MONOCHROMATIC PATHS AND MONOCHROMATIC SETS OF ARCS IN 3-QUASITRANSITIVE DIGRAPHS

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Abstract

We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path is called monochromatic if all of its arcs are coloured alike. A set $N$ of vertices of $D$ is called a kernel by monochromatic paths if for every pair of vertices of $N$ there is no monochromatic path between them and for every vertex $v \notin N$ there is a monochromatic path from $v$ to $N$. We denote by $A^+(u)$ the set of arcs of $D$ that have $u$ as the initial vertex. We prove that if $D$ is an $m$-coloured 3-quasitransitive digraph such that for every vertex $u$ of $D$, $A^+(u)$ is monochromatic and $D$ satisfies some colouring conditions over one subdigraph of $D$ of order 3 and two subdigraphs of $D$ of order 4, then $D$ has a kernel by monochromatic paths.

Keywords: $m$-coloured digraph, 3-quasitransitive digraph, kernel by monochromatic paths, $\gamma$-cycle, quasi-monochromatic digraph.

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1. Introduction

For general concepts we refer the reader to [3]. A kernel $N$ of a digraph $D$ is an independent set of vertices of $D$ such that for every $w \in V(D) \setminus N$ there exists an arc from $w$ to $N$. A digraph $D$ is called kernel perfect digraph when every induced subdigraph of $D$ has a kernel. We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path is called monochromatic if all of its arcs are coloured alike. A set $N$ of vertices of $D$ is called a kernel by monochromatic paths if for every pair of vertices there is no monochromatic path between them and for every vertex $v$ not in $N$ there is a monochromatic path from $v$ to some vertex in $N$. The closure of $D$, denoted $\mathcal{C}(D)$, is the $m$-coloured digraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, $A(\mathcal{C}(D)) = A(D) \cup \{uv \text{ with colour } i \mid \text{ there exists a } uv\text{-monochromatic path of colour } i \text{ contained in } D\}$. Notice that for any digraph $D$, $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular Richardson [14, 15]; Duchet and Meyniel [6]; Duchet [4, 5]; Galeana-Sánchez and V. Neumann-Lara [9, 10]. The concept of kernel by monochromatic paths is a generalization of the concept of kernel and it was introduced by Galeana-Sánchez [7]. In that work she obtained some sufficient conditions for an $m$-coloured tournament $T$ to have a kernel by monochromatic paths. More information about $m$-coloured digraphs can be found in [8]. In [16] Sands et al. have proved that any 2-coloured digraph has a kernel by monochromatic paths. In particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let $T$ be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must $D$ have a kernel by monochromatic paths? (An $m$-coloured digraph $D$ is called quasi-monochromatic if with at most one exception all of its arcs are coloured alike). In [13] Shen Minggang proved that under the additional assumption that every transitive tournament of order 3 is quasi-monochromatic, the answer will be yes. In [7] it was proved that if $T$ is an $m$-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then $T$ has a kernel by monochromatic paths. In [11] we give an affirmative answer for this question for quasi-transitive digraphs whenever $A^+(u)$ is monochromatic for each vertex $u$ ($A^+(u)$ is the set of arcs of $D$ that have $u$ as the initial vertex). A digraph $D$ is called quasi-transitive if whenever $(u, v) \in A(D)$ and $(v, w) \in A(D)$ then $(u, w) \in A(D)$ or $(w, u) \in A(D)$.
Quasi-transitive digraphs were introduced by Ghouila-Houri [12] and have been studied by several authors for example Bang-Jensen and Huang [1, 2].

We call a digraph \(D_n\)-quasitransitive digraph if it has the following property: If \(u, v \in V(D)\) and there is a directed \(uv\)-path of length \(n\) in \(D\), then \((u, v) \in \text{A}(D)\) or \((v, u) \in \text{A}(D)\). In this paper we study \(3\)-quasitransitive digraphs.

We denote by \(e_{T,4}\) the digraph such that \(V(e_{T,4}) = \{u, v, w, x\}\) and \(\text{A}(e_{T,4}) = \{(u, v), (v, w), (w, x), (u, x)\}\). If \(C\) is a walk we will denote by \(\ell(C)\) its length. If \(S \subseteq V(D)\) we denote by \(D[S]\) the subdigraph induced by \(S\). An arc \((u, v) \in \text{A}(D)\) is symmetrical if \((v, u) \in \text{A}(D)\). In this paper we prove that if \(D\) is an \(m\)-coloured \(3\)-quasitransitive digraph such that for every \(C_3\) (the directed cycle of length 3), \(C_4\) (the directed cycle of length 4) and \(T_4\) contained in \(D\) are quasi-monochromatic then \(D\) has a kernel by monochromatic paths.

