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MONOCHROMATIC PATHS AND MONOCHROMATIC SETS OF ARCS IN 3-QUASITRANSITIVE DIGRAPHS

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Abstract

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path is called monochromatic if all of its arcs are coloured alike. A set N of vertices of D is called a kernel by monochromatic paths if for every pair of vertices of N there is no monochromatic path between them and for every vertex $v \notin N$ there is a monochromatic path from v to N. We denote by $A^+(u)$ the set of arcs of D that have u as the initial vertex. We prove that if D is an m-coloured 3-quasitransitive digraph such that for every vertex u of D, $A^+(u)$ is monochromatic and D satisfies some colouring conditions over one subdigraph of D of order 3 and two subdigraphs of D of order 4, then D has a kernel by monochromatic paths.

Keywords: m-coloured digraph, 3-quasitransitive digraph, kernel by monochromatic paths, γ-cycle, quasi-monochromatic digraph.

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1. Introduction

For general concepts we refer the reader to [3]. A *kernel* N of a digraph D is an independent set of vertices of D such that for every $w \in V(D) \setminus N$ there exists an arc from w to N. A digraph D is called *kernel perfect digraph* when every induced subdigraph of D has a kernel. We call the digraph D an *m-coloured* digraph if the arcs of D are coloured with m colours. A directed path is called *monochromatic* if all of its arcs are coloured alike. A set N of vertices of D is called a *kernel by monochromatic paths* if for every pair of vertices there is no monochromatic path between them and for every vertex v not in N there is a monochromatic path from v to some vertex in N. The *closure of* D, denoted $\mathfrak{C}(D)$, is the m-coloured digraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) = A(D) \cup \{uv$ with colour i | there exists a uv-monochromatic path of colour i contained in $D\}$. Notice that for any digraph D, $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular Richardson [14, 15]; Duchet and Meyniel [6]; Duchet [4, 5]; Galeana-Sánchez and V. Neumann-Lara [9, 10]. The concept of kernel by monochromatic paths is a generalization of the concept of kernel and it was introduced by Galeana-Sánchez [7]. In that work she obtained some sufficient conditions for an m-coloured tournament T to have a kernel by monochromatic paths. More information about m-coloured digraphs can be found in [8]. In [16] Sands *et al.* have proved that any 2-coloured digraph has a kernel by monochromatic paths. In particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must D have a kernel by monochromatic paths? (An m-coloured digraph D is called *quasi-monochromatic* if with at most one exception all of its arcs are coloured alike). In [13] Shen Minggang proved that under the additional assumption that every transitive tournament of order 3 is quasi-monochromatic, the answer will be yes. In [7] it was proved that if T is an m-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then T has a kernel by monochromatic paths. In [11] we give an affirmative answer for this question for quasi-transitive digraphs whenever $A^+(u)$ is monochromatic for each vertex u ($A^+(u)$ is the set of arcs of D that have u as the initial vertex). A digraph D is called *quasi-transitive* if whenever $(u,v) \in A(D)$ and $(v,w) \in A(D)$ then $(u,w) \in A(D)$ or $(w,u) \in A(D)$.

Quasi-transitive digraphs were introduced by Ghouilá-Houri [12] and have been studied by several authors for example Bang-Jensen and Huang [1, 2]. We call a digraph D *n-quasitransitive digraph* if it has the following property: If $u, v \in V(D)$ and there is a directed uv-path of length n in D, then $(u, v) \in A(D)$ or $(v, u) \in A(D)$. In this paper we study 3-quasitransitive digraphs. We denote by $\widetilde{T}_4$ the digraph such that $V(\widetilde{T}_4) = \{u, v, w, x\}$ and $A(\widetilde{T}_4) = \{(u, v), (v, w), (w, x), (u, x)\}$. If C is a walk we will denote by $\ell(C)$ its length. If $S \subseteq V(D)$ we denote by $D[S]$ the subdigraph induced by S. An arc $(u, v) \in A(D)$ is *symmetrical* if $(v, u) \in A(D)$. In this paper we prove that if D is an m-coloured 3-quasitransitive digraph such that for every C_3 (the directed cycle of length 3), C_4 (the directed cycle of length 4) and $\widetilde{T}_4$ contained in D are quasi-monochromatic then D has a kernel by monochromatic paths.

