

## CHVÁTAL-ERDÖS TYPE THEOREMS

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### Abstract

The Chvátal-Erdős theorems imply that if  $G$  is a graph of order  $n \geq 3$  with  $\kappa(G) \geq \alpha(G)$ , then  $G$  is hamiltonian, and if  $\kappa(G) > \alpha(G)$ , then  $G$  is hamiltonian-connected. We generalize these results by replacing the connectivity and independence number conditions with a weaker minimum degree and independence number condition in the presence of sufficient connectivity. More specifically, it is noted that if  $G$  is a graph of order  $n$  and  $k \geq 2$  is a positive integer such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - k)/(k + 1)$ , and  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian. It is shown that if  $G$  is a graph of order  $n$  and  $k \geq 3$  is a

positive integer such that  $\kappa(G) \geq 4k^2 + 1$ ,  $\delta(G) > (n + k^2 - 2k)/k$ , and  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected. This result supports the conjecture that if  $G$  is a graph of order  $n$  and  $k \geq 3$  is a positive integer such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - 2k)/k$ , and  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected, and the conjecture is verified for  $k = 3$  and  $4$ .

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## 1. INTRODUCTION

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. Given a subset of vertices (or subgraph)  $H$  of a graph  $G$  and a vertex  $v$ , let  $d_H(v)$  denote the degree of  $v$  relative to  $H$ , and  $N_H(v)$  the neighborhood of  $v$  in  $H$ . The minimum degree, independence number, and connectivity of  $G$  will be denoted by  $\delta(G)$ ,  $\alpha(G)$ , and  $\kappa(G)$  respectively.

Two classical results of Chvátal and Erdős [2] are the following:

**Theorem 1.** *If  $G$  is a graph of order  $n \geq 3$  such that  $\kappa(G) \geq \alpha(G)$ , then  $G$  is hamiltonian.*

**Theorem 2.** *If  $G$  is a graph of order  $n \geq 3$  such that  $\kappa(G) > \alpha(G)$ , then  $G$  is hamiltonian-connected.*

The following result on the existence of hamiltonian cycles, which is an analogue of Theorem 1, will be proved. Actually, we note that Proposition 1 is an easy consequence of a result of Fraisse [5] and follows from a result of Ota [6] with the appropriate interpretation of the condition on  $\alpha$ .

**Proposition 1.** *Let  $G$  be a graph of order  $n$  and  $k \geq 2$  a positive integer such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - k - 1)/(k + 1)$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian.*

Corresponding to the hamiltonian result of Theorem 1 and an analogue to Theorem 2, we make the following conjecture.

**Conjecture 1.** Let  $G$  be a graph of order  $n$  and  $k \geq 3$  a positive integer such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - 2k)/k$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected.

We prove the following two results supporting Conjecture 1. The first result has the same degree and independence number conditions and conclusion as the conjecture, but requires a higher connectivity assumption on the graph. The second result verifies the conjecture for the cases  $k = 3$  and 4.

**Theorem 3.** Let  $G$  be a graph of sufficiently large order  $n$  and  $k \geq 3$  a positive integer such that  $\kappa(G) \geq 4k^2 + 1$ ,  $\delta(G) > (n + k^2 - 2k)/k$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected.

**Theorem 4.** Let  $G$  be a graph of sufficiently large order  $n$  and  $k = 3$  or 4 such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - 2k)/k$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected.

In Section 2 we will give a short proof of Proposition 1 and present a family of graphs which show that none of the conditions in Proposition 1 and Theorem 4 can be weakened. In Section 3 we prove the main results.

## 2. PRELIMINARY RESULTS AND SHARPNESS EXAMPLES

We begin by describing graphs  $H_i$  for  $1 \leq i \leq 5$  which demonstrate the sharpness of the conditions in Proposition 1 and Theorem 4.

