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ON UNIVERSAL GRAPHS FOR HOM-PROPERTIES

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Abstract

A graph property is any isomorphism closed class of simple graphs. For a simple finite graph H, let $\rightarrow H$ denote the class of all simple countable graphs that admit homomorphisms to H, such classes of graphs are called hom-properties. Given a graph property $\mathcal{P}$, a graph

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$G \in \mathcal{P}$ is universal in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to an induced subgraph of G. In particular, we consider universal graphs in $\rightarrow H$ and we give a new proof of the existence of a universal graph in $\rightarrow H$, for any finite graph H.

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1. Introduction

Let us denote by $\mathcal{I}$ the class of all finite simple graphs and by $\mathcal{I}(\aleph_0)$ the class of all simple countable graphs. *A graph property* $\mathcal{P}$ is any nonempty isomorphism-closed subclass of $\mathcal{I}(\aleph_0)$. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A graph property $\mathcal{P}$ is of *finite character* if a graph G has property $\mathcal{P}$ if and only if each finite vertex-induced subgraph of G has property $\mathcal{P}$. We consider graph properties of finite character only. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph.

A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking subgraphs. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. One can easily see that every hereditary property is induced-hereditary as well. On the other hand, the previous definitions yields that properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character; for example the graph property $\mathcal{Q}$ of not containing a vertex of infinite degree is induced-hereditary but not of finite character. A property $\mathcal{P}$ is called *additive* if it is closed under disjoint unions of graphs, which means that a graph has property $\mathcal{P}$ providing all its connected components have this property. The interested reader can find more details about hereditary and induced-hereditary properties in [4].

Given a graph property $\mathcal{P}$, a graph $U \in \mathcal{P}$ is said to be *universal* in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to an induced subgraph of U and every induced subgraph of U is a member of $\mathcal{P}$. R. Rado in [15] first remarked that among the countable graphs there exists a universal one, often called "the Rado graph" or "the infinite random graph" R (for details see [5]). A graph $W \in \mathcal{P}$ is called *weakly universal* in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to a subgraph of W. In practice the two notions of universality behave similarly. A universal graph is evidently weakly universal, and very often the proofs of the nonexistence of a universal graph can be made by excluding weakly universal graphs (see [6]). More information concerning universal graphs and their features can be found in [11].

For a finite graph H, the existence of a weakly universal graph $W(H)$ in the class $\to H$ was in fact shown in [13]. In [1, 2] A. Bonato gave an explicit construction of the universal (pseudo-homogeneous) graph $M(H)$ in $\to H$ as a deterministic limit of a chain of finite H-colourable graphs. In this paper we provide a new and explicit representation of a universal graph $U(H)$ in the class $\to H$. The graph is presented by codes associated to its vertices. We shall show that this graph is isomorphic to $M(H)$.

2. Hom-properties

All graphs considered in this paper are simple (without multiple edges or loops), finite or countable and we use the standard notation of [8].

A homomorphism of a graph G to a graph H is an edge-preserving mapping $f : V(G) \to V(H)$ satisfying $e = uv \in E(G)$ implies $f(e) = f(u)f(v) \in E(H)$. In this case we say that G is homomorphic to H and we write $G \to H$.

A core of a finite graph G, denoted by $C(G)$, is any subgraph G' of G such that $G \to G'$ while G fails to be homomorphic to any proper subgraph of G'. A finite graph G is called a core if G is a core of itself, so that $G \cong C(G)$. A graph G homomorphic to a given graph H is also said to be *H-colourable*. It can be easily seen that up to isomorphism every finite graph has a unique core (see [9]). A *hom-property* is any class $\to H = \{G \in \mathcal{I}(\aleph_0) | G \to H\}$. The properties $\to H$, $H \in \mathcal{I}$, are called hom-properties or colour classes (see [14]). Graph homomorphisms and their structure were extensively investigated (see [9, 12, 13, 17]), more references can be found in the survey [14] and in the book [10].

Let us mention some known results concerning hom-properties. Hom-properties can be given in various ways, for example the property $\to C_6$ is the same as the property $\to C_{38}$ and/or $\to K_2$. Let us say that a graph G *generates* the hom-property $\to H$ whenever $\to H = \to G$.

A standard way to describe hom-properties is by cores (see [13]):

Proposition 1. *For any finite graph H and its core $C(H)$ it holds $\to H =\to (C(H))$.*

The next result follows directly from the definitions:

Proposition 2. *For any graph $H \in \mathcal{I}$, the hom-property $\to H$ is hereditary and additive.*

For any graph $G \in \mathcal{I}$ with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, we define a *multiplication* $G^{::}(W_1; W_2; \dots; W_n)$ of G in the following way:

1. $V(G^{::}) = W_1 \cup W_2 \cup \cdots \cup W_n$,
2. for each $1 \leq i \leq n$: $|W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \leq i \leq j \leq n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{::})$ if and only if $\{v_i v_j\} \in E(G)$.