We will need the following results.

Theorem 1.1. *Let D be a digraph. D has a kernel by monochromatic paths if and only if $\mathfrak{C}(D)$ has a kernel.*

Theorem 1.2. *Every uv-monochromatic walk in a digraph contains a uv-monochromatic path.*

Theorem 1.3 (Berge-Duchet [4])**.** *Let D be a digraph. If every directed cycle of D contains a symmetrical arc, then D is a kernel-perfect digraph.*

2. 3-Quasitransitive Digraphs

The following lemma and remarks are about 3-quasitransitive digraphs such that for every $u \in V(D)$, $A^+(u)$ is monochromatic, and they are useful to prove our main result.

Let $T = (u_0, u_1, \ldots, u_n)$ be a path. Then we will denote the path $(u_i, u_{i+1}, \ldots, u_j)$ by (u_i, T, u_j). Here, $[x]$ represents the largest integer less or equal than x.

Lemma 2.1. *Let D be an m-coloured 3-quasitransitive digraph such that for every vertex $u \in V(D)$, $A^+(u)$ is monochromatic. If u and v are vertices of D and $T = (u = u_0, u_1, \ldots, u_n = v)$ is a uv-monochromatic path of minimum length $n \geq 3$, then $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{3, \ldots, n\}$ and $k \in \{1, \ldots, [\frac{i-1}{2}]\}$. In particular, if $\ell(T)$ is odd, then $(v, u) \in A(D)$ and if $\ell(T)$ is even, then (v, u) may be absent in D.*

Proof. Observe that if T is a uv-monochromatic path of minimum length and $\{u_i, u_j\} \subseteq V(T)$ with $i < j$ then the hypothesis that $A^+(z)$ is monochromatic for every $z \in V(D)$ implies that (u_i, T, u_j) is also a $u_i u_j$-monochromatic path of minimum length.

We will proceed by induction on $\ell(T) = n$.

When $n = 3$ then $T = (u = u_0, u_1, u_2, u_3 = v)$. Since D is a 3-quasitransitive digraph then $(u_0, u_3) \in A(D)$ or $(u_3, u_0) \in A(D)$. Since T is of minimum length we have that $(u_3, u_0) \in A(D)$.

If $n = 4$ then $T = (u = u_0, u_1, u_2, u_3, u_4 = v)$. By the case $n = 3$ $(u_3, u_0) \in A(D)$ and $(u_4, u_1) \in A(D)$.

Suppose that if $\ell(T) = n \geq 4$ then $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{3, \ldots, n\}$ and $k \in \{1, \ldots, [\frac{i-1}{2}]\}$.

Let $T = (u = u_0, u_1, \ldots, u_n, u_{n+1} = v)$ be a uv-monochromatic path of minimum length. Let $T' = (u, T, u_n)$, then T' is a uu_n-monochromatic path of minimum length. By the induction hypothesis we have that $(u_i, u_{i-(2k+1)})$ $\in A(D)$ for each $i \in \{3, \ldots, n\}$ and $k \in \{1, \ldots, [\frac{i-1}{2}]\}$. Also, let $T'' =$ (u_1, T, v), then T'' is a $u_1 v$-monochromatic path of minimum length, the induction hypothesis implies that $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{4, \ldots,$ $n+1\}$ and $k \in \{1, \ldots, [\frac{i-1}{2}]\}$. So, it is sufficient to prove that $(u_{n+1}, u_0) \in$ $A(D)$ whenever $n+1$ is odd. Assume $n+1$ is odd. We have that $\{(u_{n+1}, u_2),$ $(u_2, u_3), (u_3, u_0)\} \subseteq A(D)$, so (u_{n+1}, u_2, u_3, u_0) is a path of length 3. Since D is a 3-quasitransitive digraph then $(u_{n+1}, u_0) \in A(D)$ or $(u_0, u_{n+1}) \in A(D)$. Thus $(u_{n+1}, u_0) \in A(D)$.

Remark 2.1. Let D be an m-coloured 3-quasitransitive digraph. If every $\widetilde{T}_4$ and C_4 contained in D are at most 2-coloured then D contains no 3-coloured path of length 3.