For  $k \geq 2$ , let  $H_1(k) = K_k + (k+1)K_{(n-k)/(k+1)}$ , where  $n \equiv k \pmod{k+1}$ . Since there are  $k+1$  components in the graph  $H_1(k) - K_k$ , the graph  $H_1(k)$  is not hamiltonian. Also,  $\kappa(H_1(k)) = k$ ,  $\delta(H_1(k)) = (n + k^2 - k - 1)/(k+1)$ , and  $\alpha(H_1(k)) = k + 1 \leq \delta(H_1(k))$  for  $n$  sufficiently large.

For  $k \geq 2$ , let  $H_2(k) = K_{(n-k)/(k+1)} + ((n+1)/(k+1))K_k$  where  $n \equiv k \pmod{k+1}$ . Since there are strictly more than  $(n-k)/(k+1)$  components in  $H_2(k) - K_{(n-k)/(k+1)}$ , the graph  $H_2(k)$  is not hamiltonian. Also,  $\delta(H_2(k)) = (n + k^2 - k - 1)/(k+1)$ , and  $\alpha(H_2(k)) = (n+1)/(k+1) = \delta(H_2(k)) - (k-2)$ .

For  $k \geq 3$ , let  $H_3(k) = K_k + kK_{(n-k)/k}$ , when  $n \equiv 0 \pmod{k}$ . Since there are as many components in  $H_3(k) - K_k$  as there are vertices in  $K_k$ , the graph  $H_3(k)$  is not hamiltonian-connected. Also,  $\kappa(H_3(k)) = k$ ,  $\delta(H_3(k)) = (n + k^2 - 2k)/k$ , and  $\alpha(H_3(k)) = k \leq \delta(H_3(k))$  for  $n$  sufficiently large.

For  $k \geq 3$ , let  $H_4(k) = K_{n/k} + (n/k)K_{k-1}$  where  $n \equiv 0 \pmod k$ . The graph  $H_4(k)$  is not hamiltonian-connected, since there are as many components in  $H_4(k) - K_{n/k}$  as there are vertices in  $K_{n/k}$ . Also,  $\delta(H_4(k)) = (n+k^2-2k)/k$ , and  $\alpha(H_4(k)) = n/k = \delta(H_4(k)) - (k-2)$ .

For  $(n+1)/3 < \delta(G) < n/2$ , the graph  $H_5(\delta) = K_\delta + (\overline{K}_\delta \cup K_{n-2\delta})$  is  $\delta$ -connected, not hamiltonian and not hamiltonian-connected, and  $\alpha(H_5(\delta)) = \delta(H_5(\delta)) + 1$ .

The graph  $H_1(k)$  implies the minimum degree condition of Theorem 1 cannot be decreased for  $k \geq 2$ . Note that the graph  $H_2(k-1)$  satisfies the relationship  $\delta(H_2(k-1)) = \alpha(H_2(k-1)) + k - 3$ , and all of the other conditions of Theorem 1, so the conditions cannot be decreased in Theorem 1 for  $k \geq 3$ . The graph  $H_5(\delta)$  verifies the sharpness of Theorem 1 when  $k = 2$ .

The graph  $H_3(k)$  implies that the minimum degree conditions of Conjecture 1 and Theorem 4 cannot be decreased. Note that the graph  $H_4(k-1)$  satisfies the relationship  $\delta(H_4(k-1)) \geq \alpha(H_4(k-1)) + k - 3$  and all of the other conditions of Conjecture 1, so the conditions cannot be decreased in Conjecture 1 and also Theorem 4 for  $k \geq 4$ . The graph  $H_5(\delta)$  verifies the sharpness of Conjecture 1 and Theorem 4 when  $k = 3$ .

The above examples verify the sharpness of Theorem 1, Conjecture 1 and Theorem 4 when  $n$  satisfies the appropriate congruence relative to  $k$ . Analogous examples, which are less symmetric, can be described for general  $n$ .

