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THE SET CHROMATIC NUMBER OF A GRAPH

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Abstract

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. For a vertex v of G, the neighborhood color set $\mathrm{NC}(v)$ is the set of colors of the neighbors of v. The coloring c is called a set coloring if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every pair u, v of adjacent vertices of G. The minimum number of colors required of such a coloring is called the set chromatic number $\chi_s(G)$ of G. The set chromatic numbers of some well-known classes of graphs are determined and several bounds are established for the set chromatic number of a graph in terms of other graphical parameters.

Keywords: neighbor-distinguishing coloring, set coloring, neighborhood color set.

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1. Introduction

Many methods have been introduced in graph theory to distinguish all of the vertices of a graph or to distinguish every two adjacent vertices in a graph. Several of these methods involve graph colorings or graph labelings. In particular, with a given edge coloring c of G, each vertex of G can be labeled with the set of colors of its incident edges. If distinct vertices have distinct labels, then c is a vertex-distinguishing edge coloring (see [2, 4]); while if every two adjacent vertices have distinct labels, then c is a neighbor-distinguishing edge coloring (see [1]).

If all of the vertices of a graph G of order n are distinguished as a result of a vertex coloring of G, then of course n colors are needed to accomplish this. On the other hand, if the goal is only to distinguish every two adjacent vertices in G by a vertex coloring, then this can be accomplished by means of a proper coloring of G and the minimum number of colors needed to do this is the *chromatic number* $\chi(G)$ of G. There are, however, other methods that can be used to distinguish every two adjacent vertices in G by means of vertex colorings which may require fewer than $\chi(G)$ colors.

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be assigned the same color. For a set $S \subseteq V(G)$, define the set $c(S)$ of colors assigned to the vertices of S by

$$c(S) = \{c(v) : v \in S\}.$$

For a vertex v in a graph G, let $N(v)$ be the neighborhood of v (the set of all vertices adjacent to v in G). The *neighborhood color set* $\mathrm{NC}(v) = c(N(v))$ is the set of colors of the neighbors of v. The coloring c is called *set neighbor-distinguishing* or simply a *set coloring* if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every pair u, v of adjacent vertices of G. The minimum number of colors required of such a coloring is called the *set chromatic number* of G and is denoted by $\chi_s(G)$. We refer to the book [3] for graph theory notation and terminology not described in this paper.

For a graph G with chromatic number k, let c be a proper k-coloring of G. Suppose that u and v are adjacent vertices of G. Since $c(u) \in \mathrm{NC}(v)$ and $c(u) \notin \mathrm{NC}(u)$, it follows that $\mathrm{NC}(u) \neq \mathrm{NC}(v)$. Hence every proper k-coloring of G is also a set k-coloring of G. Therefore, for every graph G,

$$\chi_s(G) \leq \chi(G). \tag{1}$$

Observe that if G is a connected graph of order n, then $\chi_s(G) = 1$ if and only if $\chi(G) = 1$ (in this case $G = K_1$) and $\chi_s(G) = n$ if and only if $\chi(G) = n$ (in this case $G = K_n$). Thus if G is a nontrivial connected graph of order n that is not complete, then

$$2 \leq \chi_s(G) \leq n - 1. \tag{2}$$

To illustrate these concepts, we consider the graph $G = C_5 + K_1$ (the wheel of order 6). The chromatic number of G is $\chi(G) = 4$. In fact, the set chromatic number of G is $\chi_s(G) = 3$. Figure 1 shows a set 3-coloring of G and so $\chi_s(G) \leq 3$. We now show that $\chi_s(G) \geq 3$. Suppose that there is a set 2-coloring c of G using the colors 1 and 2. Consider a triangle in G induced by three vertices v_1, v_2, v_3 of G. Since at least two of these three vertices are colored the same, we may assume that two of these vertices are assigned the color 1. Thus $\mathrm{NC}(v_i) = \{1\}$ or $\mathrm{NC}(v_i) = \{1, 2\}$ for each i $(1 \leq i \leq 3)$. This implies, however, that there are two adjacent vertices having the same neighborhood color set, which contradicts our assumption that c is a set coloring. Thus $\chi_s(G) = 3$, as claimed.

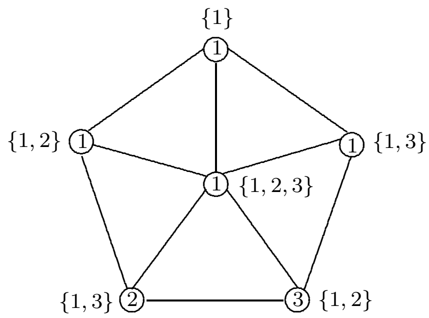

Figure 1. A set coloring of a graph.

The following observation will be useful to us.

Observation 1.1. *If u and v are two adjacent vertices in a graph G such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every set coloring c of G. Furthermore, if $S = N(u) - \{v\} = N(v) - \{u\}$, then $\{c(u), c(v)\} \not\subseteq c(S)$.*

2. The Set Chromatic Numbers of Some Classes of Graphs

Since every nonempty bipartite graph has chromatic number 2, the following is an immediate consequence of (1) and (2).

Observation 2.1. *If G is a nonempty bipartite graph, then $\chi_s(G) = 2$.*

In fact, if G is a nonempty graph, then $\chi_s(G) = 2$ if and only if G is bipartite, as we show next. We may restrict our attention to connected graphs.