Remark 2.2. Let D be an m-coloured digraph such that for every vertex $u \in V(D)$ $A^+(u)$ is monochromatic and D contains no 3-coloured C_3. If (u, u_1, u_2, v) is a 3-coloured walk then $u \neq u_1, u \neq u_2, u \neq v, u_1 \neq u_2$ and $u_2 \neq v$.

3. The Main Result

Definition 3.1. Let D be an m-coloured digraph. A γ-cycle in D is a sequence of distinct vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$

1. There is a u_iu_{i+1}-monochromatic path and
2. There is no $u_{i+1}u_i$-monochromatic path.

The addition over the indices of the vertices of γ are modulo $n+1$. And we say that the length of γ is $n+1$.

Theorem 3.2. *Let D be an m-coloured 3-quasitransitive digraph such that for every vertex u of D, $A^+(u)$ is monochromatic. If every C_3, C_4 and $\widetilde{T}_4$ contained in D is quasi-monochromatic, then there are no γ-cycles in D.*

Proof. We will proceed by contradiction. Suppose that $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ is a γ-cycle in D of minimum length. The definition of γ-cycle implies that for every $i \in \{0, \ldots, n\}$ there exist a u_iu_{i+1}-monochromatic path in D namely T_i, (we may assume that T_i is of minimum length) and there is no $u_{i+1}u_i$-monochromatic path in D (notation mod$(n+1)$). So we have $(u_{i+1}, u_i) \notin A(D)$ and by Remark 2.1 $\ell(T_i)$ is even or $\ell(T_i) = 1$ for every $i \in \{0, \ldots, n\}$. Now we have the following assertions.

1. $\ell(\gamma) \geq 3$. If $\ell(\gamma) = 2$ then $\gamma = (u_0, u_1, u_0)$ and this implies that there is a u_1u_0-monochromatic path, contradicting the definition of γ-cycle.

2. There is an index $i \in \{0, \ldots, n\}$ such that T_i and T_{i+1} have different colours. Otherwise $T_0 \cup T_1 \cup \cdots \cup T_n$ contains a u_0u_n-monochromatic path, a contradiction. Suppose w.l.o.g. that T_0 is coloured 1 and T_1 is coloured 2.

3. There is no u_2u_0-monochromatic path in D. Suppose by contradiction that $T = (u_2 = x_0, x_1, \ldots, x_t = u_0)$ is a u_2u_0-monochromatic path of minimum length in D. Then:

3.1. T is neither coloured 1 nor 2. This follows from the facts that T_0 is coloured 1, T_1 is coloured 2 and there is no u_2u_1-monochromatic path and u_1u_0-monochromatic path either.

3.2. $\ell(T_0) \geq 4$ and $\ell(T_1) \geq 4$.
If $\ell(T_0) = 1 = \ell(T_1)$, then $C = (u_0, u_1, u_2, x_1)$ is a 3-coloured u_0x_1-walk of length 3. So by Remark 2.2 we have that C is a 3-coloured u_0x_1-path of length 3 contradicting the Remark 2.1.

If $\ell(T_0) = 2$ and $\ell(T_1) = 1$, let $T_0 = (u_0, y, u_1)$, then $C = (y, u_1, u_2, x_1)$ is a 3-coloured walk of length 3. It follows from Remark 2.2 that C is a 3-coloured path of length 3 contradicting the Remark 2.1.

If $\ell(T_0) = 2 = \ell(T_1)$ then we may consider $T_0 = (u_0, y, u_1)$ and $T_1 = (u_1, z, u_2)$. We have that $z \notin V(T_0)$ so $T_0 \cup (u_1, z)$ (it will denote (u_0, y, u_1, z))

is a path of length 3. Since D is a 3-quasitransitive digraph $(u_0, z) \in A(D)$ or $(z, u_0) \in A(D)$. If $(z, u_0) \in A(D)$ then it is coloured 2 ($A^+(z)$ is coloured 2) and this implies that (u_0, y, u_1, z, u_0) is a C_4 that is not quasi-monochromatic, a contradiction. So $(u_0, z) \in A(D)$ and it is coloured 1 ($A^+(u_0)$ is coloured 1). Let $C = (u_0, z, u_2, x_1)$. Then C is a 3-coloured walk of length 3. By Lemma 2.2 we have that C is a 3-coloured path of length 3 contradicting the Remark 2.1.

