{moth, } H\}$-free.

Proof. Let $G$ be a threshold graph. If $G$ contains a moth graph or $H$ as an induced sub graph, then $E_4(G)$ contains a $2K_2$ and hence it cannot be threshold.

Conversely, suppose that $G$ is a $\{\text{moth, } H\}$-free threshold graph. Since $G$ is threshold, $dilw(G) = 1$ and hence $\gamma(G) = 1$. So $G$ must have a universal vertex $u$.

If at most two vertices in $N(u)$ are of degree greater than one, then $E_4(G)$ cannot contain an induced $2K_2$, $C_4$ or $P_4$.

Now let $k$, $k \geq 3$ vertices in $N(u)$ are of degree greater than one.

Claim: There exist three vertices $u_1, u_2, u_3$ such that the vertex $u_2$ is adjacent to $u_1$ and $u_3$.

If $k = 3$, this claim holds true. If $k > 3$, let $u_1, u_2, u_3$ and $u_4$ be four vertices of degree greater than one in $N(u)$ such that $u_1$ is adjacent to $u_2$ and $u_3$ is adjacent to $u_4$. Since $G$ is threshold, it can not contain an induced $2K_2$ and hence $u_3$ or $u_4$ must be adjacent to $u_1$ or $u_2$. Let $u_3$ be adjacent to $u_1$. Then $u_2, u_1, u_3, u_4$ forms an induced $P_4$ which is not possible since $G$ is threshold. In this case, if $u_4$ is adjacent to $u_2$, then $G$ contains an induced $C_4$ which is again not possible. Hence the claim.

Further if $u_1$ and $u_3$ are adjacent, the vertex $u$ can have at most one more neighbour since $G$ is $H$-free. In this case also $E_4(G)$ is threshold since it is $\{2K_2, C_4, P_4\}$-free. On the other hand if $u_1$ and $u_3$ are not adjacent,
then since $G$ is moth-free, the vertex $u$ can have at most one more neighbour. In this case also $E_4(G)$ is threshold.

**Remark.** Let $G$ be a connected graph such that $E_4(G)$ is a cograph. Then $\gamma(G) \leq 4$, which follows from Theorem 4 and the fact that the domination number of cographs is at most two.

**Theorem 8.** Let $G$ be a connected graph. Then

1. $E_4(G)$ is a weakly geodetic graph if and only if $G$ is $\{\text{paw, 4-pan}\}$-free.
2. $E_4(G)$ is a geodetic graph if and only if $G$ is $\{C_{2n} : n > 2\} \cup \{4$-pan$\} \cup \{2n - 1 : n > 1\}$-free.
3. $E_4(G)$ is a block graph if and only if $G$ is $\{\text{paw, 4-pan}\} \cup \{C_n : n \geq 5\}$-free.

**Proof.** 1. If $G$ contains a paw in which $C_3 = (u_1, u_2, u_3)$ and $a$ is a pendant vertex attached to $u_1$, then in $E_4(G)$, $d(a u_1, u_2 u_3) = 2$, but they have two common neighbours $u_1 u_2$ and $u_1 u_3$. Similarly if $G$ contains a 4-pan in which $C_4 = (u_1, u_2, u_3, u_4)$ and $a$ is a pendant vertex attached to $u_1$, then in $E_4(G)$, $d(a u_1, u_3 u_4) = 2$, but they have two neighbours $u_1 u_2$ and $u_1 u_4$.

Conversely, suppose that $G$ is a $\{\text{paw, 4-pan}\}$-free graph. If $G$ is an acyclic graph, there exists a unique shortest path joining any two vertices in $E_4(G)$. Thus $E_4(G)$ is weakly geodetic.

Next suppose that $G$ contains cycles.

If $g(G) = 3$ then $G$ contains a $C_3 = (u_1, u_2, u_3)$.

**Claim.** $G$ is a cograph.

Suppose that $G$ contains an induced $P_4 = (v_1, v_2, v_3, v_4)$. Let $u_1 \neq v_1$. Consider a shortest path $(u_1, a_1, a_2, \ldots, a_k, v_1)$ joining $u_1$ and $v_1$. Since $G$ is paw free, $a_1$ must be adjacent to at least one more $u_i$, $i = 2, 3$. Proceeding like this, $v_1$ and then $v_2$ must be adjacent to at least two $u_i$’s. This implies that $v_1$ and $v_2$ must have a common neighbour among the $u_i$’s. Let it be $u_1$. Then $(v_1, u_1, v_2)$ form a $C_3$. Since $G$ is paw-free, $v_3$ must be adjacent to at least one of $v_1$ and $u_1$. But, since $(v_1, v_2, v_3, v_4)$ is an induced $P_4$, $v_3$ must be adjacent to $u_1$. Then $(v_1, u_1, v_3)$ will form a $C_3$ in $G$. Again since $G$ is paw-free, $v_4$ must be adjacent to $u_1$. Now, consider $(v_1, u_1, v_2)$ with the edge $u_1 - v_4$. Since $G$ is paw-free, $v_4$ must be adjacent to $v_1$ or $v_2$, which is a contradiction.
If \( g(G) = 4 \), then \( G \) contains a \( C_4 = (u_1, u_2, u_3, u_4) \). If \( G = C_4 \), then \( E_4(G) = K_4 \). If there exists a vertex \( v_1 \) in \( G \) which is adjacent to \( u_1 \), \( v_1 \) must be adjacent to \( u_3 \) also since \( G \) is 4-pan-free. Similarly if there exists a vertex \( v_2 \) which is adjacent to \( u_2 \), \( v_2 \) must be adjacent to \( u_4 \). If there exists a vertex \( v_1' \) which is adjacent to \( v_1 \), it must be adjacent to both \( u_2 \) and \( u_4 \). Hence \( G \) is a complete bipartite graph. Since \( g(G) = 4 \), \( G \) is paw-free. Again by Theorem 5, \( E_4(G) \) is complete, and hence \( G \) is a weakly geodetic graph.