We will need the following results.

**Theorem 1.1.** Let \(D\) be a digraph. \(D\) has a kernel by monochromatic paths if and only if \(\mathcal{C}(D)\) has a kernel.

**Theorem 1.2.** Every \(uv\)-monochromatic walk in a digraph contains a \(uv\)-monochromatic path.

**Theorem 1.3** (Berge-Duchet [4]). Let \(D\) be a digraph. If every directed cycle of \(D\) contains a symmetrical arc, then \(D\) is a kernel-perfect digraph.

## 2. 3-Quasitransitive Digraphs

The following lemma and remarks are about \(3\)-quasitransitive digraphs such that for every \(u \in V(D)\), \(A^+(u)\) is monochromatic, and they are useful to prove our main result.

Let \(T = (u_0, u_1, \ldots, u_n)\) be a path. Then we will denote the path \((u_i, u_{i+1}, \ldots, u_j)\) by \((u_i, T, u_j)\). Here, \([x]\) represents the largest integer less or equal than \(x\).

**Lemma 2.1.** Let \(D\) be an \(m\)-coloured \(3\)-quasitransitive digraph such that for every vertex \(u \in V(D)\), \(A^+(u)\) is monochromatic. If \(u\) and \(v\) are vertices of \(D\) and \(T = (u = u_0, u_1, \ldots, u_n = v)\) is a \(uv\)-monochromatic path of minimum length \(n \geq 3\), then \((u_i, u_{i+(2k+1)}) \in \text{A}(D)\) for each \(i \in \{3, \ldots, n\}\) and \(k \in \{1, \ldots, \lceil\frac{n-1}{2}\rceil\}\). In particular, if \(\ell(T)\) is odd, then \((v, u) \in \text{A}(D)\) and if \(\ell(T)\) is even, then \((v, u)\) may be absent in \(D\).
**Proof.** Observe that if $T$ is a $uv$-monochromatic path of minimum length and $\{u_i, u_j\} \subseteq V(T)$ with $i < j$ then the hypothesis that $A^+(z)$ is monochromatic for every $z \in V(D)$ implies that $(u_i, T, u_j)$ is also a $u_iu_j$-monochromatic path of minimum length.

We will proceed by induction on $\ell(T) = n$.

When $n = 3$ then $T = (u = u_0, u_1, u_2, u_3 = v)$. Since $D$ is a 3-quasitransitive digraph then $(u_0, u_3) \in A(D)$ or $(u_3, u_0) \in A(D)$. Since $T$ is of minimum length we have that $(u_3, u_0) \in A(D)$.

If $n = 4$ then $T = (u = u_0, u_1, u_2, u_3, u_4 = v)$. By the case $n = 3$ $(u_3, u_0) \in A(D)$ and $(u_4, u_1) \in A(D)$.

Suppose that if $\ell(T) = n \geq 4$ then $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{3, \ldots, n\}$ and $k \in \{1, \ldots, \left[\frac{n-1}{2}\right]\}$.

Let $T = (u = u_0, u_1, \ldots, u_n, u_{n+1} = v)$ be a $uv$-monochromatic path of minimum length. Let $T' = (u, T, u_n)$, then $T'$ is a $uu_n$-monochromatic path of minimum length. By the induction hypothesis we have that $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{3, \ldots, n\}$ and $k \in \{1, \ldots, \left[\frac{n-1}{2}\right]\}$. Also, let $T'' = (u_1, T, v)$, then $T''$ is a $u_1v$-monochromatic path of minimum length, the induction hypothesis implies that $(u_i, u_{i-(2k+1)}) \in A(D)$ for each $i \in \{4, \ldots, n + 1\}$ and $k \in \{1, \ldots, \left[\frac{n-1}{2}\right]\}$. So, it is sufficient to prove that $(u_{n+1}, u_0) \in A(D)$ whenever $n + 1$ is odd. Assume $n + 1$ is odd. We have that $(u_{n+1}, u_2)$, $(u_2, u_3), (u_3, u_0) \subseteq A(D)$, so $(u_{n+1}, u_2, u_3, u_0)$ is a path of length 3. Since $D$ is a 3-quasitransitive digraph then $(u_{n+1}, u_0) \in A(D)$ or $(u_0, u_{n+1}) \in A(D)$. Thus $(u_{n+1}, u_0) \in A(D)$.