If $\ell(T_0) = 1$ and $\ell(T_1) = 2$, let $T_1 = (u_1, z, u_2)$ and consider $C = (x_{t-1}, u_0, u_1, z)$. Then C is a 3-coloured walk. Remark 2.2 imply that C is a 3-coloured path of length 3, contradicting the Remark 2.1.

We conclude that $\ell(T_0) \geq 4$ and $\ell(T_1) \geq 4$.

Let $T_0 = (u_0 = y_0, y_1, \ldots, y_\ell = u_1)$ and $T_1 = (u_1 = z_0, z_1, \ldots, z_k = u_2)$ with $\ell \geq 4$ and $k \geq 4$.

3.3. $\ell(T) \geq 3$.

Suppose by contradiction that $\ell(T) < 3$.

If $\ell(T) = 1$ then $C = (z_{k-1}, u_2, u_0, y_1)$ is a 3-coloured walk. Remark 2.2 implies that C is a 3-coloured path of length 3 but this is a contradiction with the Remark 2.1. If $\ell(T) = 2$ then $C_1 = (z_{k-1}, u_2) \cup T$ is a $z_{k-1}u_0$-path of length three. Since D is a 3-quasitransitive digraph then $(z_{k-1}, u_0) \in A(D)$ or $(u_0, z_{k-1}) \in A(D)$. If $(z_{k-1}, u_0) \in A(D)$ then it is coloured 2 and $D[\{z_{k-1}, u_2, x_1, u_0\}]$ contains a $\widetilde{T}_4$ which is not quasi-monochromatic, a contradiction. If $(u_0, z_{k-1}) \in A(D)$ then it is coloured 1 and (u_0, z_{k-1}, u_2, x_1) is a 3-coloured path of length three, a contradiction to Remark 2.1. We conclude that $\ell(T) \geq 3$.

3.4. $(u_0, u_2) \notin A(D)$.

Proceeding by contradiction, suppose that $(u_0, u_2) \in A(D)$. Since T_0 is coloured 1 then (u_0, u_2) is coloured 1. By Lemma 2.1 (remember that $\ell(T_i)$ is even) we have that $(u_2, z_1) \in A(D)$, so it is coloured 3. Then (u_0, u_2, z_1, z_2) is a path of length 3 that is 3-coloured, but this is a contradiction with Remark 2.1.

3.5. $\ell(T_0) \geq 4$, $\ell(T_1) \geq 4$, $\ell(T) \geq 4$ and $\ell(T)$ is even.

(3.3) implies that $\ell(T) \geq 3$. Since T is a u_2u_0-monochromatic path of minimum length $(u_2, u_0) \notin A(D)$ and by assertion (3.4) $(u_0, u_2) \notin A(D)$. So it follows from Lemma 2.1 that $\ell(T)$ is even.

Now, Lemma 2.1 implies that $(u_0, x_1) \in A(D)$, and it is coloured 1. Then (z_{k-1}, u_2, x_1, x_2) is a path of length 3. Since D is a 3-quasitransitive digraph $(z_{k-1}, x_2) \in A(D)$ or $(x_2, z_{k-1}) \in A(D)$. If $(z_{k-1}, x_2) \in A(D)$ it is coloured 2 and $D[\{z_{k-1}, u_2, x_1, x_2\}]$ contains a $\widetilde{T}_4$ that is not quasi-monochromatic. So $(x_2, z_{k-1}) \in A(D)$ and it is coloured 3. Then (u_0, x_1, x_2, z_{k-1}) is a u_0z_{k-1}-path of length 3. Since D is a 3-quasitransitive digraph then $(u_0, z_{k-1}) \in A(D)$ or $(z_{k-1}, u_0) \in A(D)$. If $(u_0, z_{k-1}) \in A(D)$ then it is coloured 1, so $D[\{u_0, x_1, x_2, z_{k-1}\}]$ contains a $\widetilde{T}_4$ that is not quasi-monochromatic, a contradiction. We may assume that $(z_{k-1}, u_0) \in A(D)$, so it is coloured 2. Then (u_0, x_1, x_2, z_{k-1}) is a C_4 that is not quasi-monochromatic, a contradiction.