Proposition 2.2. *If G is a connected graph with $\chi(G) \geq 3$, then $\chi_s(G) \geq 3$.*

Proof. Assume, to the contrary, that there exists a connected graph G with $\chi(G) \geq 3$ for which there exists a set 2-coloring $c : V(G) \to \{1,2\}$. Since $\chi(G) \geq 3$, it follows that G contains an odd cycle $C : v_1, v_2, \ldots, v_\ell, v_1$, where $\ell \geq 3$ is an odd integer.

Consider the (cyclic) color sequence

$$s: \; c(v_1), c(v_2), \ldots, c(v_\ell), c(v_1).$$

By a *block* of s, we mean a maximal subsequence of s consisting of terms of the same color. First, we claim that s cannot contain a block with an even number of terms; for suppose, without loss of generality, that $c(v_\ell) = 2$, $c(v_i) = 1$ for $1 \leq i \leq a$, where a is an even integer with $2 \leq a \leq \ell - 1$, and $c(v_{a+1}) = 2$. Thus $\mathrm{NC}(v_i) \in \{\{1\}, \{1,2\}\}$ for $1 \leq i \leq a$. Since $\mathrm{NC}(v_1) = \{1,2\}$ and c is a set coloring, it follows that

$$\mathrm{NC}(v_i) = \begin{cases} \{1\} & \text{if } i \text{ is even,} \\ \{1,2\} & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq a$. However, this implies that $\mathrm{NC}(v_a) = \{1\}$, which is impossible since $c(v_{a+1}) = 2$.

Hence either

(i) $c(v_i) = 1$ for all i $(1 \leq i \leq \ell)$ or

(ii) s contains an even number of blocks each of which has an odd number of terms.

If (i) occurs, then $\mathrm{NC}(v_i) \in \{\{1\}, \{1,2\}\}$ for $1 \leq i \leq \ell$. Since ℓ is odd, there is an integer j $(1 \leq j \leq \ell)$ such that $\mathrm{NC}(v_j) = \mathrm{NC}(v_{j+1})$, which is impossible. If (ii) occurs, then ℓ is even, which is also impossible. ■

The following three corollaries are immediate consequences of (1), Observation 2.1, and Proposition 2.2.

Corollary 2.3. *A nonempty graph G has set chromatic number* 2 *if and only if G is bipartite.*

Corollary 2.4. *If G is a* 3*-chromatic graph, then $\chi_s(G) = 3$.*

Corollary 2.5. *For each integer $n \geq 3$, $\chi_s(C_n) = \chi(C_n)$.*

We have seen that $\chi_s(K_n) = n$ for $n \geq 1$. We now determine the set chromatic number of a class of graphs that are related to K_n. For a graph H, its *corona* $\text{cor}(H)$ is that graph obtained by adding a pendant edge at each vertex of H. For an integer $n \geq 2$ and an integer t ($0 \leq t \leq n$), let $G_{n,t}$ denote the graph of order $n + t$ obtained from K_n with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by adding t new vertices $u_1, u_2, \ldots, u_t$ (if $t \geq 1$) and joining each u_i to v_i for $1 \leq i \leq t$. Therefore, $G_{n,0} = K_n$ while $G_{n,n} = \text{cor}(K_n)$. We show that $\chi_s(G_{n,t}) = n$ for all t ($0 \leq t \leq n$). It is convenient to introduce some notation. For each integer k, let

$$\mathbb{N}_k = \{1, 2, \ldots, k\}.$$

Proposition 2.6. *For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.*

Proof. The result follows immediately if $n = 2$ or $t = 0$, so we assume that $n \geq 3$ and $1 \leq t \leq n$. Since $\chi(G_{n,t}) = n$, we have $\chi_s(G_{n,t}) \leq n$ by (1). Suppose that $\chi_s(G_{n,t}) = k \leq n - 1$ and let there be given a set k-coloring of $G_{n,t}$ using the colors in $\mathbb{N}_k$. Permuting colors if necessary, we can obtain a set k-coloring $c : V(G_{n,t}) \to \mathbb{N}_k$ such that $c(V(K_n)) = \mathbb{N}_\ell$ for some $\ell \leq k$. Let X be the subset of $V(K_n)$ such that for every $x \in X$ there exists a vertex $y \in X - \{x\}$ for which $c(y) = c(x)$. Since c uses at most $n - 1$ colors, $|X| \geq 2$ and, furthermore, since each of the vertices in $V(K_n) - X$ receives a unique color, $n - |X| + 1 \leq \ell$. For each $x \in X$, either

(i) $\text{NC}(x) = \mathbb{N}_\ell$ or

(ii) $\text{NC}(x) = \mathbb{N}_\ell \cup \{c(u)\}$ if u is the end-vertex adjacent to x and $c(u) \notin \mathbb{N}_\ell$.

Since at most one of the $|X|$ vertices can have the neighborhood color set $\mathbb{N}_\ell$, at least $|X| - 1$ colors not in $\mathbb{N}_\ell$ are needed to color the end-vertices so that the vertices in X have distinct neighborhood color sets, that is,

$$k \geq \ell + |X| - 1 \geq n,$$

which is a contradiction. Therefore, $\chi_s(G_{n,t}) = n$. ■

We now determine the set chromatic number of every complete multipartite graph.