Before starting our main proofs, for convenience we describe additional notation. Given a positive integer  $p$ ,  $\sigma_p(G) = \min\{d(v_1) + d(v_2) + \dots + d(v_p) : \{v_1, v_2, \dots, v_p\}$  is an independent set of  $G\}$ . Given a positive integer  $\lambda$ , a cycle  $C$  in a graph  $G$  is a  $D_\lambda$ -cycle if each component of  $G - C$  has fewer than  $\lambda$  vertices. With this notation, we can state the following result of Fraïsse in [5].

**Theorem A.** *If  $G$  is a  $k$ -connected graph of order  $n \geq 3$  with  $\sigma_{k+1}(G) \geq n + k(k-1)$ , then  $G$  contains a  $D_k$ -cycle.*

**Proof (of Proposition 1).** Select a maximal length cycle  $C$  of  $G$  that is a  $D_k$ -cycle. Theorem A implies that such a cycle exists. If  $C$  is hamiltonian, then the proof is complete, so assume not. Select a vertex  $v$  in one of the components, say  $H$ , of  $G - C$ . Since  $|H| \leq k-1$ , we know  $d_H(v) \leq k-2$ , and so  $d_C(v) \geq \delta(G) - k + 2$ . Let  $S = N_C(v)$ , and let  $S^+$  denote the successors of  $S$  on  $C$  for some orientation of  $C$ . Since  $C$  is a maximal length  $D_k$ -cycle,

the set  $S^* = S^+ \cup \{v\}$  is an independent set with at least  $\delta(G) - k + 3$  vertices. This is a contradiction, since  $\alpha(G) \leq \delta(G) - k + 2$ . This completes the proof of Proposition 1. ■

Proposition 1 is also a direct consequence of the main result of Ota [6], and is related to the following corollary of the main result of Ota [6].

**Theorem B.** *Let  $G$  be a  $k$ -connected graph of order  $n(n \geq 3)$  with  $\alpha(G) \leq (n + 1)/(k + 1) + 1$ . If  $\sigma_{k+1}(G) \geq n + k^2 - k$ , then  $G$  is hamiltonian.*

### 3. MAIN RESULTS

**Theorem 3.** *Let  $G$  be a graph of sufficiently large order  $n$  and  $k \geq 3$  a positive integer such that  $\kappa(G) \geq 4k^2 + 1$ ,  $\delta(G) > (n + k^2 - 2k)/k$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected.*

**Proof.** Suppose that  $G$  is not hamiltonian-connected, and select distinct vertices  $x, y \in V(G)$  and let  $P$  be a path in  $G$  from  $x$  to  $y$  with the maximum number of vertices, say  $m < n$ . Let  $t = 4k^2 + 1$ , so  $\kappa(G) \geq t$ .

We would like to show that each vertex of  $H = G - P$  may be adjacent to at most  $\alpha(G)$  vertices of  $P$ . Suppose a vertex  $v \in H$  has a set  $S$  of at least  $\alpha(G) + 1$  adjacencies in  $P$ . Since, for either orientation of  $P$ , at most one vertex of  $S$  will not have a successor, we know  $|S^+| \geq |S| - 1$ . Because  $P$  is a path of maximum length,  $S^+ \cup \{v\}$  must be an independent set of order at least  $\alpha(G) + 1$ , which is a contradiction. Thus,  $\delta(H) \geq \delta(G) - \alpha(G) \geq k - 2$ .

Let  $s$  be the cardinality of a maximum length cycle  $C \subseteq H$ , if a cycle exists. If  $H$  has no cycle, let  $s = 1$ .

