

## DECOMPOSITIONS OF MULTIGRAPHS INTO PARTS WITH THE SAME SIZE

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### Abstract

Given a family  $\mathcal{F}$  of multigraphs without isolated vertices, a multigraph  $M$  is called  $\mathcal{F}$ -decomposable if  $M$  is an edge disjoint union of multigraphs each of which is isomorphic to a member of  $\mathcal{F}$ . We present necessary and sufficient conditions for existence of such decompositions if  $\mathcal{F}$  consists of all multigraphs of size  $q$  except for one. Namely, for a multigraph  $H$  of size  $q$  we find each multigraph  $M$  of size  $kq$ , such that every partition of the edge set of  $M$  into parts of cardinality  $q$  contains a part which induces a submultigraph of  $M$  isomorphic to  $H$ .

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### 1. INTRODUCTION

We consider finite undirected multigraphs without loops and isolated vertices. Given a family  $\mathcal{F}$  of multigraphs, an  $\mathcal{F}$ -decomposition of a multigraph  $M$  is a collection of submultigraphs which partition the edge set  $E(M)$  of  $M$  and are all isomorphic to members of  $\mathcal{F}$ . If such a decomposition exists,  $M$  is called  $\mathcal{F}$ -decomposable; and also  $H$ -decomposable if  $H$  is the only member of  $\mathcal{F}$ .

If  $M$  is a multigraph, then  $V(M)$  and  $E(M)$  stand for the vertex set and edge set of  $M$ , respectively. Cardinalities of those sets, denoted by  $v(M)$  and  $e(M)$ , are called the *order* and *size* of  $M$ , respectively. For  $E \subset E(M)$ ,

$M[E]$  denotes the submultigraph of  $M$  induced by  $E$ . The number of edges incident to a vertex  $x$  in  $M$ , denoted by  $\text{val}_M(x)$ , is called the *valency* of  $x$ , whilst the number of neighbours of  $x$  in  $M$ , denoted by  $\deg_M(x)$ , is called the *degree* of  $x$ . As usual  $\Delta(M)$  stands for the maximum valency among vertices of  $M$ . For any two vertices  $x, y$  of  $M$ , let  $p_M(x, y)$  denote the number of edges joining  $x$  and  $y$ . We call  $p_M(x, y)$  the *multiplicity* of an edge  $xy$  in  $M$ .  $\Pi(M)$  stands for the maximum multiplicity among edges of  $M$ . Edges joining the same vertices are called *parallel edges* (if they are distinct).

The multipath of length  $k$  with edge multiplicities  $m_1, \dots, m_k$  is denoted by  $P(m_1, \dots, m_k)$ . Note that  $P(1) = P_2 = K_2$  is the simple path on two vertices and  $P(1, 1) = P_3$  is the simple path on three vertices. The multistar of order  $k+1$  with edge multiplicities  $m_1, \dots, m_k$  is denoted by  $S(m_1, \dots, m_k)$ . Note that  $S(1, 1, 1) = K_{1,3}$  is a simple 3-star. The multicycle of length  $k$  with edge multiplicities  $m_1, \dots, m_k$  is denoted by  $C(m_1, \dots, m_k)$ . Note that  $C(1, 1, 1) = C_3$  is a 3-cycle. The multiplicity 0 is allowed in these cases. Note that  $S(1, 0, 1)$  is a path. The union of two disjoint multigraphs  $M$  and  $H$  is denoted by  $M \dot{\cup} H$  and the union of  $k$  disjoint copies of a multigraph  $H$  is denoted by  $kH$ . For a multigraph  $H$ , denote by  $H^{+e}$  the set of all multigraphs which we obtain from  $H$  by adding an edge. Note that  $K_2^{+e} = \{C_2, P_3, 2K_2\}$ .

Given a multigraph  $M$ , let  $G(M)$  denote a graph which we obtain from  $M$  by removal of all edges of the maximal family of pairwise edge-disjoint copies of  $C_2$ , i.e.,  $V(G(M)) := V(M)$  and  $E(G(M)) := \{xy : p_M(x, y) \equiv 1 \pmod{2}\}$ .

In [1] there are provided necessary and sufficient conditions for a multigraph  $M$  to be  $\{H_1, H_2\}$ -decomposable, where  $H_1, H_2$  are any two multigraphs out of  $C_2, P_3$  and  $2K_2$ . One of the results follows.

**Theorem 1** ([1]). *A multigraph  $M$  is  $\{C_2, 2K_2\}$ -decomposable if and only if each of the following five conditions holds:*

- (1)  $e(M) \equiv 0 \pmod{2}$ ;
- (2)  $\text{val}_M(x) + \deg_{G(M)}(x) \leq e(M)$  for every  $x \in V(M)$ ;
- (3) if  $xy \in E(G(M))$  then  $\text{val}_M(x) + \text{val}_M(y) - p_M(x, y) < e(M)$ ;
- (4) if  $yx, xz \in E(G(M))$ , then  $1 + \text{val}_M(x) + p_M(y, z) < e(M)$ ;
- (5)  $M$  is different from each of the (forbidden) multigraphs shown in Figure 1.