Proposition 2.7. *For every complete k-partite graph G, $\chi_s(G) = k$.*

Proof. By (1), $\chi_s(G) \leq k$. Assume that the statement is false. Then there is a smallest positive integer k for which there exists a complete k-partite graph G with $\chi_s(G) \leq k - 1$. Necessarily, $k \geq 4$. Suppose that the partite sets of G are $V_1, V_2, \ldots, V_k$. Let there be given a set $(k-1)$-coloring $c : V(G) \to \mathbb{N}_{k-1}$ of G. We claim that for each partite set V_i $(1 \leq i \leq k)$ the coloring $c_i = c|_{V(G)-V_i}$ is a set coloring of $G - V_i$, which is a complete $(k-1)$-partite graph. In order to see that this is the case, let u and v be adjacent vertices in $G - V_i$. In G we have $\mathrm{NC}_c(u) \neq \mathrm{NC}_c(v)$. Since

$$\mathrm{NC}_c(u) = \mathrm{NC}_{c_i}(u) \cup c(V_i) \quad \text{and} \quad \mathrm{NC}_c(v) = \mathrm{NC}_{c_i}(v) \cup c(V_i),$$

it follows that $\mathrm{NC}_{c_i}(u) \neq \mathrm{NC}_{c_i}(v)$. This implies that the coloring c_i of $G - V_i$ is a set coloring, as claimed. Since $\chi_s(G - V_i) = k - 1$, it follows that $c(V(G) - V_i) = \mathbb{N}_{k-1}$. Thus $\mathrm{NC}_c(x) = \mathbb{N}_{k-1}$ for every vertex x of V_i. Since the partite set V_i was chosen arbitrarily, $\mathrm{NC}_c(x) = \mathbb{N}_{k-1}$ for every vertex x of G, which is impossible. ■

By Proposition 2.7, the complete k-partite graph $K_{1,1,\ldots,1,n-(k-1)}$ has set chromatic number k, giving the following result.

Corollary 2.8. *For each pair k, n of integers with $2 \leq k \leq n$, there is a connected graph G of order n with $\chi_s(G) = k$.*

It is well known that the chromatic number of a graph G is at least as large as its clique number $\omega(G)$, which is the largest order of a clique (a complete subgraph) in G. The following observation will be useful to us.

Observation 2.9. *Let G be a graph of order $n \geq 2$. Then $\chi(G) = n - 1$ if and only if $\omega(G) = n - 1$.*

Proposition 2.10. *For a connected graph G of order $n \geq 3$,*

$$\chi_s(G) = n-1 \quad \textit{if and only if} \quad \chi(G) = n-1.$$

Proof. If $\chi_s(G) = n-1$, then $G \neq K_n$ and so the result immediately follows by (1). For the converse, assume that $\chi(G) = n-1$. Then by Observation 2.9, $\omega(G) = n-1$ and so G is obtained from K_{n-1} by adding a new vertex u and joining u to some (but not all) vertices of K_{n-1}. Assume, to the contrary, that $\chi_s(G) = k \leq n-2$ and let there be given a set k-coloring of G using the colors in $\mathbb{N}_k$. Permuting the colors if necessary, we can obtain a set k-coloring $c : V(G) \to \mathbb{N}_k$ such that $c(V(K_{n-1})) = \mathbb{N}_\ell$, where $1 \leq \ell \leq k$. Since $\ell < n-1$, some vertices in K_{n-1} are colored the same. Let $X \subseteq V(K_{n-1})$ such that for each $x \in X$, there exists a vertex $y \in X - \{x\}$ such that $c(y) = c(x)$. Hence $|X| \geq 2$. Since each of the remaining $n-1-|X|$ vertices in K_{n-1} receives a unique color, it follows that $n - |X| \leq \ell$. For each $x \in X$, either

(i) $\mathrm{NC}(x) = \mathbb{N}_\ell$ or
(ii) $\mathrm{NC}(x) = \mathbb{N}_\ell \cup \{c(u)\}$ if $x \in N(u)$ and $c(u) \notin \mathbb{N}_\ell$.

This implies that $|X| \leq 2$. Hence $|X| = 2$ and so $\ell = n-2$. Then $k = \ell + 1$ (since $c(u) \notin \mathbb{N}_\ell$) and

$$n-2 = \ell = k-1 \leq n-3,$$

which is impossible. ■

By Proposition 2.10 and its proof, a connected graph G of order $n \geq 3$ has $\chi_s(G) = n-1$ if and only if $G = (K_{n-1-k} \cup K_1) + K_k$ for some integer k with $1 \leq k \leq n-2$.

Corollary 2.11. *If G is a connected graph of order n such that $\chi(G) \in \{1, 2, 3, n-1, n\}$, then $\chi_s(G) = \chi(G)$.*

3. Lower Bounds for the Set Chromatic Number

We have already observed that $\chi_s(G) \leq \chi(G)$ for every graph G. There is also a lower bound for the set chromatic number of a graph in terms of its chromatic number.