**Claim 1.**  $s < 2(m - 1)/(t - 1)$ .

**Proof.** Since the conditions of Theorem 3 imply that  $G$  is hamiltonian, so there is a path between  $x$  and  $y$  that contains at least  $(n + 1)/2$  vertices. Thus, we have  $m \geq \delta(G)$ , since  $\delta(G) \geq (n + 1)/2$  would imply hamiltonian-connected by the result of Dirac ([3]). Assume  $s < t = 4k^2 + 1$ . Then, since  $m \geq \delta(G) \geq (n + k^2 + k - 1)/(k + 1)$ , for  $n$  sufficiently large,  $s < 2(m - 1)/(t - 1)$ . Thus, assume  $s \geq t$ . There exist  $t$  vertex-disjoint paths between  $C$  and  $P$ . Two of these paths, say  $Q_1$  and  $Q_2$ , have end vertices in  $P$  with at most  $(m - 1)/(t - 1) - 1$  vertices of  $P$  between them. The end vertices of  $Q_1$  and  $Q_2$  on  $C$  have at least  $(s - 2)/2$  vertices between

them in one direction around the cycle  $C$ . The maximality of  $P$  implies that  $(s-2)/2 + 2 \leq (m-1)/(t-1) - 1$ . Therefore,  $s < 2(m-1)/(t-1)$ . This completes the proof of the claim. ■

**Claim 2.** The order of the path  $P$  is given by  $m \geq kn/(k+1)$ .

**Proof.** Our proof is by contradiction. Since the longest cycle in  $H$  has length  $s$ , the endvertices of a longest path in  $H$  have degree less than  $s$ . Therefore, there exist two vertices, say  $u$  and  $v$ , such that  $d_H(u), d_H(v) \leq s$ . Furthermore, since  $\delta(H) \geq k-2$ , this path, denoted by  $Q$ , from  $u$  to  $v$  in  $H$  has at least  $k-1$  vertices.

Since  $P$  is a maximum length path, no vertex of  $H$  can be adjacent to two consecutive vertices of  $P$ . Let  $U = N_P(u), V = N_P(v), W = U \cap V$  and, without loss of generality, assume that  $|U| \leq |V|$ . Thus, there are  $|U| + |V| - |W|$  vertices of  $P$  adjacent to either  $u$  or  $v$ , and there are the same number, or possibly one more or one less of “open” intervals of  $P$  with no adjacencies to either  $u$  or  $v$ . Since the path  $P$  is of maximum length, each “open” interval contains at least one vertex, and those intervals between a vertex in  $U$  and a vertex in  $V$ , which will we call “long” intervals, contain at least  $k-1$  vertices. There are at least  $|W| - 1$  such “long” intervals. Thus,

$$(1) \quad \begin{aligned} m &\geq |U| + |V| - |W| + (k-1)(|W| - 1) + (|U| + |V| - 2|W|) \\ &= 2|U| + 2|V| + (k-4)|W| - k + 1. \end{aligned}$$

All of the vertices in  $(U \cup V)^+ \cup \{u\}$  are independent, since any edge between vertices in this set would contradict the fact that  $P$  was chosen of maximum length. This implies that  $|U| + |V| - |W| \leq \alpha(G) \leq \delta(G) - k + 2$ . Therefore  $(\delta(G) - d_H(u)) + (\delta(G) - d_H(v)) - \delta(G) + k - 2 \leq |W|$ , and so

$$|W| \geq \delta(G) - 4(m-1)/(t-1) + k.$$

Hence, by Equation 1, we get

$$\begin{aligned} m &\geq 4(\delta(G) - s + 1) + (k-4)|W| - k + 1 \\ &\geq 4(\delta(G) - 2(m-1)/(t-1) + 1) \\ &\quad + (k-4)(\delta(G) - 4(m-1)/(t-1) + k) - k + 1. \end{aligned}$$

Using the bounds  $m < kn/(k+1)$  and  $\delta(G) \geq (n+k^2-2k)/k$  in the previous

equation gives

$$(t + 4k - 9)kn/(k + 1) > (t - 1)(n + k^2 - 2k) + (t - 1)(k^2 - 5k + 5) + 4k - 8.$$

However, this implies  $t \leq 4k^2 - 8k + 1$  which is a contradiction. Finally we may conclude that  $|P| = m \geq kn/(k + 1)$  completing the proof of the claim. ■