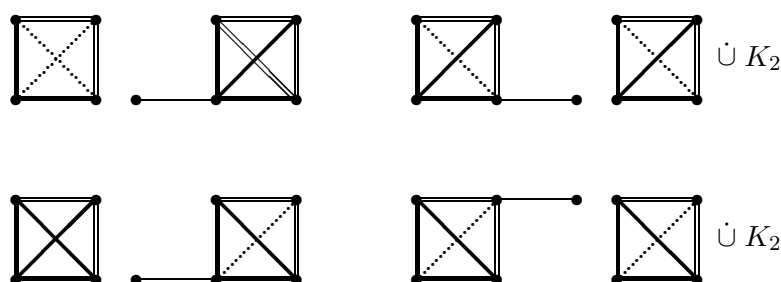


Figure 1. Forbidden multigraphs

Table 1. Codes in Figure 1.

edge :	bold	thin	doubled	dotted
multiplicity :	odd	1	even $\geq 2$	even $\geq 0$

As it can be seen the assertion is not trivial. So, the aim of this paper is to extend the previous result. For a positive integer  $k$ , let  $\mathcal{M}(k)$  be the set of all mutually non-isomorphic multigraphs of size  $k$ . For a multigraph  $H \in \mathcal{M}(k)$ , denote by  $\tilde{H}$  the set  $\mathcal{M}(k) - \{H\}$ . As  $\mathcal{M}(2) = \{C_2, P_3, 2K_2\}$ , Theorem 1 characterizes  $\tilde{P}_3$ -decomposable multigraphs. In the paper we characterize  $\tilde{H}$ -decomposable multigraphs where  $H$  is an arbitrary multigraph.

## 2. $\tilde{H}$ -DECOMPOSABLE MULTIGRAPHS

For a given multigraph  $H$  we define the family  $\mathcal{F}(H)$  as follows. A multigraph  $M$  belongs to  $\mathcal{F}(H)$  iff  $e(M) \equiv 0 \pmod{e(H)}$  and every partition of  $E(M)$  into parts of cardinality  $e(H)$  contains a part which induces a submultigraph of  $M$  isomorphic to  $H$ .

According to the definition of  $\mathcal{F}(H)$  we have immediately

**Proposition 1.** *A multigraph  $M$  is  $\tilde{H}$ -decomposable if and only if*

$$e(M) \equiv 0 \pmod{e(H)} \quad \text{and} \quad M \notin \mathcal{F}(H).$$

The previous characterization of  $\tilde{H}$ -decomposable multigraphs may be useful only for a multigraph  $H$  whose forbidden set  $\mathcal{F}(H)$  is described. Evidently,

$\mathcal{F}(K_2)$  includes all multigraphs. However, the forbidden sets of the others multigraphs are not so large, but a multigraph  $H$  is usually (except for some special cases) the only member of  $\mathcal{F}(H)$ . Next we describe the forbidden sets of exceptional multigraphs.

**Theorem 2.** *A multigraph  $M$  belongs to  $\mathcal{F}(P(k))$  if and only if  $e(M) \equiv 0 \pmod{k}$  and  $\Pi(M) > \frac{k-1}{k}e(M)$ .*

**Proof.** The condition  $e(M) \equiv 0 \pmod{k}$  is obvious. If there is an edge  $xy$  of a multigraph  $M$  with multiplicity  $p_M(x, y) > \frac{k-1}{k}e(M)$  then any partition of  $E(M)$  into parts of cardinality  $k$  contains a part consisting of  $k$  parallel edges joining  $x$  and  $y$ . It induces a submultigraph of  $M$  isomorphic to  $P(k)$ . On the other hand, if  $\Pi(M) \leq \frac{k-1}{k}e(M)$  then there is a partition of  $E(M)$  into parts of cardinality  $k$  such that any part contains at most  $k-1$  parallel edges. So,  $M \notin \mathcal{F}(P(k))$ . ■

**Theorem 3.** *A multigraph  $M$  belongs to  $\mathcal{F}(kK_2)$  if and only if  $e(M) \equiv 0 \pmod{k}$  and the number of odd size components of  $M$  is greater than  $\frac{k-2}{k}e(M)$ .*

**Proof.** The condition  $e(M) \equiv 0 \pmod{k}$  is obvious. It is proved in [1] that a multigraph is  $\{C_2, P_3\}$ -decomposable if and only if each its component has an even number of edges. Thus, if we remove an appropriate edge from each odd size component of a multigraph  $M$ , we get a  $\{C_2, P_3\}$ -decomposable multigraph. Therefore, the maximum number of mutually edge-disjoint pairs of adjacent edges in  $M$  is  $\frac{1}{2}(e(M) - c)$ , where  $c$  denotes the number of odd size components. If  $c > \frac{k-2}{k}e(M)$ , then  $M$  contains less than  $e(M)/k$  edge-disjoint pairs of adjacent edges. So, any partition of  $E(M)$  into parts of cardinality  $k$  contains a part consisting of  $k$  edges lying in distinct components. It induces a submultigraph of  $M$  isomorphic to  $kK_2$ . On the other hand, if  $c \leq \frac{k-2}{k}e(M)$  then there is a partition of  $E(M)$  into parts of cardinality  $k$  such that any part contains two adjacent edges. Thus,  $M \notin \mathcal{F}(kK_2)$ . ■