**Remark 2.1.** Let $D$ be an $m$-coloured 3-quasitransitive digraph. If every $\bar{T}_4$ and $C_4$ contained in $D$ are at most 2-coloured then $D$ contains no 3-coloured path of length 3.

**Remark 2.2.** Let $D$ be an $m$-coloured digraph such that for every vertex $u \in V(D)$ $A^+(u)$ is monochromatic and $D$ contains no 3-coloured $C_3$. If $(u, u_1, u_2, v)$ is a 3-coloured walk then $u \neq u_1, u \neq u_2, u \neq v, u_1 \neq u_2$ and $u_2 \neq v$.

### 3. THE MAIN RESULT

**Definition 3.1.** Let $D$ be an $m$-coloured digraph. A $\gamma$-cycle in $D$ is a sequence of distinct vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$
1. There is a $u_iu_{i+1}$-monochromatic path and
2. There is no $u_{i+1}u_i$-monochromatic path.

The addition over the indices of the vertices of $\gamma$ are modulo $n+1$. And we say that the length of $\gamma$ is $n+1$.

**Theorem 3.2.** Let $D$ be an $m$-coloured $3$-quasitransitive digraph such that for every vertex $u$ of $D$, $A^+(u)$ is monochromatic. If every $C_3$, $C_4$ and $T_4$ contained in $D$ is quasi-monochromatic, then there are no $\gamma$-cycles in $D$.

**Proof.** We will proceed by contradiction. Suppose that $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ is a $\gamma$-cycle in $D$ of minimum length. The definition of $\gamma$-cycle implies that for every $i \in \{0, \ldots, n\}$ there exist a $u_iu_{i+1}$-monochromatic path in $D$ namely $T_i$, (we may assume that $T_i$ is of minimum length) and there is no $u_{i+1}u_i$-monochromatic path in $D$ (notation $\text{mod}(n+1)$). So we have $(u_{i+1}, u_i) \notin A(D)$ and by Remark 2.1 $\ell(T_i)$ is even or $\ell(T_i) = 1$ for every $i \in \{0, \ldots, n\}$. Now we have the following assertions.

1. $\ell(\gamma) \geq 3$. If $\ell(\gamma) = 2$ then $\gamma = (u_0, u_1, u_0)$ and this implies that there is a $u_1u_0$-monochromatic path, contradicting the definition of $\gamma$-cycle.
2. There is an index $i \in \{0, \ldots, n\}$ such that $T_i$ and $T_{i+1}$ have different colours. Otherwise $T_0 \cup T_1 \cup \cdots \cup T_n$ contains a $u_0u_n$-monochromatic path, a contradiction. Suppose w.l.o.g. that $T_0$ is coloured $1$ and $T_1$ is coloured $2$.
3. There is no $u_2u_0$-monochromatic path in $D$. Suppose by contradiction that $T = (u_2 = x_0, x_1, \ldots, x_t = u_0)$ is a $u_2u_0$-monochromatic path of minimum length in $D$. Then:

   3.1. $T$ is neither coloured $1$ nor $2$. This follows from the facts that $T_0$ is coloured $1$, $T_1$ is coloured $2$ and there is no $u_2u_1$-monochromatic path and $u_1u_0$-monochromatic path either.

   3.2. $\ell(T_0) \geq 4$ and $\ell(T_1) \geq 4$.

   If $\ell(T_0) = 1 = \ell(T_1)$, then $C = (u_0, u_1, u_2, x_1)$ is a $3$-coloured $u_0x_1$-walk of length $3$. So by Remark 2.2 we have that $C$ is a $3$-coloured $u_0x_1$-path of length $3$ contradicting the Remark 2.1.

   If $\ell(T_0) = 2$ and $\ell(T_1) = 1$, let $T_0 = (u_0, y, u_1)$, then $C = (y, u_1, u_2, x_1)$ is a $3$-coloured walk of length $3$. It follows from Remark 2.2 that $C$ is a $3$-coloured path of length $3$ contradicting the Remark 2.1.

   If $\ell(T_0) = 2 = \ell(T_1)$ then we may consider $T_0 = (u_0, y, u_1)$ and $T_1 = (u_1, z, u_2)$. We have that $z \notin V(T_0)$ so $T_0 \cup (u_1, z)$ (it will denote $(u_0, y, u_1, z)$)
is a path of length 3. Since D is a 3-quasitransitive digraph \((u_0, z) \in A(D)\) or \((z, u_0) \in A(D)\). If \((z, u_0) \in A(D)\) then it is coloured 2 \((A^+(z)\) is coloured 2) and this implies that \((u_0, y, u_1, z, u_0)\) is a \(C_4\) that is not quasi-monochromatic, a contradiction. So \((u_0, z) \in A(D)\) and it is coloured 1 \((A^+(u_0)\) is coloured 1). Let \(C = (u_0, z, u_2, x_1)\). Then \(C\) is a 3-coloured walk of length 3. By Lemma 2.2 we have that \(C\) is a 3-coloured path of length 3 contradicting the Remark 2.1.