Assume that  $P$  is not a hamiltonian path. Select a longest path  $Q$  in  $H$ , say with  $q$  vertices and with end vertices  $u$  and  $v$ . Note that since  $\delta(H) \geq k - 2$ , we get  $q \geq k - 1$ . Recall that each vertex of  $H$  has at most  $\alpha(G)$  adjacencies in  $P$ . In fact,  $|N_P(Q)| \leq \alpha(G)$  for the same reason. Assume that  $s = d_H(u) \geq d_H(v)$ , thus  $u$  has  $s \geq k - 2$  adjacencies on  $Q$ . Denote the predecessors of these  $s$  vertices by  $\{u = u_1, u_2, \dots, u_s\}$ . Between  $v$  and any of the vertices  $u_i$  for  $1 \leq i \leq s$ , there is a path with  $q$  vertices, and between  $u_i$  and  $u_j$  for  $i \neq j$  there is a path with at least  $(s + 1)/2$  vertices. There is no loss of generality in assuming that  $d_H(u_i) \leq s$  for all  $i$ , and so each vertex of  $S = \{u_1, u_2, \dots, u_s, v\}$  has at least  $\delta(G) - s$  adjacencies in  $P$ .

As before, let  $U = N_P(u), V = N_P(v), W = U \cap V$ , and assume that  $|U| \leq |V|$ . Since  $d_P(u), d_P(v) \geq (n/k) + k - 2 - s$  and  $\alpha(G) \leq n/k$ , we have  $|W| = |U| + |V| - |U \cup V| \geq 2((n/k) + k - 2 - s) - n/k = (n/k) + 2k - 4 - 2s$ . This implies

$$\begin{aligned} n - s - 1 &\geq n - q \geq (s + 2)|W| + (|U| - |W|) + (|V| - |W|) - 1 \\ &\geq m \geq (s + 2)((n/k) + 2k - 4 - 2s) + 2(s + 2 - k) - 1. \end{aligned}$$

Therefore, we know

$$2s^2 - ((n/k) + 2k - 5)s + n - (2n)/k - 2k + 4 \geq 0.$$

However, for  $k - 2 \leq s \leq n/(2k) - 1$  the previous inequality is contradicted, so we can assume that  $s > n/(2k) - 1$ . We have previously shown in Claim 1 that  $s \leq n/(2k^2)$ . Therefore, the assumption that the path  $P$  was not a hamiltonian path between  $x$  and  $y$  is contradicted. This completes the proof of Theorem 3. ■

The conditions of Theorem 3 are sharp except for the condition on the connectivity  $\kappa(G)$ , but for small values of  $k$ , we prove Theorem 4 which uses the sharp condition for  $\kappa(G)$ .

**Theorem 4.** *Let  $G$  be a graph of sufficiently large order  $n$  and  $k = 3$  or  $4$  such that  $\kappa(G) \geq k$ ,  $\delta(G) > (n + k^2 - 2k)/k$ . If  $\delta(G) \geq \alpha(G) + k - 2$ , then  $G$  is hamiltonian-connected.*

**Proof.** Suppose that  $G$  is not hamiltonian-connected, and select distinct vertices  $x$  and  $y$  and let  $P$  be a path of  $G$  from  $x$  to  $y$  with a maximum number of vertices, say  $m < n$ . Let  $H = G - P$ . If  $\kappa(G) \geq 4k^2 + 1$ , then the proof is complete by Theorem 3, so we can assume that  $k \leq \kappa(G) \leq 4k^2$ .

We would like to show each vertex of  $H$  can be adjacent to at most  $\alpha(G)$  vertices of  $P$ . Otherwise, a vertex  $v \in H$  has a set  $S$  of at least  $\alpha(G) + 1$  adjacencies in  $P$ . Note that  $|S^+| \geq |S| - 1$ , since at most one vertex of  $S$  will not have a successor. Since  $P$  is a path of maximum length, this implies that  $S^+ \cup \{v\}$  is an independent set of order at least  $\alpha(G) + 1$ , a contradiction. Thus,  $\delta(H) \geq \delta(G) - \alpha(G) \geq k - 2$ .