Proposition 3.1. *For every graph G,*

$$\chi_s(G) \geq \lceil \log_2(\chi(G) + 1) \rceil.$$

Proof. Since this is true if $1 \leq \chi(G) \leq 3$, we may assume that $\chi(G) \geq 4$. Let $\chi_s(G) = k$ and let there be given a set k-coloring of G using the colors in $\mathbb{N}_k$. Thus $\mathrm{NC}(x) \subseteq \mathbb{N}_k$ for every vertex x of G. Since $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every two adjacent vertices u and v of G, it follows that $\mathrm{NC}(x)$ can be considered as a color for each $x \in V(G)$, that is, the coloring c of G defined by $c(x) = \mathrm{NC}(x)$ for $x \in V(G)$ is a proper coloring of G. Since there are $2^k - 1$ nonempty subsets of $\mathbb{N}_k$, it follows that c uses at most $2^k - 1$ colors. Thus $\chi(G) \leq 2^k - 1$ or $\chi(G) + 1 \leq 2^k$. Thus $\chi_s(G) = k \geq \lceil \log_2(\chi(G) + 1) \rceil$, as desired. ■

By Corollary 2.11, the lower bound for the set chromatic number of a graph G in Proposition 3.1 is sharp if $\chi(G) \in \{1, 2\}$. If $\chi(G) = 3$, then $\chi_s(G) = 3 > \lceil \log_2(3 + 1) \rceil = 2$ and so this bound is not sharp in this case.

The Grötzsch graph G^* of Figure 2 is known to have chromatic number 4. A set 3-coloring of G^* is also given in Figure 2 and so $\chi_s(G^*) \leq 3$. By Proposition 2.2, $\chi_s(G^*) \geq 3$. Thus $\chi_s(G^*) = 3$. Since $\lceil \log_2(\chi(G^*) + 1) \rceil = \lceil \log_2 5 \rceil = 3$, the lower bound for $\chi_s(G^*)$ is attained in this case.

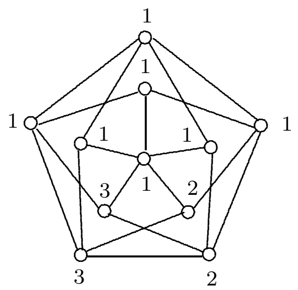

Figure 2. A set 3-coloring of the Grötzsch graph.

While $\chi(G) \geq \omega(G)$ for every graph G, the clique number is not a lower bound for the set chromatic number of a graph.

Proposition 3.2. *For every graph G,*

$$\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil \,. \tag{3}$$

Proof. If $\omega(G) = 2$, then $\chi_s(G) \geq 2$; while if $\omega(G) = 3$, then $\chi_s(G) \geq 3$. Thus we may assume that $\omega(G) = \omega \geq 4$. Let H be a clique of order ω in G with $V(H) = \{v_1, v_2, \ldots, v_\omega\}$. Suppose that $\chi_s(G) = k$ and let $c : V(G) \to \mathbb{N}_k$ be a set k-coloring of G. We consider two cases, according to whether there are two vertices in $V(H)$ colored the same or no two vertices in $V(H)$ are assigned the same color.

Case 1. There are two vertices in $V(H)$ colored the same, say $c(v_1) = c(v_2) = 1$.
Then $1 \in \mathrm{NC}(v_i)$ for $1 \leq i \leq \omega$. Since there are exactly 2^{k-1} subsets of $\mathbb{N}_k$ containing 1, it follows that $\omega \leq 2^{k-1}$ and so $k - 1 \geq \log_2 \omega$. Therefore, (3) holds.

Case 2. No two vertices in $V(H)$ are colored the same.
Then ω distinct colors are used for the vertices in $V(H)$ and so $\omega \leq k$. Since $\omega \geq 4$, it follows that

$$k \geq \omega > 1 + \lceil \log_2 \omega(G) \rceil .$$

Again, (3) holds. ■

The lower bound for the set chromatic number of a graph in Proposition 3.2 is sharp. To see this, we construct a connected graph G with $\omega(G) = 2^{k-1}$ and $\chi_s(G) = k$ for each integer $k \geq 2$. We start with the complete graph $H = K_{2^{k-1}}$ of order 2^{k-1}, where $V(H) = \{v_1, v_2, \ldots, v_{2^{k-1}}\}$. Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\mathbb{N}_{k-1}$, where $S_1 = \emptyset$. For each integer i with $2 \leq i \leq 2^{k-1}$, we add $|S_i|$ pendant edges at the vertex v_i, obtaining the connected graph G with $\omega(G) = 2^{k-1}$. It remains to show that $\chi_s(G) = k$. By Proposition 3.2, $\chi_s(G) \geq k$. Define a k-coloring of G by assigning

(i) the color k to each vertex of H and

(ii) the colors in S_i to the $|S_i|$ end-vertices adjacent to v_i for $2 \leq i \leq 2^{k-1}$.

Figure 3 shows the graph G for $k = 4$ and the corresponding 4-coloring. Thus $\mathrm{NC}(v_i) = S_i \cup \{k\}$ for $1 \leq i \leq 2^{k-1}$. Hence $|\mathrm{NC}(v_i)| \geq 2$ for $2 \leq i \leq 2^{k-1}$ and $|\mathrm{NC}(x)| = 1$ for each end-vertex x of G. This implies that every two adjacent vertices in G have different neighborhood color sets. Consequently, c is a set k-coloring of G and so $\chi_s(G) = k$.

