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Note

PAIRS OF FORBIDDEN CLASS OF SUBGRAPHS CONCERNING $K_{1,3}$ AND P_6 TO HAVE A CYCLE CONTAINING SPECIFIED VERTICES

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Abstract

In [3], Faudree and Gould showed that if a 2-connected graph contains no $K_{1,3}$ and P_6 as an induced subgraph, then the graph is hamiltonian. In this paper, we consider the extension of this result to cycles passing through specified vertices. We define the families of graphs which are extension of the forbidden pair $K_{1,3}$ and P_6, and prove that the forbidden families implies the existence of cycles passing through specified vertices.

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1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2].

For a family $\{H_1, H_2, \ldots, H_k\}$ of graphs, a graph G is called an $H_1 H_2 \cdots H_k$-free graph if G contains no induced subgraphs isomorphic to any H_i with $i = 1, 2, \ldots, k$. A cycle containing all vertices of a graph is called hamiltonian cycle. There exists a close relation between forbidden subgraphs and hamiltonicity. In fact, for $k = 1$, we can easily see that a P_3-free graph is a hamiltonian, where P_i is a path of length $i - 1$ (this is known that a X-free graph is a hamiltonian implies that $X = P_3$ [3]). For $k = 2$, Faudree and Gould dertermined all pairs of such forbidden subgraphs (not contain P_3), and proved that one of the graphs in the pair must be isomorphic to $K_{1,3}$. For convenience, we denote $C = K_{1,3}$.

Theorem A (Faudree and Gould [3]**.** *Let G be a 2-connected graph of order $n \geq 10$, and let X is one of the ten graphs, which are called as $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N, W$. If G is CX-free, then G is hamiltonian.*

On the other hand, many hamiltonian conditions are generalized to sufficient conditions for graphs to contain cycles passing through all specified vertices (see [1, 5]). The purpose of this paper is to generalize Theorem A. For cases $X \in \{C_3, Z_1, Z_2, Z_3, B, N\}$, such a generalization have already been done (for C_3, B, N in [4], for Z_1, Z_2, Z_3 in [6]). In this paper, we focus on the generalization for case $X = P_6$. (Note that P_i-free graph ($i = 4, 5, 6$) is P_6-free.) For a generalization, we define the following two families of graphs.

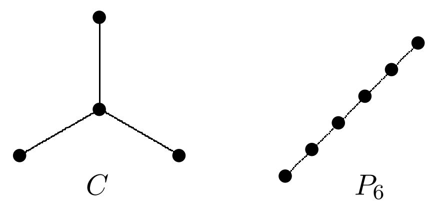

Figure 1. C and P_6

Let G be a graph and $S \subseteq V(G)$. We define a family of graphs $C(S)$ as following. For any $F \in C(S)$, F satisfies the following properties.

(1) F consists of three paths P_1, P_2 and P_3 such that they have only one common vertex x and $V(F) = V(P_1) \cup V(P_2) \cup V(P_3)$ (we call x *the root of* F),

(2) for any $i \in \{1, 2, 3\}$, the end vertex of P_i which is not x is contained in S (we call such a vertex of P_i *a leaf of* F),

(3) for any $i \in \{1, 2, 3\}$, internal vertices of P_i are contained in $V(G) \backslash S$ and

(4) $E(F) = E(P_1) \cup E(P_2) \cup E(P_3)$.

We also define a family of graphs $P_6(S)$ as following. For any $F \in P_6(S)$, F satisfies the following properties,

(1) F is a path of length at least 5 and

(2) the end vertices of F are contained in S.

If there exist no induced subgraph contained in $C(S) \cup P_6(S)$, we call G a *$CP_6(S)$-free graph.*

In this paper, we prove the following theorem.

Theorem 1. *Let G be a 2-connected graph and $S \subseteq V(G)$. If G is $CP_6(S)$-free, then G contains a cycle D such that $S \subseteq V(D)$.*

We let $\mathcal{T}(G)$ denote the set of triples $(T; A, B)$ of subsets of $V(G)$ such that $\{T, A, B\}$ is a partiton of $V(G)$, $|T| = 2$, $A \neq \emptyset, B \neq \emptyset$, and no edge of G joins a vertex in A and a vertex in B.

For a $CP_6(S)$-free graph G with $S \subseteq V(G)$, the following two lemmas hold.

Lemma 1. *For any $x \in V(G) \setminus S$, $G \setminus x$ is a $CP_6(S)$-free graph.*

Lemma 2. *Let $T = \{x, y\}$ be a 2-cut set of G with $(T; A, B) \in \mathcal{T}(G)$. If $A \subseteq V(G) \setminus S$, then $(G \setminus A) + xy$ is a 2-connected $CP_6(S)$-free graph.*

***Proof of Lemmas* 1 *and* 2.** We can easily obtain Lemma 1. For every two vertices in $V((G \setminus A) + xy)$, there exists a cycle which contains them, because G is 2-connected. Therefore $(G \setminus A) + xy$ is 2-conneted.

Suppose that $(G \setminus A) + xy$ contains an induced subgraph $F \in C(S) \cup P_6(S)$. If $xy \in E(G)$ or $xy \notin E(F)$, G contains F, a contradiction. Hence $xy \notin E(G)$ and $xy \in E(F)$. Let $P \subseteq G \setminus B$ be a shortest path which joins x and y. Now $(F \setminus xy) \cup P$ is an element of $C(S) \cup P_6(S)$, and is an induced subgraph of G, a contradiction. ■

2. Proof of Theorem 1

Let G be a 2-connected graph and $S \subseteq V(G)$. Suppose G is $CP_6(S)$-free and there is no cycle D with $S \subseteq V(D)$. Take such a graph G and $S \subseteq V(G)$ as

(1) $|V(G)|$ is as small as possible,

(2) $|V(G) \setminus S|$ is as small as possible, subject to (1), and

(3) $|E(G)|$ is as large as possible, subject to (1) and (2).