We next show that  $|P| \geq (k - 1)n/k + k - 1$ , so assume not. Select a minimal cutset  $S$  of  $G$ , and let  $C_1, C_2, \dots, C_t$  be the components of  $G - S$ . Thus,  $k \leq |S| = s \leq 4k^2$ , and we can assume that  $|C_1| \geq |C_2| \geq \dots \geq |C_t|$ .

First consider the case  $k = 3$ , and so  $\delta(G) \geq n/3 + 2$ ,  $\alpha(G) \leq \delta(G) - 1$ , and  $\kappa(G) \leq 36$ . If  $t \geq 4$ , then  $|C_t| \leq (n - s)/t$ , and for any vertex  $v \in C_t$ ,  $d(v) \leq (n - s)/t + s - 1 < n/3$ , a contradiction. Therefore,  $t \leq 3$ . If  $t = 3$ , then  $n/3 + s - 6 \geq |C_1| \geq |C_2| \geq |C_3| \geq n/3 + 3 - s$ , and  $\delta(C_i) \geq n/3 - s + 2$  for each  $i$ . If  $s = 3$ , then for any vertex  $v$  in  $C_3$ ,  $d(v) \leq (n - 3)/3 + 2$ , a contradiction. Thus,  $s \geq 4$ . Each of the graphs  $C_i$  are nearly complete, and are hamiltonian-connected even after the deletion of any small number of vertices. Also, there is an  $s$ -matching between  $S$  and each of the components  $C_i$ , since  $S$  is a minimal cut set. Since  $s \geq 4$ , independent of the location of the vertices  $x$  and  $y$ , it is an easy and straightforward case analysis to show that there is a path  $P$  from  $x$  to  $y$  containing all of the vertices of  $G - S$  and either 2, 3 or 4 vertices of  $S$ . Thus, in this case,  $|P| \geq n - s + 2 > 2n/3 + 2$ .

We now consider the case when  $t = 2$ . Hence,  $n/3 + 3 - s \leq |C_2| \leq |C_1| \leq 2n/3 - 3$ , and  $\delta(C_i) \geq n/3 + 2 - s$  for each  $i$ . The component  $C_2$  is hamiltonian-connected, but if  $|C_1| \leq 2n/3 + 3 - 2s$ , the component  $C_1$  is also hamiltonian-connected. Consider the case when  $C_1$  is hamiltonian-connected. If one of  $x$  or  $y$  is not in  $C_1$ , then it is straightforward to form a path from  $x$  to  $y$  using all of the vertices of  $C_1$  and  $C_2$  along with 2 or 3 vertices of  $S$ . This implies  $|P| \geq n - s + 2 > 2n/3 + 2$ . If  $x$  and  $y$  are both in  $C_1$  then there is a hamiltonian path  $Q$  in  $C_1$  from  $x$  to  $y$ . There



is also a matching with  $s$  edges between  $S$  and  $Q$ . Using two of these  $s$  edges, whose end vertices in  $Q$  are of minimum distance apart on  $Q$ , along with a hamiltonian path of  $C_2$  gives a path  $P$  from  $x$  to  $y$  of length at least  $n - (|C_1| - s)/(s - 1) - (s - 2) \geq n - (2n/3 - 3 - 3s)/(s - 1) - s + 2 > n/3 + 2$ . Thus, we can assume that  $C_1$  is not hamiltonian-connected, and so  $|C_1| \geq 2n/3 + 4 - 2s$ .

If  $\kappa(C_1) \leq 2$ , then there is a cut set, say  $S'$ , with  $|S'| = 1$  or  $2$ , such that  $C_1 - S'$  has two components, say  $C'_1$  and  $C''_1$ . The minimum degree in each component is at least  $n/3 - s$ , so each of these components, and also  $C_2$  is nearly complete. Also, there are  $s - 2$  independent edges between  $S$  and each of the components  $C'_1$  and  $C''_1$  and  $s$  independent edges between  $S$  and  $C_2$ . Hence, just as in the case when there were 3 components of  $G - S$ , it is an easy and straightforward case analysis to find a path  $P$  with at least  $n - s + 2$  vertices from  $x$  to  $y$ , independent of the location of  $x$  and  $y$ . Therefore, we can assume that  $\kappa(C_1) \geq 3$ .