**Theorem 4.** *A multigraph  $M$  belongs to  $\mathcal{F}(K_{1,2})$  if and only if  $e(M)$  is even and at least one of the following four conditions holds:*

- (1) *there is  $x \in V(M)$  such that  $\text{val}_M(x) + \deg_{G(M)}(x) > e(M)$ ;*
- (2) *there is  $xy \in E(G(M))$  such that  $\text{val}_M(x) + \text{val}_M(y) - p_M(x, y) = e(M)$ ;*

- (3) there are  $yx, xz \in E(G(M))$  such that  $1 + \text{val}_M(x) + p_M(y, z) \geq e(M)$ ;
- (4)  $M$  is one of the multigraphs shown in Figure 1.

For  $k \geq 3$ ,  $M \in \mathcal{F}(K_{1,k})$  if and only if  $e(M) \equiv 0 \pmod{k}$  and there is a vertex  $x \in V(M)$  such that  $\text{val}_M(x) + \deg_{G(M)}(x) > 2\frac{k-1}{k}e(M)$ .

**Proof.** The first equivalence follows immediately from Theorem 1.

The condition  $e(M) \equiv 0 \pmod{k}$  is obvious. If  $M \notin \mathcal{F}(K_{1,k})$ , then there is a partition of  $E(M)$  into  $e(M)/k$  parts of cardinality  $k$  such that no part induces a submultigraph of  $M$  isomorphic to  $K_{1,k}$ . Hence, for any vertex  $x$ , every part contains either an edge not incident to  $x$  or two parallel edges incident to  $x$ . Therefore, the sum of the number of edges not incident to  $x$  and the number of edge-disjoint pairs of parallel edges incident to this vertex is at least  $e(M)/k$ , i.e.,  $(e(M) - \text{val}_M(x)) + (\text{val}_M(x) - \deg_{G(M)}(x))/2 \geq e(M)/k$ . This implies the inequality  $\text{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$ .

On the other hand, assume that  $M$  is a multigraph of size  $kt$  such that  $\text{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$  for every  $x \in V(M)$ . Evidently,  $M$  is not isomorphic to  $K_{1,k}$  if  $t = 1$ . For  $t \geq 2$ , consider a multigraph  $H := M \dot{\cup} mK_2$ , where  $m = (k-2)t$ . Clearly,  $e(H) = e(M) + m = 2(k-1)t$ . By Theorem 1, one can easily check that  $H$  is a  $\{C_2, 2K_2\}$ -decomposable multigraph. Therefore, there exist  $t = (k-1)t - m$  edge-disjoint pairs of edges  $e_i^1, e_i^2 \in E(M)$  such that  $M[\{e_i^1, e_i^2\}]$  is isomorphic to either  $C_2$ , or  $2K_2$ , for every  $i \in \{1, \dots, t\}$ . Thus, there is a partition of  $E(M)$  into parts of cardinality  $k$  such that the  $i$ -th part contains edges  $e_i^1$  and  $e_i^2$ . Clearly, none of these parts induces a submultigraph isomorphic to  $K_{1,k}$  and so  $M \notin \mathcal{F}(K_{1,k})$ . ■

In the next proofs we will use an induction, so the following description of forbidden sets will be very useful.

For a given multigraph  $H$  and each positive integer  $n$  we define the family  $\mathcal{F}_n(H)$  recursively as follows.  $H$  is the only member of  $\mathcal{F}_1(H)$ . For  $n > 1$  a multigraph  $M$  belongs to  $\mathcal{F}_n(H)$  iff  $e(M) = ne(H)$  and for every subset  $E \subset E(M)$  of cardinality  $|E| = e(H)$  it holds either  $M[E]$  is isomorphic to  $H$  or  $M[E(M) - E]$  is isomorphic to a member of  $\mathcal{F}_{n-1}(H)$ . According to the definition of  $\mathcal{F}(H)$  we have immediately

**Lemma 1.** Let  $H$  and  $M$  be multigraphs such that  $e(M) = ne(H)$ . The multigraph  $M$  belongs to  $\mathcal{F}(H)$  if and only if it belongs to  $\mathcal{F}_n(H)$ .

Put  $\mathcal{F}^*(H) := \cup_{i \geq 2} \mathcal{F}_i(H)$ . Then  $\mathcal{F}(H) = \cup_{i \geq 1} \mathcal{F}_i(H) = \mathcal{F}_1(H) \cup \mathcal{F}^*(H)$ . Note that the conditions  $\mathcal{F}_2(H) \neq \emptyset$ ,  $\mathcal{F}^*(H) \neq \emptyset$ ,  $|\mathcal{F}(H)| > 1$  are mutually equivalent.

We will often use the following auxiliary assertion.