We conclude that there is no u_2u_0-monochromatic path in D.

4. $\ell(\gamma) \geq 4$. It follows from (1) and (3).

5. There is no u_0u_2-monochromatic path in D.
Assume that there exists a u_0u_2-monochromatic path in D. Then $\gamma_1 = (u_0, u_2, u_3, \ldots, u_n, u_0)$ would be a γ-cycle such that $\ell(\gamma_1) < \ell(\gamma)$ contradicting the choice of γ.

6. If T_i and T_{i+1} have different colours then there is no $u_{i+2}u_i$-monochromatic path and there is no u_iu_{i+2}-monochromatic path either.
This follows the same way as (3) and (5).

7. If T_i and T_{i+1} have different colours and $\ell(T_i) = 1$, for some $i \in \{0, \ldots, n\}$, then $\ell(T_{i+1}) = 1$.

W.l.o.g. suppose that $\ell(T_0) = 1$. Suppose by contradiction that $\ell(T_1) \geq 2$. If $\ell(T_1) = 2$, let $T_1 = (u_1, z, u_2)$. In this case (u_0, u_1, z, u_2) is a u_0u_2-path of length 3. Since D is a 3-quasitransitive digraph then $(u_0, u_2) \in A(D)$ or $(u_2, u_0) \in A(D)$, contradicting (5) or (3) respectively. We may assume that $\ell(T_1) > 2$. Let $T_1 = (u_1 = z_0, z_1, \ldots, z_k = u_2)$. Then (u_0, u_1, z_1, z_2) is a u_0z_2-path of length 3. Since D is a 3-quasitransitive digraph $(u_0, z_2) \in A(D)$ or $(z_2, u_0) \in A(D)$. If $(u_0, z_2) \in A(D)$ then it is coloured 1 and $D[\{u_0, u_1, z_1, z_2\}]$ contains a $\widetilde{T}_4$ that is not quasi-monochromatic, a contradiction. If $(z_2, u_0) \in A(D)$ then it is coloured 2 and (u_1, z_1, z_2, u_0) is a u_1u_2-monochromatic path contradicting that γ is a γ-cycle. We conclude that $\ell(T_1) = 1$.

8. If T_i and T_{i+1} have different colours and $\ell(T_i) = 1$ then T_{i+2} is coloured with the same colour of T_i.

W.l.o.g. suppose that $i = 0$, T_0 is coloured 1 and T_1 is coloured 2. $\ell(T_0) = 1$ and assertion (7) imply that $\ell(T_1) = 1$. Let $T_2 = (u_2, x_1, \ldots, x_t = u_3)$. Then $C = (u_0, u_1, u_2, x_1)$ is a u_0x_1-walk of length 3. The definition of γ-cycle implies that $x_1 \neq u_1$ and from assertion (3) we obtain that $x_1 \neq u_0$. So C is a u_0x_1-path of length 3. Since D is a 3-quasitransitive digraph

$(u_0, x_1) \in A(D)$ or $(x_1, u_0) \in A(D)$. From the hypothesis that every C_4 and $\widetilde{T}_4$ in D is quasi-monochromatic, then the arc between x_1 and u_0 and (u_2, x_1) have the same colour. If $(x_1, u_0) \in A(D)$ then (u_2, x_1, u_0) is a u_2u_0-monochromatic path contradicting assertion (3). We may assume that $(u_0, x_1) \in A(D)$. Then (u_0, x_1) and (u_2, x_1) are coloured 1. Hence T_2 is coloured 1.

To conclude the proof of the theorem we will analyze 5 possible cases.

Case 1. Suppose that $\ell(T_0) = 1$.
Applying assertions (7) and (8) repeatedly we have that $\ell(T_i) = 1$ for every $i \in \{0, \ldots, n\}$, T_i is coloured 1 if i is even and T_i is coloured 2 if i is odd. This implies that $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ is a 2-coloured cycle in D such that the colours of its arcs are alternated, so n is odd.