Since  $C_1$  is 3-connected, by a result of Dirac [3], there is a cycle  $C$  in  $C_1$  of length at least  $2n/3 + 4 - 2s$ , and there are  $s$  independent paths from  $S$  to  $C$ . Select two end vertices of these  $s$  paths that have a minimum distance between them on  $C$ . If  $x$  and  $y$  are not in  $C_1$ , then a path from  $x$  to  $y$  can be formed using all of the vertices of  $C_2$  (since  $C_2$  is hamiltonian-connected), at least two vertices of  $S$ , and all of the vertices of  $C$  except for possibly  $(|C| - s)/s$ . Thus the path  $P$  will have at least  $(n/3 + 3 - s) + 2 + (s - 1)(2n/3 + 4 - 2s)/s \geq 7n/9 + 7 - 3s > 2n/3 + 2$  vertices. If  $x$  and  $y$  are in  $C_1$ , then by a result of Enomoto in [4] there is a path between  $x$  and  $y$  with at least  $2n/3 + 4 - 2s$  vertices. Thus, just as in the case of the cycle, a path with at least  $7n/9 + 8 - 5s/2 > 2n/3 + 2$  vertices can be formed. If  $x \in C_1$  and  $y \notin C_1$ , then using a path  $Q$  from  $x$  to some vertex  $z$  in  $C_1$  with a least  $2n/3 + 4 - 2s$  vertices, a path between  $x$  and  $y$  can be formed using all of the vertices of  $Q$  and  $C_2$ , thereby using more than  $2n/3 + 2$  vertices. This completes the proof of the claim that there is a path from  $x$  to  $y$  with at least  $2n/3 + 2$  vertices.

Let  $P$  be a path between  $x$  and  $y$  of maximum length  $m$ , and let  $H = G - P$ . Select a path  $Q$  with a maximum number of vertices, say  $q$ , in  $H$  with end vertices  $u$  and  $v$ . Without loss of generality, let  $s = d_H(u) \geq d_H(v)$ , which means  $u$  has  $s$  adjacencies in  $Q$  and  $q \geq s + 1$ . Denote the predecessors of these  $s$  vertices by  $S' = \{u = u_1, u_2, \dots, u_s\}$ , and let  $S = S' \cup \{v\}$ . Observe that no vertex of  $H$  can be adjacent to two consecutive vertices of  $P$ , since  $P$  is a maximum length path. Let  $U = N_P(u)$ ,  $V = N_P(v)$ ,  $W = U \cap V$ , and

so  $|U|, |V| \geq \delta(G) - s + 1$ . There are  $|U| + |V| - |W|$  vertices of  $P$  adjacent to either  $u$  or  $v$ , and there are the same number or possibly one more or one less “open” interval of  $P$  with no adjacencies to either  $u$  or  $v$ . Since the path  $P$  is of maximal length, each of the “open” intervals will have at least one vertex, and those intervals between a vertex in  $U$  and a vertex in  $V$ , which will we call “long” intervals, will have at least  $q$  vertices.

If  $U = V = W$ , then  $|W| \geq n/3 + 3 - q$ , and there will be at least  $|W| - 1$  “long” intervals. Hence,

$$n - q \geq m \geq (q + 1)(n/3 + 2 - q) + 1.$$

This implies the inequality  $q^2 - (n/3 + 2)q + 2n/3 - 3 \geq 0$ . However, for  $2 \leq q \leq n/3$ , this gives a contradiction. If  $q = 1$ , then  $u$  has at least  $\delta(G)$  adjacencies in  $P$ , which implies the existence of an independent set of order  $\delta(G) > \alpha(G)$ , a contradiction.