**Lemma 2.** *Let  $H$  and  $M$  be multigraphs such that  $M \in \mathcal{F}_2(H)$ . Let  $E$  be a subset of  $E(M)$  such that  $M[E]$  is isomorphic to no submultigraph of  $H$ .*

*If  $|E| = e(H) - 1$ , then there is a multigraph  $H^* = M[E(M) - E] \in H^{+e}$  such that all edges of  $H^*$  have the same multiplicities and pairs of degrees of their end vertices.*

*If  $|E| < e(H) - 1$ , then  $H \in \{kK_2, K_{1,k}, P(k)\}$ .*

**Proof.** Let  $H^*$  be a submultigraph of  $M$  induced by  $E(M) - E$ . If we remove any edge from  $H^*$  (in the case  $|E| = e(H) - 1$ ), we get a multigraph isomorphic to  $H$ . So, all edges of  $H^*$  are equivalent and  $H^*$  has the required properties. Similarly, for  $|E| < e(H) - 1$ , if we remove any  $e(H) - |E| \geq 2$  edges from  $H^*$ , we get a multigraph isomorphic to  $H$ . Thus, all pairs of edges of  $H^*$  are equivalent in this case. Evidently,  $H^* \in \{P(t), K_{1,t}, tK_2\}$ , where  $t = e(H^*)$ . As  $H$  is a submultigraph of  $H^*$ , we get the assertion. ■

**Theorem 5.**  $\mathcal{F}^*(P(3, 2)) = \{P(7, 3)\}$  and  $\mathcal{F}^*(P(3) \dot{\cup} P(2)) = \{P(7) \dot{\cup} P(3)\}$ .

**Proof.** Suppose that  $M \in \mathcal{F}_2(P(3, 2))$ . Then  $e(M) = 10$  and  $\Pi(M) \geq 5$  because otherwise there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality five such that  $\Pi(M[E_i]) \leq 2$ ,  $i \in \{1, 2\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $P(3, 2)$ , a contradiction. According to Lemma 2, we get a multigraph  $H^*$  isomorphic to  $P(3, 3)$ , if we remove four parallel edges from  $M$ . It is easy to see that  $M = P(7, 3)$ , i.e.,  $\mathcal{F}_2(P(3, 2)) = \{P(7, 3)\}$ .

Now suppose that  $M \in \mathcal{F}_3(P(3, 2))$ . Then  $e(M) = 15$  and  $\Pi(M) \geq 7$  because  $P(7, 3)$  is a submultigraph of  $M$ . Let  $xy$  be an edge of  $M$  such that  $p_M(xy) = \Pi(M)$ . If we remove five parallel edges joining  $x$  and  $y$  from  $M$ , we get a multigraph isomorphic to  $P(7, 3)$ . Thus,  $M \in \{P(12, 3), P(8, 7)\}$ . However, it is easy to see that neither  $P(12, 3)$  nor  $P(8, 7)$  belongs to  $\mathcal{F}_3(P(3, 2))$ . Therefore,  $\mathcal{F}_3(P(3, 2)) = \emptyset$  and consequently  $\mathcal{F}_i(P(3, 2)) = \emptyset$  for every  $i \geq 3$ .

The second equality can be proved in the same manner, details are left to the reader. ■

**Theorem 6.**  $\mathcal{F}^*(P_4) = \{C(3, 1, 1, 1)\}$ .

**Proof.** Assume that  $M \in \mathcal{F}_2(P_4)$ . As  $M$  contains just six edges it is not difficult to check that  $M$  is not a simple graph (i.e.,  $\Pi(M) > 1$ ). According to Lemma 2, a multigraph which is obtained from  $M$  by deleting two parallel edges is a 4-cycle. Now it is easy to see that  $C(3, 1, 1, 1)$  is the only member of  $\mathcal{F}_2(P_4)$ .

Suppose that  $M \in \mathcal{F}_3(P_4)$ . As  $C(3, 1, 1, 1)$  is a submultigraph of  $M$ ,  $\Pi(M) \geq 3$ . If we remove any triple of parallel edges from  $M$ , we must obtain a multigraph isomorphic to  $C(3, 1, 1, 1)$ . So, there are two edge-disjoint triples of parallel edges in  $M$ . The only multigraph satisfying the previous two conditions is  $C(6, 1, 1, 1)$ , but it does not belong to  $\mathcal{F}_3(P_4)$ . Therefore,  $\mathcal{F}_3(P_4) = \emptyset$  and consequently  $\mathcal{F}_i(P_4) = \emptyset$  for every  $i \geq 3$ . ■

**Theorem 7.** For the multigraph  $P(k) \dot{\cup} K_2$  it holds:

$$\begin{aligned} \mathcal{F}(P(2) \dot{\cup} K_2) &= \{P(r) \dot{\cup} P(s) \dot{\cup} P(t) : 0 \leq r \equiv 0, s \equiv 1, t \equiv 2 \pmod{3}\} \cup \\ &\quad \{P(r, 1) \dot{\cup} P(t) : 0 \leq r \equiv 0, t \equiv 2 \pmod{3}\}, \\ \mathcal{F}(P(3) \dot{\cup} K_2) &= \{P(r) \dot{\cup} P(s) : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(P(k) \dot{\cup} K_2) &= \{P(r) \dot{\cup} K_2 : 3 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