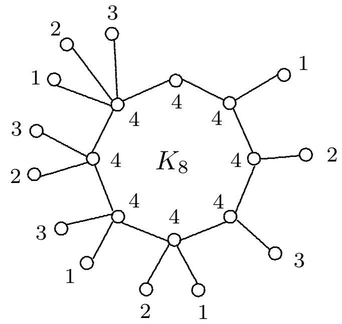

Figure 3. A set 4-coloring of a graph G with $\chi_s(G) = 1 + \lceil \log_2 \omega(G) \rceil$.

4. Vertex or Edge Deletions and the Set Chromatic Number

For the graph G of Figure 4(a), $\chi(G) = \omega(G) = 4$. By Proposition 3.2, $\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil = 3$. The set 3-coloring of G in Figure 4(b) shows that $\chi_s(G) = 3$. The graph $G - x_2$ is shown in Figure 4(c) together with a set 4-coloring. Observe that the graph $G - x_2$ is isomorphic to the graph $G_{4,3}$ described prior to Proposition 2.6. In fact, $\chi_s(G - x_2) = \chi_s(G_{4,3}) = 4$ by Proposition 2.6.

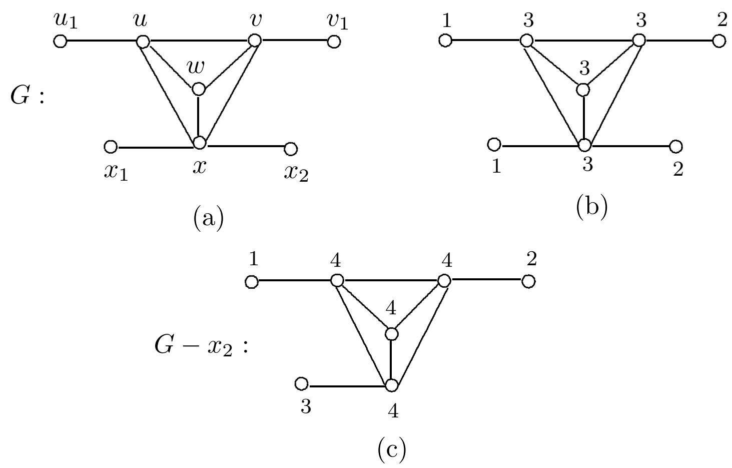

Figure 4. A set 3-coloring of a graph G and a set 4-coloring of $G - x_2$.

The preceding example shows that it is possible for a graph G to contain a vertex v such that the set chromatic number of $G - v$ is greater than the set chromatic number of G. If $G = C_5$, then $\chi_s(G - v) = 2 = \chi_s(G) - 1$ for every vertex v of G. If $G = C_5 + K_1$ where v is the central vertex of G, then $\chi_s(G - v) = 3 = \chi_s(G)$. Therefore, for each $i \in \{-1, 0, 1\}$, there exists a graph G containing a vertex v such that $\chi_s(G - v) = \chi_s(G) + i$. In fact, $\chi_s(G - v)$ can exceed $\chi_s(G)$ by more than 1. Prior to showing this, we introduce additional notation. For integers a and b with $a < b$, let

$$[a..b] = \{x \in \mathbb{Z} : \ a \leq x \leq b\}.$$

In particular, $[1..b] = \mathbb{N}_b$.

Let G be a graph of order $n = 11$ and clique number $\omega(G) = 8$ constructed from K_8 with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$ by adding three pairwise nonadjacent vertices u_1, u_2, u_3 and joining v_i and u_j as follows: Let $S_1 = \emptyset$, $S_2 = \{1\}$, $S_3 = \{2\}$, $S_4 = \{1, 2\}$, and $S_i = S_{i-4} \cup \{3\}$ for $5 \leq i \leq 8$. For $1 \leq i \leq 8$ and $1 \leq j \leq 3$, $v_i u_j \in E(G)$ if and only if $j \in S_i$ (see Figure 5). By Proposition 3.2, $\chi_s(G) \geq 1 + \lceil \log_2 8 \rceil = 4$, while the coloring $c_1 : V(G) \to \mathbb{N}_4$ of G defined by

$$c_1(v) = \begin{cases} i & \text{if } v = u_i \ (1 \leq i \leq 3), \\ 4 & \text{otherwise} \end{cases}$$

is a set 4-coloring. Therefore, $\chi_s(G) = 4$.

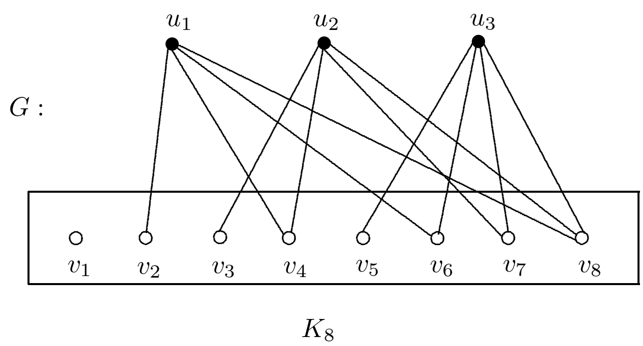

Figure 5. A graph G with $\chi_s(G - u_3) = \chi_s(G) + 3$.