Finally, Let \( g(G) = k \), \( k > 4 \). Let \( (u_1, u_2, u_3, \ldots, u_k) \) be a \( C_k \) in \( G \). Then \( E_4(G) \) also contains a \( C_k \). This \( C_k \) is not a part of any clique in \( E_4(G) \) and hence \( b(E_4(G)) \leq k \). Since \( G \) does not contain any \( C_4 \), two vertices in \( E_4(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent. Thus \( E_4(G) \) cannot contain a \( b \)-cycle of length less than \( k \) and so \( b(E_4(G)) = k \) where \( k > 4 \). We know that a graph \( G \) is weakly geodetic if and only if \( b(G) \geq 5 \). Thus \( E_4(G) \) is a weakly geodetic graph.

2. Let \( E_4(G) \) be a geodetic graph. If \( G \) contains a 4-pan, there exists more than one shortest path joining two vertices in \( E_4(G) \) as proved earlier. If \( G \) contains a \( C_{2n} = (u_1, u_2, \ldots, u_{2n}) \), then \( u_1u_2 \) and \( u_{n+1}u_{n+2} \) in \( E_4(G) \) are connected by more than one shortest path and hence \( E_4(G) \) is not geodetic. If \( G \) contains a \((2n-1)\)-pan in which \( C_{2n-1} = (u_1, u_2, \ldots, u_{2n-1}) \) and \( a \) is a pendant vertex attached to \( u_1 \), then \( au_1 \) and \( u_nu_{n+1} \) in \( E_4(G) \) are connected by more than one shortest path and hence \( E_4(G) \) is not geodetic.

Conversely, assume that \( G \) is \( \{4\text{-pan}, C_{2n}, (2n-1)\text{-pan}\} \)-free. If \( G \) is an acyclic graph there exists a unique shortest path joining any two vertices in \( E_4(G) \) and hence is geodetic. So consider the graphs \( G \) containing cycles.

Let \( g(G) = 3 \). Since \( G \) is paw-free, \( E_4(G) \) is complete and hence is geodetic. If \( g(G) = 4 \), \( E_4(G) \) is complete since \( G \) is 4-pan-free and thus geodetic. If \( g(G) = 2n - 1, n > 2 \), then \( G \) contains a \( C_{2n-1} = (u_1, u_2, \ldots, u_{2n-1}) \). If \( G = C_{2n-1} \), then \( E_4(G) = C_{2n-1} \) and hence geodetic. If \( a \) is a vertex attached to \( u_1 \), since \( G \) is \( (2n-1)\)-pan-free, \( a \) must be adjacent to at least one more \( u_i \). But this is impossible since \( g(G) = 2n - 1 \). Since \( G \) is \( C_{2n}\)-free, \( g(G) \neq 2n, n > 2 \). Hence in all the cases, it follows that \( E_4(G) \) is geodetic.

3. Let \( E_3(G) \) be a block graph. If \( G \) contains a paw in which \( C_3 = u_1, u_2, u_3 \) and \( a \) is the pendant vertex adjacent to \( u_1 \), then \( E_4(G) \) contains a \( C_4 = (au_1, u_1u_2, u_2u_3, u_3u_1) \) which is not a part of any clique. Thus \( b(E_4(G)) \leq 4 \). Similarly if \( G \) contains a 4-pan, in which \( C_4 =
$\{u_1, u_2, u_3, u_4\}$ and $a$ is a pendant vertex adjacent to $u_1$, then $E_4(G)$ contains a $C_4 = (au_1, u_1u_2, u_3u_4, u_4u_1)$ which is not a part of any clique and hence $b(E_4(G)) \leq 4$. If $G$ contains a $C_n$, $n > 4$, then $E_4(G)$ also contains a $C_n$, $n > 4$. This $C_n$ forms a $b$-cycle and hence $b(E_4(G)) \leq n$ and hence $E_4(G)$ is not a block graph.

Conversely, suppose that $G$ is $\{\text{paw, 4-pan}\} \cup \{C_n : n > 4\}$-free. If $G$ is an acyclic graph, then $E_4(G)$ cannot contain a $b$-cycle and hence is a block graph. Now, consider the graphs $G$ containing cycles. Since $G$ is $\{C_n : n \geq 5\}$-free, $g(G) = 3$ or $4$. But since $G$ is $\{\text{paw, 4-pan}\}$-free, $E_4(G)$ is complete as proved earlier and thus is a block graph.

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