The sets $W_1, W_2, \dots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \dots, v_n$, respectively. The condition 4 immediately yields that $W_1, W_2, \dots, W_n$ are independent sets and any two vertices belonging to the same multivertex have identical neighbourhoods. Furthermore, it is not difficult to see that $G^{::}$ is homomorphic to G. In order to emphasize the structure of $G^{::}$ we also use the notation $G^{::}(W_1, W_2, \dots, W_n)$.

Let us recall some important properties of multiplications presented in [12, 13]:

Lemma 1. *Let $G^{::}$ be a multiplication of a graph G. If w, w^* are two distinct vertices belonging to the same multivertex W of $G^{::}$, then there exists a homomorphism $\psi : G \longrightarrow G - w^*$.*

The multiplication operation strongly copies the structure of the original graph H. This can be expressed in the language of uniquely H-colourable graphs. This concept was introduced in [17]. We say that a graph G is *uniquely H-colourable* if there is a surjective homomorphism φ from G to H, such that any other homomorphism from G to H is the composition $\varphi \circ \alpha$ of φ and an automorphism α of H.

According to Lemma 1 one can rather easily see the following fact.

Theorem 1. *Let H be a core. Then any multiplication $H^{::}(W_1, W_2, \dots, W_n)$ of H is uniquely H-colourable.*

3. Main Results

By the definition of $\to H$, it is easy to see that for a given finite core H, the graph $W(H) = H^{::}(W_1, W_2, \ldots, W_n)$, with $|W_i| = \aleph_0$ for $i = 1, 2, \ldots, n$ is a weakly universal graph in the class $\to H$.

In this section we shall show, how to derive a universal graph $U(H)$ in $\to H$ from the graph $W(H)$. As was already pointed out, its existence was proved by A. Bonato in [2]. Some of its properties were investigated in [1].

Consider a graph H of order d. We are going to construct a graph $U(H)$ – the candidate for a universal graph for the property $\to H$. Let η be a bijection $\eta : \mathbb{N}^d \to \mathbb{N}$. Let us denote the vertices of H by $v_1, v_2, \ldots, v_d$. For each $i = 1, 2, \ldots, d$ take a countable set of independent vertices $W_i = \{v_i^1, v_i^2, \ldots, v_i^k, \ldots\}$ and for a fixed $i \in \{1, 2, \ldots, d\}$ and for each $k \in \mathbb{N}$, let us assign to v_i^k a $d+1$-tuple $(u_1, u_2, \ldots, u_d, u_{d+1})$ such that $k = u_{d+1} = \eta(u_1, u_2, \ldots, u_d)$ (if there is no danger of confusion we shall write and use $v_i^k = (u_1, u_2, \ldots, u_d, u_{d+1})$). One can immediately see that in such a way the vertices in W_i obtain different ordered $(d+1)$-tuples, while the codes of v_i^k and v_j^k are the same.

Now we are going to describe the structure of the universal graph $U = U(H)$ in $\to H$. Put $V(U) = W_1 \cup W_2 \cup \cdots \cup W_d$. If $u = v_i^r = (u_1, u_2, \ldots, u_d, u_{d+1}) \in W_i$ and $u' = v_j^s = (u'_1, u'_2, \ldots, u'_d, u'_{d+1}) \in W_j$ are vertices of U, then uu' is an edge of U if and only if $i < j$ and $v_i v_j \in E(H)$ and 2^r occurs in the unique base 2 expansion of u'_i (the i-th element of the code of the vertex $u' \in W_j$). Note that for each i the set W_i is independent.

Now we are going to prove the main result. The proof of the theorem follows the idea of the proof of Rado in [15] (see also [3]).

Theorem 2. *Let H be a graph. Then $U(H)$ is an universal graph for the property $\to H$.*

Proof. Let us fix a positive integer $k \in \{2, 3, \ldots, d\}$ and for $j = \{1, 2, \ldots, k-1\}$ let $A_j, B_j \subseteq W_j$ be arbitrary finite disjoint sets. We shall show that there exists a vertex $w \in W_k$ such that w is joined to all vertices from A_j's but it is joined to no vertex from B_j's. This property provides a variation of the property called "e.c. – existentially closed" (see e.g. [3]). We referred it briefly EC*. For each $j = 1, 2, \ldots, k-1$ let us put $z_j = \max\{u_{d+1} : u = (u_1, u_2, \ldots, u_{d+1}) \in A_j \cup B_j\}$. Now define

$$a_j = \begin{cases} 2^{z_j+1} + \sum_{u \in A_j} 2^{u_{d+1}} & \text{for all } j \in \{1, \dots, k-1\}, \\ 0 & \text{for } k \le j \le d, \\ \eta(a_1, a_2, \dots, a_d) & \text{for } j = d+1. \end{cases}$$

We claim that for each $j \in \{1, \dots, k-1\}$ the vertex $w \in W_k$ with the code $(a_1, a_2, \dots, a_{d+1})$ is joined to all vertices from A_j but with no vertex from B_j providing that $v_j v_k \in E(H)$. Indeed, by the definition of $U = U(H)$ and the construction of the code of w, the vertex w is joined to each vertex of A_j. To see, that w is joined to no vertex of B_j, note that for all vertices $u' = (u'_1, u'_2, \dots, u'_{d+1})$ of B_j $2^{u'_{d+1}}$ is not in the base 2 expansion of a_j.