We will prove by induction that $(u_0, u_i) \in A(D)$ for every odd i, $i \in \{1, \ldots, n\}$. For $i = 1$, $(u_0, u_1) \in A(D)$, since γ is a cycle. Suppose that $(u_0, u_{2k-1}) \in A(D)$ for $i = 2k - 1$, where $k \geq 1$. Now, we will prove that $(u_0, u_{2k+1}) \in A(D)$. We have that $\{(u_0, u_1), (u_0, u_{2k-1}), (u_{2k}, u_{2k+1})\}$ are coloured 1 and (u_{2k-1}, u_{2k}) is coloured 2. Let $T = (u_0, u_{2k-1}, u_{2k}, u_{2k+1})$. Then T is a u_0u_{2k+1}-path of length 3. Since D is a 3-quasitransitive digraph $(u_0, u_{2k+1}) \in A(D)$ or $(u_{2k+1}, u_0) \in A(D)$. Hence $D[V(T)]$ contains a $\widetilde{T}_4$ or a C_4. Since every $\widetilde{T}_4$ and C_4 contained in D is quasi-monochromatic then the arc between u_0 and u_{2k+1} is coloured 1. If $(u_{2k+1}, u_0) \in A(D)$ then $(u_{2k}, u_{2k+1}, u_0, u_{2k-1})$ is a $u_{2k}u_{2k-1}$-monochromatic path, contradicting the definition of γ-cycle, so $(u_0, u_{2k+1}) \in A(D)$. We conclude that $(u_0, u_i) \in A(D)$ for every odd $i \in \{1, \ldots, n\}$. Since n is odd $(u_0, u_n) \in A(D)$, but this contradicts the definition of γ-cycle.

Case 2. Suppose that $\ell(T_0) = 2$ and $\ell(T_1) = 1$.
Let $T_0 = (u_0, x, u_1)$. Then $C = T_0 \cup T_1$ is a walk of length 3. Assertion (5) implies that $x \neq u_2$, so C is a path of length 3. Since D is a 3-quasitransitive digraph $(u_0, u_2) \in A(D)$ or $(u_2, u_0) \in A(D)$. In any case we obtain a contradiction to assertion (5) or (3) respectively.

Case 3. $\ell(T_0) = 2$ and $\ell(T_1) \geq 2$.
Let $T_0 = (u_0, x, u_1)$ and $T_1 = (u_1, y_1, y_2, \ldots, y_t = u_2)$ where, $t \geq 2$. Then $C = T_0 \cup (u_1, y_1)$ is a path of length 3. Since D is a 3-quasitransitive digraph then $(u_0, y_1) \in A(D)$ or $(y_1, u_0) \in A(D)$. So, $D[V(C)]$ contains a C_4 or a $\widetilde{T}_4$, by the hypothesis it should be quasi-monochromatic. Then

the arc between u_0 and y_1 is coloured 1. Hence $(y_1, u_0) \notin A(D)$ ($A^+(y_1)$ is coloured 2) and $(u_0, y_1) \in A(D)$. Also $C' = (x, u_1, y_1, y_2)$ is a path of length 3, $(y_2, x) \in A(D)$ and it is coloured 2. Now, $D[\{u_0, y_1, y_2, x\}]$ contains a $\widetilde{T}_4$ that is not quasi-monochromatic, a contradiction.

Case 4. $\ell(T_0) \geq 4$ and $\ell(T_1) = 1$.
Let $T_0 = (u_0, x_1, x_2, \ldots, x_{t-1}, x_t = u_1)$ with $t \geq 4$. We have $C = (x_{t-2}, x_{t-1}, x_t = u_1, u_2)$ is a path of length 3 (the definition of γ-cycle implies that there is no u_2u_1-monochromatic path). Since D is a 3-quasitransitive digraph $(x_{t-2}, u_2) \in A(D)$ or $(u_2, x_{t-2}) \in A(D)$. So, $D[V(C)]$ contains a $\widetilde{T}_4$ or a C_4, by hypothesis it should be quasi-monochromatic. Then the arc between x_{t-2} and u_2 is coloured 1. If $(u_2, x_{t-2}) \in A(D)$ then $(u_2, x_{t-2}, x_{t-1}, u_1)$ is a u_2u_1-monochromatic path contradicting the definition of γ-cycle. So $(x_{t-2}, u_2) \in A(D)$. Hence $(u_0, x_1, \ldots, x_{t-2}, u_2)$ is a u_0u_2-monochromatic path contradicting assertion (5).