For the graph G of Figure 5, let $H = G - u_3$. We claim that $\chi_s(H) = 7$. First observe that the coloring $c_2 : V(H) \to \mathbb{N}_7$ of H defined by

$$c_2(v) = \begin{cases} 1 & \text{if } v = v_i \ (5 \le i \le 8), \\ 1+i & \text{if } v = v_i \ (1 \le i \le 4), \\ 5+i & \text{if } v = u_i \ (i = 1, 2) \end{cases}$$

is a set 7-coloring and so $\chi_s(H) \le 7$. Assume, to the contrary, that there exists a set ℓ-coloring of H using the colors in $\mathbb{N}_\ell$ for some $\ell \le 6$. Permuting the colors if necessary, we can obtain a set ℓ-coloring $c_3 : V(H) \to \mathbb{N}_\ell$ of H such that $c_3(V(K_8)) = \mathbb{N}_{\ell'}$ for some integer ℓ' with $1 \le \ell' \le \ell$. Since $\ell' < 8$, some vertices in K_8 are colored the same. Let X be the subset of $V(K_8)$ such that for each $x \in X$, there exists a vertex $y \in X - \{x\}$ with $c_3(y) = c_3(x)$. Since each vertex of $V(K_8) - X$ receives a unique color and at least one additional color is used for the vertices in X, it follows that $(8 - |X|) + 1 = 9 - |X| \le \ell' \le 6$ and so $|X| \ge 3 > 2$.

The remaining $\ell - \ell'$ colors are used for the two vertices u_1 and u_2, implying that $\ell - \ell' \le 2$. Also, since each vertex $x \in X$ must have a unique neighborhood color set containing $\mathbb{N}_{\ell'}$ as a subset, the set $\mathrm{NC}(x) - \mathbb{N}_{\ell'}$ is a unique subset of $[\ell' + 1..\ell]$. Therefore, $2 < |X| \le 2^{\ell - \ell'} \le 2^2$, implying that $\ell - \ell' = 2$ and so $\ell' \le 4$. However, since $|X| \le 4$,

$$5 \le 9 - |X| \le \ell' \le 4,$$

which is impossible.

Therefore, $\chi_s(H) = 7$, as claimed, and so $\chi_s(G - u_3) = \chi_s(G) + 3$. In fact, $\chi_s(G - u_i) = \chi_s(G) + 3$ for each vertex u_i $(1 \le i \le 3)$. Observe for the graph G of Figure 5 that $\deg_G u_i = 4$ for each i $(1 \le i \le 3)$. In general, we have the following result.

Theorem 4.1. *If v is a vertex of a graph G, then*

$$\chi_s(G) - 1 \le \chi_s(G - v) \le \chi_s(G) + \deg v.$$

Proof. First, we verify the lower bound for $\chi_s(G - v)$. Suppose that $\chi_s(G - v) = k$. Let $c_1 : V(G - v) \to \mathbb{N}_k$ be a set k-coloring of $G - v$. Then the coloring c_1' of G defined by

$$c_1'(x) = \begin{cases} c_1(x) & \text{if } x \ne v, \\ k+1 & \text{if } x = v \end{cases}$$

is a set coloring of G using $k+1$ colors. Therefore, $\chi_s(G) \leq k+1 = \chi_s(G-v)+1$.

Next, we show that $\chi_s(G-v) \leq \chi_s(G)+\deg v$. Suppose that $\chi_s(G) = \ell$ and $\deg v = d$, where $N(v) = \{v_1, v_2, \ldots, v_d\}$. Let $c_2 : V(G) \to \mathbb{N}_\ell$ be a set ℓ-coloring of G. Then the coloring c_2' of $G-v$ defined by

$$c_2'(x) = \begin{cases} c_2(x) & \text{if } x \notin N(v), \\ \ell + i & \text{if } x = v_i \ (1 \leq i \leq d) \end{cases}$$

is a set coloring of $G-v$, using at most $\ell+d$ colors. Therefore, $\chi_s(G-v) \leq \ell + d = \chi_s(G) + \deg v$. ■

We have already seen that the lower bound for $\chi(G-v)$ given in Theorem 4.1 is sharp. To see that the upper bound in Theorem 4.1 is sharp, let $n = 2k \geq 4$. We construct a graph G of order $2n$ from K_n with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by adding n new vertices $u_1, u_2, \ldots, u_{n-1}, w$ and joining

(i) u_i to v_i for $1 \leq i \leq n-1$ and

(ii) w to v_i for $k+1 \leq i \leq n-1$.

Hence $\deg w = k-1$ and, furthermore, $G-w$ is isomorphic to the graph $G_{n,n-1}$ described prior to Proposition 2.6. Since $\chi_s(G-w) = n = 2k$, it follows by Theorem 4.1 that

$$\chi_s(G) \geq \chi_s(G-w) - \deg w = k+1.$$

Furthermore, since the coloring $c : V(G) \to \mathbb{N}_{k+1}$ defined by

$$c(v) = \begin{cases} i & \text{if } v \in \{u_i, u_{k+i}\} \ (1 \leq i \leq k-1), \\ k & \text{if } v \in \{u_k, w\}, \\ k+1 & \text{otherwise} \end{cases}$$

is a set $(k+1)$-coloring of G, it follows that $\chi_s(G) \leq k+1$ and so $\chi_s(G) = k+1$. Consequently,

$$\chi_s(G-w) = \chi_s(G) + \deg w,$$

establishing the sharpness of the upper bound in Theorem 4.1.