It remains to prove that for any countable graph G belonging to the class $\to H$ there exists a graph G', induced subgraph of U, isomorphic to the graph G.

Since the property $\to H$ is additive and hereditary we can represent a countable graph G in $\to H$ as a limit of finite graphs from $\to H$ (see [16, 3]). Thus it is sufficient to provide the embeddings of all finite graphs to $U = U(H)$. Let us remark here that the property $\to H$ is of finite character, hence the compactness can also be used (see [7]). In order to prove that if G is a fixed finite graph belonging to $\to H$ then there exists a graph G', the induced subgraph of U, isomorphic to G we follow the idea of the proof of Theorem 6.7 of [3] and we omit some technical details.

It is obvious that K_1 is and induced subgraph of U. Now let G be a finite graph belonging to $\to H$. Then there exists a homomorphism $\varphi : G \to H$. For an arbitrary vertex $v \in V(G)$ the graph $S = G - v$ has order smaller than G and therefore, according to the induction hypothesis, there exists an induced subgraph S' of the graph U that is isomorphic to S. Moreover, it is not difficult to see, that there exists such an embedding that a vertex $u \in V(G)$ with $\varphi(u) = j$ is mapped to a vertex of $W_j \subseteq V(U)$.

According to the labeling of the vertices of H (see the description of the construction above), let k be the largest index such that $v_i \in V(H)$ $(i = 1, 2, \dots, d)$ is an image of some vertex of G, i.e., $k = \max\{i : \varphi(x) = v_i, x \in V(G)\}$. Let us choose a vertex $u^* \in V(G)$ such that $\varphi(u) = k$ (note that the set of vertices of G with $\varphi(u) = k$ is independent in G). According to the previous, using EC* property and taking an appropriate vertex of $W_k \subseteq V(U)$ we can now extend the embedding of the graph $S = G - u^*$ to an embedding of the whole graph G of U and the proof is complete. ■

A. Bonato in [1] investigated universal pseudo-homogeneous graphs, that were defined in the following way:

Definition 1. Let $\mathcal{C}$ be a class of countable graphs closed under isomorphisms. A countable graph $M \in \mathcal{C}$ is called *universal pseudo-homogeneous* if there is a subclass $\mathcal{C}'$ of finite graphs from $\mathcal{C}$ such that:

(PH1) The graph M embeds each graph in C' as an induced subgraph.

(PH2) Each finite $S \leq M$ is contained in $T \leq M$ with $T \in \mathcal{C}'$.

(PH3) For each $G \leq M$ with $G \in \mathcal{C}'$ and for each graph $H \in \mathcal{C}'$ so that $G \leq H$, there is an $H' \leq M$ and an isomorphism $f : H \to H'$ such that f restricted to G is and identity mapping.

A. Bonato in [1, 2] proved that for each finite core graph H there is a countable universal pseudo-homogeneous H-colourable graph $M(H)$, that is unique up to isomorphism. If we consider the class of graphs that are H-colourable and as the class $\mathcal{C}'$ we take the class of finite uniquely H-colourable graphs, then we immediately have the following result.

Theorem 3. *Let H be a finite core. Then $U(H)$ is the unique universal pseudo-homogeneous graph for the property $\to H$ with respect to the family of finite uniquely H-colourable graphs.*

Proof. In order to prove the assertion of the theorem we have to verify properties (PH1)–(PH3) from Definition 1. We remind that in our case the set $\mathcal{C}'$ is the class of uniquely H-colourable graphs.

Since $U(H)$ is universal in $\to H$, the property (PH1) is evidently satisfied. As all the induced subgraphs of $U(H)$ belongs to $\to H$, the condition (PH2) is evidently satisfied as well. Now we focus on the condition (PH3). Firstly we fix the graph G. Let $G \leq X$, $X \in \to H$ and let $V(X) \setminus V(G) = \{v_1^1, \ldots, v_1^{i_1}, \ldots, v_k^1, \ldots, v_k^{i_k}\}$. Since X and G are uniquely H-colourable, we can find an extension $X' \leq U(H)$ of X. Observe that there exists a vertex $w_1^1 \in U(H)$ that is an image of v_1^1. Thus by induction hypothesis we obtain that such images exist for all vertices in $V(X) \setminus V(G)$ (we can apply similar arguments as in the proof of Theorem 2, but using also the "or" statement in the construction of $U(H)$). Indeed, by the consecutive selection of vertices with a suitable structure of neighbours (because of selection of the vertex w with respect to the structure of A_j's and B_j's) we can find the desired graph X'. ∎

Corollary 1. *Let H be a finite core and let $G \in \rightarrow H$, then $U(H) \cong U(G)$.*

Proof. Both universal graphs $U(H)$ and $U(G)$ are universal pseudo-homogeneous graphs for $\rightarrow H$ and thus they are isomorphic to the universal pseudo-homogeneous graph $M(H)$, the existence of which have been proved by A. Bonato in [1]. ■

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