If $V(G) \setminus S = \emptyset$, then by Theorem A there exists a cycle D such that $S \subseteq V(D)$. Hence $V(G) \setminus S \neq \emptyset$. Let $x \in V(G) \setminus S$. If $G \setminus x$ is a 2-connected graph, then by the minimality of $|V(G)|$ and Lemma 1, $G \setminus x$ contains a cycle D such that $S \subseteq V(D)$. Now D is also a cycle in G with $S \subseteq V(D)$, a contradiction. It follows that there exists $(T; A, B) \in \mathcal{T}(G)$ with $x \in T$. Let $T = \{x, y\}$.

Claim 1. $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$.

***Proof of Claim* 1.** Suppose $A \cap S = \emptyset$. Then by Lemma 2, $(G \setminus A) + xy$ is a 2-connected $CP_6(S)$-free graph. By the minimality of $|V(G)|$, there exists a cycle $D \subseteq (G \setminus A) + xy$ with $S \subseteq V(D)$. Since G contains no cycle passing through all the vertices of S, we may assume $xy \notin E(G)$ and $xy \in E(D)$. Take an x-y path $P \subseteq \langle A \cup \{x, y\} \rangle$. Then $(D \setminus xy) \cup P$ is a cycle passing through all the vertices of S, a contradiction. ■

Claim 2. $|A| = 1$ or $|B| = 1$.

***Proof of Claim* 2.** Suppose $|A| \geq 2$ and $|B| \geq 2$. Let G_A be a graph such that $V(G_A) = V(G \setminus A) \cup \{a\}$ and $E(G_A) = E(G \setminus A) \cup \{ax, ay\}$, where a is a new vertex. Let $S_A = (S \setminus A) \cup \{a\}$. (Note that $|V(G)| > |V(G_A)|$.)

Now we show that G_A is a $CP_6(S_A)$-free graph. Suppose that G_A is not a $CP_6(S_A)$-free graph. Let $F \in C(S_A) \cup P_6(S_A)$ be an induced subgraph in G_A. Then $V(F)$ contains a, otherwise G contains F as an induced subgraph. If $F \in C(S)$, a is a leaf of F since $d_{G_A}(a) = 2$ and $a \in S_A$. On the other hand, suppose $F \in P_6(S)$ and a is an endvertex of S. Therefore we can see $|\{x, y\} \cap V(F)| = 1$. Let $\{z\} = \{x, y\} \cap V(F)$. Take an $(A \cap S)$-z path P in $\langle A \cup \{z\} \rangle$ as $|V(P)|$ is as small as possible. Then $(F \setminus a) \cup P$ belongs to $C(S) \cup P_6(S)$, and is an induced subgraph in G, a contradiction. If $F \in P_6(S)$ and a is not an endvertex of S, then $V(F)$ contains x and y.

Let P' is x-y path in G and is an induced subgraph in G. Then $F \cup P'$ also belongs to $P_6(S)$, a contradiction.

By the minimality of $|V(G)|$, there exists a cycle D_A in G_A such that $S_A \subseteq V(D_A)$. Hence we obtain an x-y path P_A in $G \setminus A$ such that $S \cap B \subseteq V(P_A)$. By the same argument as above, there exists an x-y path P_B in $G \setminus B$ which satisfies $S \cap A \subseteq V(P_B)$. Now $P_A \cup P_B$ consists a cycle with $S \subseteq V(P_A \cup P_B)$, a contradiction. ■

Without loss of generality, we may assume $|A| = 1$. Let $A = \{a\}$. Since $A \cap S \neq \emptyset$, we have $a \in S$.

Claim 3. $G + xy$ is a $CP_6(S)$-free graph.

***Proof of Claim* 3.** Since G is $CP_6(S)$-free, we may assume $xy \notin E(G)$. Let $G' = G + xy$. Suppose that G' has an induced subgraph $F \in C(S) \cup P_6(S)$. If $xy \notin E(F)$, F is an induced subgraph in G, a contradiction. Hence we may assume that $E(F)$ contains xy. Since $ax, ay \in E(G)$ and $d_G(a) = 2$, we have $a \notin F$. Then $F \setminus xy$ is disconnected. Suppose that $F \in C(S)$. Let F' be a component of $F \setminus xy$ such that F' contains a root of F. Let $\{z\} = V(F') \cap \{x, y\}$. Then $F' + za$ belongs to $C(S)$, and is an induced subgraph of G, a contradiction. Next, suppose that $F \in P_6(S)$. Then $(F \setminus xy) \cup \{xa, ya\}$ belongs to $P_6(S)$, and is an induced subgraph of G, a contradiction. ■

By Claim 3 and the maximality of $|E(G)|$, we may assume $xy \in E(G)$. Let $S' = S \cup \{x\}$.

Claim 4. G is a $CP_6(S')$-free graph.

***Proof of Claim* 4.** Suppose that G contains $F \in C(S') \cup P_6(S')$. Hence we may assume that F contains x as a leaf or an endvertex of F. If $y \in V(F)$, then $a \notin V(F)$, and hence $(F \setminus x) + ay \in C(S) \cup P_6(S)$, a contradiction. If $y \notin V(F)$, then $F + ax \in C(S) \cup P_6(S)$, also a contradiction. ■

By Claim 4 and the minimality of $|V(G) \setminus S|$, there exists a cycle D with $S' \subseteq V(D)$. Since $S \subseteq S'$, D contains all vertices of S, which contradicts the assumption of G and completes the proof of Theorem 1. ■

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