**Proof.** Suppose that  $M \in \mathcal{F}_n(P(2) \dot{\cup} K_2)$  for  $n \geq 2$ . Then  $e(M) = 3n$  and  $\Pi(M) \geq 1 + n$  because otherwise there is a partition of  $E(M)$  into disjoint parts  $E_1, \dots, E_n$  of cardinality three such that  $\Pi(M[E_i]) \leq 1$ ,  $i \in \{1, \dots, n\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $P(2) \dot{\cup} K_2$ , a contradiction. If we remove any triple of parallel edges (of multiplicity  $\Pi(M)$ ) from  $M$ , we must obtain a multigraph isomorphic to a member of  $\mathcal{F}_{n-1}(P(2) \dot{\cup} K_2)$ . Thus, for  $n = 2$ ,  $M \in \{P(1) \dot{\cup} P(5), P(4) \dot{\cup} P(2), P(1, 3, 2), P(3, 2) \dot{\cup} P(1), P(3, 1) \dot{\cup} P(2), P(3) \dot{\cup} P(1) \dot{\cup} P(2)\}$ . Now, it is not difficult to check that  $\mathcal{F}_2(P(2) \dot{\cup} K_2) = \{P(3) \dot{\cup} P(1) \dot{\cup} P(2), P(1) \dot{\cup} P(5), P(4) \dot{\cup} P(2), P(3, 1) \dot{\cup} P(2)\}$ . Similarly, using induction for  $n \geq 3$ , we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

**Theorem 8.** For the multigraph  $P(k, 1)$  it holds:

$$\begin{aligned} \mathcal{F}(P(2, 1)) &= \{C(r, s, t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\} \cup \\ &\quad \{S(r, s, t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\} \cup \\ &\quad \{P(1, s, t) : s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\}, \\ \mathcal{F}(P(3, 1)) &= \{P(r, s) : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(P(k, 1)) &= \{P(r, 1) : 4 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

**Proof.** Suppose that  $M \in \mathcal{F}_n(P(2,1))$  for  $n \geq 2$ . Then  $e(M) = 3n$  and  $\Pi(M) \geq 1 + n$  because otherwise there is a partition of  $E(M)$  into disjoint parts  $E_1, \dots, E_n$  of cardinality three such that  $\Pi(M[E_i]) \leq 1$ ,  $i \in \{1, \dots, n\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $P(2,1)$ , a contradiction. If we remove any triple of parallel edges (of multiplicity  $\Pi(M)$ ) from  $M$ , we must obtain a multigraph isomorphic to a member of  $\mathcal{F}_{n-1}(P(2,1))$ . Thus, for  $n = 2$ ,  $M \in \{P(5,1), P(4,2), P(3) \dot{\cup} P(2,1), P(3,2,1), P(2,1,3), S(3,2,1), C(1,2,3)\}$ . Now, it is not difficult to check that  $\mathcal{F}_2(P(2,1)) = \{P(5,1), P(4,2), P(3,2,1), S(3,2,1), C(1,2,3)\}$ . Similarly, using induction for  $n \geq 3$ , we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

**Theorem 9.**  $\mathcal{F}^*(K_{1,2} \dot{\cup} K_{1,3}) = \{K_{1,3} \dot{\cup} K_{1,7}\}$ .

**Proof.** Suppose that  $M \in \mathcal{F}_2(K_{1,2} \dot{\cup} K_{1,3})$ . Then  $e(M) = 10$  and by Lemma 2  $M$  contains no parallel edges, no triangle and no 3-matching. Assume that  $\Delta(M) \leq 3$ . Then there is an equitable 4-edge-coloring of  $M$  (see [2]) and so there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality five ( $E_i$  consists of edges having two distinct colors) such that  $\Delta(M[E_i]) \leq 2$ ,  $i \in \{1, 2\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $K_{1,2} \dot{\cup} K_{1,3}$ , a contradiction. Therefore,  $\Delta(M) \geq 4$ . According to Lemma 2, a multigraph which is obtained from  $M$  by deleting four edges incident to a maximum degree vertex is isomorphic to  $2K_{1,3}$ . It is easy to see that  $M = K_{1,3} \dot{\cup} K_{1,7}$ , i.e.,  $\mathcal{F}_2(K_{1,2} \dot{\cup} K_{1,3}) = \{K_{1,3} \dot{\cup} K_{1,7}\}$ .