We now consider how the set chromatic number of a connected graph G is affected by deleting an edge from G. Consider the connected graph G of Figure 6(a) and the three edges $e_{-1} = v_1v_2$, $e_0 = u_2u_3$, and $e_1 = u_4v_5$

in G. For the three graphs $G - e_i$ for $i \in \{-1, 0, 1\}$, observe that $\omega(G) = \omega(G - e_0) = \omega(G - e_1) = 5$ and $\omega(G - e_{-1}) = 4$. Hence $\chi_s(H) \geq 4$ for $H \in \{G, G - e_0, G - e_1\}$ and $\chi_s(G - e_{-1}) \geq 3$ by Proposition 3.2. The colorings given in Figure 6 show that

$$\chi_s(G) = \chi_s(G - e_0) = 4 \text{ and } \chi_s(G - e_{-1}) = 3.$$

We now show that $\chi_s(G - e_1) = 5$. Since $\chi(G - e_1) = 5$, it suffices to verify that $\chi_s(G - e_1) \neq 4$. Assume, to the contrary, that c is a set 4-coloring of $G - e_1$. For the graph $F = (G - e_1) - e_0$, it was shown in Proposition 2.6 that $\chi_s(F) = 5$, that is, c is not a set coloring of F. Note that

$$\mathrm{NC}_{G-e_1}(x) = \mathrm{NC}_F(x)$$

for every $x \in V(G - e_1) - \{u_2, u_3\}$ and so we may assume that

$$\mathrm{NC}_F(v_2) = \mathrm{NC}_F(u_2) = \{c(v_2)\} = \{1\}.$$

However, this implies that $\mathrm{NC}_{G-e_1}(v_1) = \mathrm{NC}_{G-e_1}(v_2) = \{1\}$, contradicting the fact that c is a set coloring of $G - e_1$. Therefore, $\chi_s(G - e_1) = 5$. Hence for $-1 \leq i \leq 1$,

$$\chi_s(G - e_i) = \chi_s(G) + i.$$

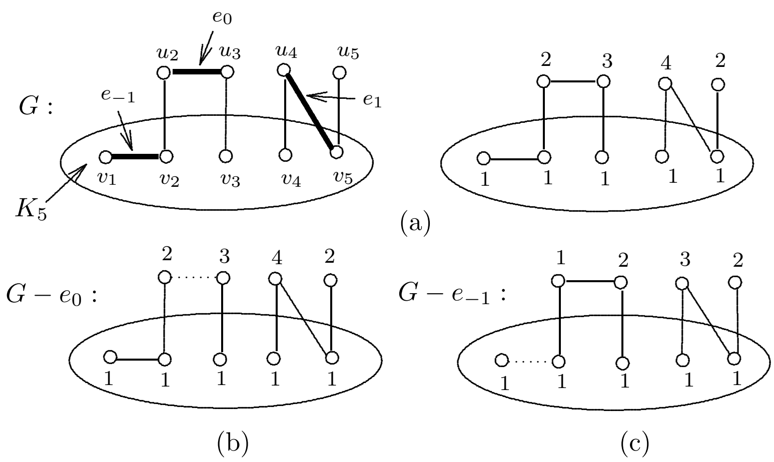

Figure 6. Graphs G and $G - e_i$ with $\chi_s(G - e_i) = \chi_s(G) + i$ for $i \in \{-1, 0\}$.

Next, we show that for every graph G and an edge e in G, the difference between $\chi_s(G)$ and $\chi_s(G-e)$ cannot exceed 2.

Proposition 4.2. *If e is an edge of a graph G, then*

$$|\chi_s(G) - \chi_s(G-e)| \leq 2.$$

Proof. Let $e = uv$. First, we verify that $\chi_s(G) - \chi_s(G-e) \leq 2$. Suppose that $\chi_s(G-e) = k$ and let $c_1 : V(G-e) \to \mathbb{N}_k$ be a set k-coloring of $G-e$. Then observe that the coloring c_1' of G defined by

$$c_1'(x) = \begin{cases} k+1 & \text{if } x = u, \\ k+2 & \text{if } x = v, \\ c_1(x) & \text{otherwise} \end{cases}$$

is a set coloring of G that using at most $k+2$ colors. Therefore, $\chi_s(G) \leq k+2 = \chi_s(G-e)+2$ and so $\chi_s(G) - \chi_s(G-e) \leq 2$. To verify that $\chi_s(G-e) - \chi_s(G) \leq 2$, suppose that $\chi_s(G) = \ell$ and consider a set ℓ-coloring $c_2 : V(G) \to \mathbb{N}_\ell$ of G. Then the coloring c_2' defined by

$$c_2'(x) = \begin{cases} \ell+1 & \text{if } x = u, \\ \ell+2 & \text{if } x = v, \\ c_2(x) & \text{otherwise} \end{cases}$$

is a set coloring of $G-e$ using at most $\ell+2$ colors. Thus, $\chi_s(G-e) \leq \ell+2 = \chi_s(G)+2$. ■

We are unaware of a graph G and an edge e of G such that $|\chi_s(G) - \chi_s(G-e)| = 2$. Nevertheless, we conclude by presenting a sufficient condition that $|\chi_s(G) - \chi_s(G-e)| \leq 1$ for an edge $e = uv$ that is not a bridge in a graph G in terms of the distance between u and v in G. For a vertex v in a graph G, let $N_G[v] = N_G(v) \cup \{v\}$ be the *closed neighborhood* of v in G.