Case 5. $\ell(T_0) \geq 4$ and $\ell(T_1) \geq 2$.
Let $T_0 = (u_0, x_1, x_2, \ldots, x_{t-1}, x_t = u_1)$ and $T_1 = (u_1, y_1, y_2, \ldots, y_\ell = u_2)$. Then $C = (x_{t-2}, x_{t-1}, x_t = u_1, y_1)$ is an $x_{t-2}y_1$-path of length 3 (Remark 2.1). Since D is a 3-quasitransitive digraph then $(x_{t-2}, y_1) \in A(D)$ or $(y_1, x_{t-2}) \in A(D)$. Then $D[V(C)]$ contains a $\widetilde{T}_4$ or a C_4, by hypothesis it should be quasi-monochromatic. Then the arc between x_{t-2} and y_1 is coloured 1. Hence $(y_1, x_{t-2}) \notin A(D)$ ($A^+(y_1)$ is coloured 2), $(x_{t-2}, y_1) \in A(D)$ and it is coloured 1. Also, $C' = (x_{t-1},\ u_1, y_1, y_2)$ is a $x_{t-1}y_2$-path of length 3. Then $(x_{t-1}, y_2) \in A(D)$ or $(y_2, x_{t-1}) \in A(D)$. Since every $\widetilde{T}_4$ and C_4 is quasi-monochromatic, we have that $(y_2, x_{t-1}) \in A(D)$ and it is coloured 2. Then $D[\{x_{t-2}, y_1, y_2, x_{t-1}\}]$ contains a $\widetilde{T}_4$ that is not quasi-monochromatic, a contradiction.

We conclude that D contains no γ-cycles.

Theorem 3.3. *Let D be an m-coloured 3-quasitransitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every C_3, C_4 and $\widetilde{T}_4$ contained in D is quasi-monochromatic, then $\mathfrak{C}(D)$ is a kernel-perfect digraph.*

Proof. By Theorem 1.3 we will prove that every cycle in $\mathfrak{C}(D)$ contains a symmetrical arc. Let C a cycle in $\mathfrak{C}(D)$. Assume for a contradiction, that C has no symmetrical arcs. Then C is a γ-cycle in D contradicting Theorem 3.2.

Corollary 3.4. *Let D be an m-coloured 3-quasitransitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every C_3, C_4 and $\widetilde{T}_4$ contained in D is quasi-monochromatic, then D has a kernel by monochromatic paths.*

Corollary 3.5. *Let T be an m-coloured tournament such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every C_3, C_4 and $\widetilde{T}_4$ contained in D is quasi-monochromatic, then T has a kernel by monochromatic paths.*

Corollary 3.6. *Let D be an m-coloured bipartite tournament such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every C_4 and $\widetilde{T}_4$ contained in D is quasi-monochromatic, then D has a kernel by monochromatic paths.*

Remark 3.1. The condition that D contains no C_3 3-coloured in Theorem 3.3 cannot be dropped. Let D_n be the digraph obtained from D_{n-1} (D_0 is a 3-coloured C_3) by adding the vertex v_n and arcs (v_n, v) for every $v \in V(D_{n-1})$, all arcs coloured with some colour j. D_n is an m-coloured 3-quasitransitive digraph with $A^+(z)$ monochromatic for every $z \in V(D_n)$, every C_4 and $\widetilde{T}_4$ are quasi-monochromatic, D_n contains a γ-cycle (C_3) and D_n has no kernel by monochromatic paths.

Remark 3.2. The condition that every C_4 of D is quasi-monochromatic in Theorem 3.2 is tight. Let D be a 3-quasitransitive digraph 2-coloured with $V(D) = \{u, v, w, x\}$ and $A(D) = \{(u, v), (v, w), (w, x), (x, u)\}$ such that $(u, v), (w, x)$ are coloured 1 and $(v, w), (x, u)$ are coloured 2. In D $A^+(z)$ is monochromatic for every $z \in V(D)$, D has a γ-cycle. Moreover, for each n we give a digraph D_n, obtained from $D_0 = D$, that satisfies all the conditions of Theorem 3.2 except the one over C_4 and has a γ-cycle. D_n is obtained from D_{n-1} by adding the vertex v_n and the arcs (v_n, x) and (v, v_n) with colours j (for some j) and 2 respectively.

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