Now suppose that  $M \in \mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3})$ . Then  $e(M) = 15$  and  $\Delta(M) \geq 7$  because  $K_{1,3} \dot{\cup} K_{1,7}$  is a submultigraph of  $M$ . Let  $x$  be a vertex of  $M$  such that  $\deg_M(x) = \Delta(M)$ . If we remove five edges incident to  $x$  from  $M$ , we get a multigraph isomorphic to  $K_{1,3} \dot{\cup} K_{1,7}$ . Thus,  $M$  is a disjoint union of two multistars. However, it is easy to see that none of such multigraphs belongs to  $\mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3})$ . Therefore,  $\mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3}) = \emptyset$  and consequently, for every  $i \geq 3$ ,  $\mathcal{F}_i(K_{1,2} \dot{\cup} K_{1,3}) = \emptyset$ . ■

**Lemma 3.** Let  $M$  be a connected multigraph of size at least 4. Then there are edges  $e_1, e_2, e_3 \in E(M)$  satisfying

- (1)  $M[\{e_1, e_2, e_3\}]$  is either a matching or a connected submultigraph of  $M$ ;
- (2)  $M[E(M) - \{e_1, e_2, e_3\}]$  is a connected multigraph.



**Proof.** Suppose that  $M$  is a counterexample. Let  $T$  be a spanning tree of  $M$ . Evidently,  $M[E(M) - (E \cup \{e\})]$  is a connected multigraph for any pendant edge  $e$  of  $T$  and any set  $E \subseteq E(M) - E(T)$ . The multigraph  $M$  satisfies the following conditions.

**A.** *There is no pendant edge of  $T$  adjacent to two distinct edges of  $E(M) - E(T)$ .*

Suppose to the contrary that  $e_1, e_2 \in E(M) - E(T)$  are two distinct edges adjacent to a pendant edge  $e_3$  of  $T$ . Clearly, the multigraphs  $M[\{e_1, e_2, e_3\}]$  and  $M[E(M) - \{e_1, e_2, e_3\}]$  are connected, a contradiction.

**B.** *There is no edge of  $E(M) - E(T)$  adjacent to two pendant edges of  $T$ .* Assume that  $e_1 \in E(M) - E(T)$  is an edge adjacent to two pendant edges  $e_2, e_3$  of  $T$ . Moreover, assume that  $e_1$  is incident to a pendant vertex of  $T$  (if there exists such edge). If  $M[E(M) - \{e_1, e_2, e_3\}]$  is a disconnected multigraph then there is an edge  $e_4$  whose end vertices are pendant vertices of  $T$  (end vertices of  $e_2$  and  $e_3$ ). Thus, edges  $e_1$  and  $e_4$  are adjacent to a pendant edge of  $T$ , contrary to **A**. Therefore,  $M[\{e_1, e_2, e_3\}]$  and  $M[E(M) - \{e_1, e_2, e_3\}]$  are connected multigraphs, a contradiction.

**C.** *There is no 3-matching consisting of pendant edges of  $T$ .*

Suppose that  $e_1, e_2, e_3$  are three independent pendant edges of  $T$ . By **B**, there is no edge of  $E(M) - E(T)$  adjacent to any two pendant edges of  $\{e_1, e_2, e_3\}$ . Thus,  $M[E(M) - \{e_1, e_2, e_3\}]$  is a connected multigraph and  $M[\{e_1, e_2, e_3\}]$  is a matching, a contradiction.

**D.** *There are not three mutually adjacent pendant edges of  $T$ .*

Assume to the contrary that  $e_1, e_2, e_3$  are three adjacent pendant edges of  $T$ . By **B**, there is no edge of  $E(M) - E(T)$  adjacent to any two pendant edges of  $\{e_1, e_2, e_3\}$ . Therefore,  $M[E(M) - \{e_1, e_2, e_3\}]$  is a connected multigraph. The multigraph  $M[\{e_1, e_2, e_3\}]$  is also connected in this case, a contradiction.

**E.** *There are not two adjacent pendant edges of  $T$ .*

Suppose that  $e_1, e_2$  are two adjacent pendant edges of  $T$ . By **B**, there is no edge of  $E(M) - E(T)$  adjacent to  $e_1$  and  $e_2$ . If  $e_3$  is another edge of  $T$  adjacent to both of  $e_1, e_2$ , then  $M[\{e_1, e_2, e_3\}]$  and  $M[E(M) - \{e_1, e_2, e_3\}]$  are connected multigraphs, a contradiction.

**F.** *There is no edge of  $E(M) - E(T)$  adjacent to a pendant edge of  $T$ .*

According to **C** and **E**, the tree  $T$  is a path. Assume that  $e_1 \in E(M) - E(T)$  is an edge adjacent to a pendant edge  $e_2$  of  $T$ . By **A**, there is no other edge of  $E(M) - E(T)$  adjacent to  $e_2$ . If  $e_3$  is the edge of  $T$  adjacent to  $e_2$ , then  $M[\{e_1, e_2, e_3\}]$  and  $M[E(M) - \{e_1, e_2, e_3\}]$  are connected multigraphs, a contradiction.