In general, if  $s$  is small, then  $u$  and  $v$  will have nearly identical neighborhoods in  $P$ . More specifically,  $|U \cup V| \leq \alpha(G) \leq \delta(G) - 1$  to avoid an independent set with more than  $\alpha(G)$  vertices. Since  $|U|, |V| \geq \delta(G) - s$ , this implies that  $|U \cap V| \geq \delta(G) - 2s + 1$ . An immediate consequence of this is that there are  $\delta(G) - 2s$  vertex disjoint intervals of  $P$  (between the common adjacencies of  $u$  and  $v$  on  $P$ ) each with at least  $s + 2$  vertices. This implies

$$n \geq (s + 1) + (\delta(G) - 2s)(s + 2) + 1 \geq (s + 1) + (n/3 - 2s + 2)(s + 2) + 1,$$

which is a contradiction for  $s \leq 3$ . Thus, we assume  $q \geq 5$  and  $s \geq 4$ . When  $U \neq V$ , we get

$$\begin{aligned} n - q \geq m &\geq |U| + |V| - |W| + q|W| + (|U| + |V| - 2|W|) - 1 \\ &= 2|U| + 2|V| + (q - 3)|W| - 1. \end{aligned}$$

Since  $n > 2|U| + 2|V|$ , we know  $|U| < n/4$  and  $s \geq n/3 + 2 - n/4 > n/12$ , for otherwise this gives a contradiction.

Let  $R$  be the set of  $r$  vertices of  $P$  with at least two adjacencies in  $S$ . The interval between two adjacencies of distinct vertices of  $S$  will have at least  $(s+1)/2$  vertices. If  $r \geq \delta(G) - s$ , then there will be at least  $\delta(G) - s - 1$  distinct intervals of  $P$  with at least  $(s+1)/2$  vertices with no adjacencies in  $Q$ , and one of the intervals will have at least  $q$  such vertices. This implies that

$$n - q \geq m \geq (\delta(G) - s - 2)(s + 1)/2 + q + \delta(G) - s.$$

The previous equation implies that  $s^2 - (\delta(G) - 5)s - 3\delta(G) + 2 + 2n - 4q \geq 0$ . However, this is a contradiction for  $4 \leq s < n/3 - 3$ , so we may assume that  $r < \delta(G) - s$ .

There are at least  $r - 1$  distinct intervals of  $P$  with at least  $(s + 1)/2$  vertices with no adjacencies in  $S$  and also an additional  $s$  intervals with this same property because the predecessor and the successor of the interval are from distinct vertices of  $S$ . There are at least  $s(\delta(G) - s - r - 1)$  additional intervals with at least one vertex with no adjacencies in  $S$ . This implies that

$$n - s - 1 \geq m \geq (r - 1)(s + 3)/2 + s(s + 3)/2 + s(\delta(G) - s - r - 1)2.$$

Since the lower bound on  $m$  in the previous equation is a decreasing function of  $r$ , this implies that the extreme value when  $r = \delta(G) - s - 1$  is also a lower bound, and so  $n \geq (\delta(G) - 2)(s + 3)/2 + s + 1 > (n/3)(7/2) + 4$ , a contradiction. This completes the proof of the case  $k = 3$ . The proof of the case  $k = 4$  is identical, except the analysis to show that there is a path between  $x$  and  $y$  with at least  $3n/4 + 3$  vertices is much more tedious. This completes the proof of Theorem 4. ■

#### 4. QUESTIONS

The obvious problem is to extend Theorem 4 to all values of  $k \geq 5$  and verify Conjecture 1 when the order  $n$  of the graph is sufficiently large. It is also desirable to be able to drop the  $n$  sufficiently large condition.

Many degree conditions that imply a graph is hamiltonian have analogues that imply much more, such as panconnected, hamiltonian ordered, etc. Are there similar analogues for the Chvátal-Erdős type conditions?

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