Proposition 4.3. *If $e = uv$ is an edge of a graph G that is not a bridge such that $d_{G-e}(u,v) \geq 4$, then*

$$|\chi_s(G) - \chi_s(G-e)| \leq 1.$$

Proof. We first verify that $\chi_s(G) - \chi_s(G-e) \le 1$. Suppose that $\chi_s(G-e) = k$ and let $c_1 : V(G-e) \to \mathbb{N}_k$ be a set k-coloring of $G-e$. We show that the coloring c_1' defined by

$$c_1'(x) = \begin{cases} c_1(x) & \text{if } x \neq u, \\ k+1 & \text{if } x = u \end{cases}$$

is a set coloring of G that uses at most $k+1$ colors. Observe that $\mathrm{NC}_{c_1'}(x) = \mathrm{NC}_{c_1}(x)$ for every $x \in V(G) - N_G[u]$, while $k+1 \in \mathrm{NC}_{c_1'}(x)$ for every $x \in N_G(u)$. Let x, y be a pair of adjacent vertices in G. If $\{x, y\} \not\subseteq N_G(u)$, then $\mathrm{NC}_{c_1'}(x) \neq \mathrm{NC}_{c_1'}(y)$. Hence we may assume that $\{x, y\} \subseteq N_G(u)$. Note that $v \notin \{x, y\}$ since $d_{G-e}(u, v) > 2$. Thus, $\{x, y\} \subseteq N_{G-e}(u)$. Since $\mathrm{NC}_{c_1}(x) \neq \mathrm{NC}_{c_1}(y)$ and $c_1(u) \in \mathrm{NC}_{c_1}(x) \cap \mathrm{NC}_{c_1}(y)$, there exists a color $i^* \in \mathbb{N}_k - \{c_1(u)\}$ that belongs to exactly one of $\mathrm{NC}_{c_1}(x)$ and $\mathrm{NC}_{c_1}(y)$, say $i^* \in \mathrm{NC}_{c_1}(x) - \mathrm{NC}_{c_1}(y)$. Then $i^* \in \mathrm{NC}_{c_1'}(x) - \mathrm{NC}_{c_1'}(y)$. Hence c_1' is a set coloring of G and so $\chi_s(G) \le k+1 = \chi_s(G-e)+1$.

To verify that $\chi_s(G-e) \le \chi_s(G)+1$, suppose that $\chi_s(G) = \ell$ and let $c_2 : V(G) \to \mathbb{N}_\ell$ be a set ℓ-coloring of G. We show that the coloring c_2' defined by

$$c_2'(x) = \begin{cases} c_2(x) & \text{if } x \notin \{u, v\}, \\ \ell+1 & \text{if } x \in \{u, v\} \end{cases}$$

is a set coloring of $G-e$ using at most $\ell+1$ colors. Observe that $\mathrm{NC}_{c_2'}(x) = \mathrm{NC}_{c_2}(x)$ for every $x \in V(G) - (N_{G-e}[u] \cup N_{G-e}[v])$, while $\ell+1 \in \mathrm{NC}_{c_2'}(x)$ for every $x \in N_{G-e}(u) \cup N_{G-e}(v)$. Suppose that x, y is a pair of adjacent vertices in $G-e$. If $\{x, y\} \not\subseteq N_{G-e}(u) \cup N_{G-e}(v)$, then $\mathrm{NC}_{c_2'}(x) \neq \mathrm{NC}_{c_2'}(y)$. On the other hand, since $d_{G-e}(u, v) \ge 4$, no vertex in $N_{G-e}(u)$ is adjacent to a vertex in $N_{G-e}(v)$. Hence if $\{x, y\} \subseteq N_{G-e}(u) \cup N_{G-e}(v)$, then either $\{x, y\} \subseteq N_{G-e}(u)$ or $\{x, y\} \subseteq N_{G-e}(v)$, say the former. Since $\mathrm{NC}_{c_2}(x) \neq \mathrm{NC}_{c_2}(y)$ and $c_2(u) \in \mathrm{NC}_{c_2}(x) \cap \mathrm{NC}_{c_2}(y)$, there is a color $j^* \in \mathbb{N}_\ell - \{c_2(u)\}$ that belongs to exactly one of $\mathrm{NC}_{c_2}(x)$ and $\mathrm{NC}_{c_2}(y)$, say $j^* \in \mathrm{NC}_{c_2}(x) - \mathrm{NC}_{c_2}(y)$. Then $j^* \in \mathrm{NC}_{c_2'}(x) - \mathrm{NC}_{c_2'}(y)$. Therefore, c_2' is a set coloring of $G-e$ and so $\chi_s(G-e) \le \ell+1 = \chi_s(G)+1$. ■

According to the proof of Proposition 4.3, if there is a graph G with an edge $e = uv$ having the property that $|\chi_s(G) - \chi_s(G-e)| = 2$, then $d_{G-e}(u, v) \le 3$. In particular, if $\chi_s(G) - \chi_s(G-e) = 2$, then $d_{G-e}(u, v) = 2$.

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