By **A** – **F**, the tree  $T$  is a path and there is no edge of  $E(M) - E(T)$  adjacent to a pendant edge of  $T$ . If  $e_1$  is any pendant edge of  $T$  and edges  $e_1, e_2, e_3$  induce a subpath of  $T$ , then  $M[\{e_1, e_2, e_3\}]$  and  $M[E(M) - \{e_1, e_2, e_3\}]$  are connected multigraphs, a contradiction. Therefore, there is no counterexample of the assertion. ■

**Theorem 10.** *For the multigraph  $K_{1,k} \dot{\cup} K_2$  it holds:*

$$\begin{aligned} \mathcal{F}(K_{1,2} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_{1,s} \dot{\cup} K_{1,t} : r \equiv 1, s \equiv 2, t \equiv 0 \pmod{3}\} \cup \\ & \{K_{1,r} \dot{\cup} K_{1,s} \dot{\cup} K_3 : r \equiv 1, s \equiv 2 \pmod{3}\} \cup \\ & \{K_{1,r} \dot{\cup} C_5 : r \equiv 1 \pmod{3}\} \cup \\ & \{H \dot{\cup} K_{1,s} : H \in K_{1,t}^{+e}, t \equiv 0, s \equiv 2 \pmod{3}\} \cup \\ & \{H \dot{\cup} K_{1,s} : H \in K_3^{+e}, s \equiv 2 \pmod{3}\} \cup \\ & \{S(3, 1, \dots, 1) \dot{\cup} K_{1,s} : e(S(3, 1, \dots, 1)) \equiv 1, s \equiv 2 \pmod{3}\} \cup \\ & \{P(2, 1, 1) \dot{\cup} K_{1,s} : s \equiv 2 \pmod{3}\}, \\ \mathcal{F}(K_{1,3} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_{1,s} : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(K_{1,k} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_2 : 4 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

**Proof.** Suppose that  $M \in \mathcal{F}_n(K_{1,2} \dot{\cup} K_2)$ . If  $n = 2$ , then  $e(M) = 6$  and by Lemma 3  $M$  is disconnected. If  $\Pi(M) > 1$ , then according to Lemma 2, a multigraph which is obtained from  $M$  by deleting two parallel edges is isomorphic to  $2K_{1,2}$ . Now it is easy to check that  $M \in \{S(3, 1) \dot{\cup} K_{1,2}, C(2, 1, 1) \dot{\cup} K_{1,2}, S(2, 1, 1) \dot{\cup} K_{1,2}, P(2, 1, 1) \dot{\cup} K_{1,2}\}$ . If  $\Pi(M) = 1$ , then  $M$  has at most three components because otherwise there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality three such that  $M[E_1]$  is a matching and  $M[E_2]$  is either a matching or a component of  $M$ , a contradiction. If  $M$  has three components, then by Lemma 3 the size of each component is at most three. Now, it is not difficult to check that  $M$  is isomorphic to either  $K_2 \dot{\cup} K_{1,2} \dot{\cup} K_{1,3}$  or  $K_2 \dot{\cup} K_{1,2} \dot{\cup} K_3$  in this case. If  $M$  has two components, then there is a component of size at most two because otherwise the components decompose  $M$  into two connected submultigraphs of size three. If  $M$  has a component of size 1, then verifying twenty possible graphs it is not difficult to check that  $M$  is isomorphic to either  $K_{1,5} \dot{\cup} K_2$  or  $C_5 \dot{\cup} K_2$ . Similarly, if  $M$  has a component of size two, then it is not difficult to check that  $M$  is a graph belonging to  $\{H \dot{\cup} K_{1,2} : H \in K_{1,3}^{+e}\}$ . Thus, the assertion holds in this case.

If  $n \geq 3$ , then it is not difficult to see (using induction and Lemma 3) that  $M$  includes a vertex of degree at least three. Assume that  $x \in V(M)$  is a vertex of maximum degree. If we remove three (non-parallel) edges

incident to  $x$  from  $M$ , we get a multigraph belonging to  $\mathcal{F}_{n-1}(K_{1,2} \dot{\cup} K_2)$ . Using induction it is not difficult to check that the assertion holds.

Now suppose that  $M \in \mathcal{F}_n(K_{1,3} \dot{\cup} K_2)$ . If  $n = 2$ , then  $e(M) = 8$ . According to Lemma 2  $M$  contains no parallel edges, no 3-matching, no triangle and no path of length three. Now, it is not difficult to check that  $M$  is isomorphic to either  $K_2 \dot{\cup} K_{1,7}$  or  $K_{1,3} \dot{\cup} K_{1,5}$ . Evidently,  $\Delta(M) > 4$  for  $n \geq 3$ . If we remove four edges incident to a maximum degree vertex from  $M$ , we get a multigraph belonging to  $\mathcal{F}_{n-1}(K_{1,3} \dot{\cup} K_2)$ . Using induction we get the assertion.

The last equality can be proved in the same manner, details are left to the reader.  $\blacksquare$

We conclude this paper with the following result.

**Theorem 11.** *Let  $H$  be a multigraph.  $|\mathcal{F}(H)| > 1$  if and only if  $H$  is one of the following multigraphs:*

- (1)  $P(k)$ , for every positive integer  $k$ ;
- (2)  $P(k, 1)$ , for every positive integer  $k$ ;
- (3)  $P(k) \dot{\cup} K_2$ , for every positive integer  $k$ ;
- (4)  $P(3, 2)$ ;
- (5)  $P(3) \dot{\cup} P(2)$ ;
- (6)  $kK_2$ , for every positive integer  $k$ ;
- (7)  $K_{1,k}$ , for every positive integer  $k$ ;
- (8)  $K_{1,k} \dot{\cup} K_2$ , for every positive integer  $k$ ;
- (9)  $K_{1,2} \dot{\cup} K_{1,3}$ ;
- (10)  $P_4$ .

**Proof.** According to previous theorems,  $|\mathcal{F}(H)| > 1$  for every multigraph  $H$  of the list (1)–(10).

On the other hand, let us assume to the contrary that  $H$  is a multigraph such that  $|\mathcal{F}(H)| > 1$  and it does not belong to the list (1)–(10). So, there is a multigraph  $M \in \mathcal{F}_2(H)$ . Consider the following cases.

**A.**  $\Pi(H) = \Pi > 1$ . As the multigraph  $H$  is not belonging to the list (1)–(10),  $4 \leq e(H) \geq \Pi + 2$ . Moreover,  $\Pi(M) \geq 2\Pi - 1$  because otherwise there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality  $e(H)$  such that  $\Pi(M[E_i]) \leq \Pi - 1$ ,  $i \in \{1, 2\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $H$ , a contradiction. The multigraph  $H$  contains no  $\Pi + 1$  parallel edges

and so according to Lemma 2,  $e(H) = \Pi + 2$  and there is a multigraph  $H^* \in H^{+e} \cap \{P(3, 3), P(3) \dot{\cup} P(3)\}$ . Therefore,  $H \in \{P(3, 2), P(3) \dot{\cup} P(2)\}$ , i.e.,  $H$  appears in the list, a contradiction.

**B.**  $\Pi(H) = 1$ . Thus,  $H$  is a simple graph. If  $M$  contains parallel edges then according to Lemma 2,  $e(H) = 3$  and there is a simple graph  $H^* \in H^{+e}$  such that all its edges have the same pairs of degrees of their end vertices. Therefore,  $H^* \in \{K_{1,4}, C_4, 2K_{1,2}, 4K_2\}$ . Hence,  $H \in \{K_{1,3}, P_3, K_{1,2} \dot{\cup} K_2, 3K_2\}$ , a contradiction. Therefore,  $M$  is also a simple graph. Consider the following subcases.

**B1.**  $\Delta(H) = e(H) - 1$ . Then  $H$  is either  $K_3$  or a connected graph belonging to  $K_{1,k}^{+e}$ , where  $k \geq 3$ . If  $H = K_3$ , then according to Lemma 2,  $M$  contains no 2-matching. However,  $K_{1,6}$  does not belong to  $\mathcal{F}_2(K_3)$  and so  $\mathcal{F}_2(K_3) = \emptyset$ . If  $H$  is a connected graph belonging to  $K_{1,k}^{+e}$ , then according to Lemma 2,  $M$  contains no 3-matching. Thus,  $M$  is a supergraph of  $H$  having no 3-matching. It is not difficult to check that none of such graphs belongs to  $\mathcal{F}_2(H)$  and so  $\mathcal{F}_2(H) = \emptyset$ , a contradiction.

**B2.**  $\Delta(H) < e(H) - 1$  and  $\Delta(M) > \Delta(H)$ . According to Lemma 2,  $e(H) = \Delta(H) + 2$  and there is a graph  $H^* \in H^{+e} \cap \{2K_{1,3}, K_{2,3}, K_4, C_5\}$ . It is not difficult to check that  $\mathcal{F}_2(K_{2,3} - e) = \emptyset$ ,  $\mathcal{F}_2(K_4 - e) = \emptyset$  and  $\mathcal{F}_2(P_5) = \emptyset$ . Thus,  $H = K_{1,2} \dot{\cup} K_{1,3}$  appears in the list, a contradiction.

**B3.**  $\Delta(H) < e(H) - 1$  and  $\Delta(M) = \Delta(H)$ . For  $\Delta(H) \geq 3$ , there is a positive integer  $k$  such that  $\Delta(H) + 1 \leq 2k$  and  $k < \Delta(H)$ . Then there is an equitable  $2k$ -edge-coloring of  $M$  (see [2]) and so there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality  $e(H)$  ( $E_i$  consists of edges having  $k$  distinct colors) such that  $\Delta(M[E_i]) \leq k$ ,  $i \in \{1, 2\}$ , i.e.,  $M[E_i]$  is not isomorphic to  $H$ , a contradiction. Thus,  $\Delta(H) = 2$  and  $e(H) \geq 4$ . As  $\Delta(M) = 2$  there is a partition of  $E(M)$  into parts  $E_1, E_2$  of cardinality  $e(H)$  such that the size of each component of  $M[E_i]$ ,  $i \in \{1, 2\}$ , is at most two. Therefore, each component of  $H$  has at most two edges, i.e.,  $P_4$  is not a subgraph of  $H$ . According to Lemma 2 there is no appropriate graph  $M$ , a contradiction. ■

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