If \(\ell(T_0) = 1\) and \(\ell(T_1) = 2\), let \(T_1 = (u_1, z, u_2)\) and consider \(C = (x_{t-1}, u_0, u_1, z)\). Then \(C\) is a 3-coloured walk. Remark 2.2 imply that \(C\) is a 3-coloured path of length 3, contradicting the Remark 2.1.

We conclude that \(\ell(T_0) \geq 4\) and \(\ell(T_1) \geq 4\). Let \(T_0 = (u_0 = y_0, y_1, \ldots, y_{\ell} = u_1)\) and \(T_1 = (u_1 = z_0, z_1, \ldots, z_k = u_2)\) with \(\ell \geq 4\) and \(k \geq 4\).

3.3. \(\ell(T) \geq 3\).

Suppose by contradiction that \(\ell(T) < 3\).

If \(\ell(T) = 1\) then \(C = (z_{k-1}, u_2, u_0, y_1)\) is a 3-coloured walk. Remark 2.2 implies that \(C\) is a 3-coloured path of length 3 but this is a contradiction with the Remark 2.1. If \(\ell(T) = 2\) then \(C_1 = (z_{k-1}, u_2) \cup T\) is a \(z_{k-1}u_0\)-path of length three. Since \(D\) is a 3-quasitransitive digraph then \((z_{k-1}, u_0) \in A(D)\) or \((u_0, z_{k-1}) \in A(D)\). If \((z_{k-1}, u_0) \in A(D)\) then it is coloured 2 and \(D[\{z_{k-1}, u_2, x_1, u_0\}]\) contains a \(T_4\) which is not quasi-monochromatic, a contradiction. If \((u_0, z_{k-1}) \in A(D)\) then it is coloured 1 and \((u_0, z_{k-1}, u_2, x_1)\) is a 3-coloured path of length three, a contradiction to Remark 2.1. We conclude that \(\ell(T) \geq 3\).

3.4. \((u_0, u_2) \notin A(D)\).

Proceeding by contradiction, suppose that \((u_0, u_2) \in A(D)\). Since \(T_0\) is coloured 1 then \((u_0, u_2)\) is coloured 1. By Lemma 2.1 (remember that \(\ell(T_i)\) is even) we have that \((u_2, z_1) \in A(D),\) so it is coloured 3. Then \((u_0, u_2, z_1, z_2)\) is a path of length 3 that is 3-coloured, but this is a contradiction with Remark 2.1.

3.5. \(\ell(T_0) \geq 4, \ell(T_1) \geq 4, \ell(T) \geq 4\) and \(\ell(T)\) is even.

(3.3) implies that \(\ell(T) \geq 3\). Since \(T\) is a \(u_2u_0\)-monochromatic path of minimum length \((u_2, u_0) \notin A(D)\) and by assertion (3.4) \((u_0, u_2) \notin A(D)\). So it follows from Lemma 2.1 that \(\ell(T)\) is even.

Now, Lemma 2.1 implies that \((u_0, x_1) \in A(D)\), and it is coloured 1. Then \((z_{k-1}, u_2, x_1, x_2)\) is a path of length 3. Since \(D\) is a 3-quasitransitive digraph \((z_{k-1}, x_2) \in A(D)\) or \((x_2, z_{k-1}) \in A(D)\). If \((z_{k-1}, x_2) \in A(D)\) it is coloured 2
and $D[\{z_{k-1}, u_2, x_1, x_2\}]$ contains a $T_4$ that is not quasi-monochromatic. So $(x_2, z_{k-1}) \in A(D)$ and it is coloured 3. Then $(u_0, x_1, x_2, z_{k-1})$ is a $u_0z_{k-1}$-path of length 3. Since $D$ is a 3-quasitransitive digraph then $(u_0, z_{k-1}) \in A(D)$ or $(z_{k-1}, u_0) \in A(D)$. If $(u_0, z_{k-1}) \in A(D)$ then it is coloured 1, so $D[\{u_0, x_1, x_2, z_{k-1}\}]$ contains a $T_4$ that is not quasi-monochromatic, a contradiction. We may assume that $(z_{k-1}, u_0) \in A(D)$, so it is coloured 2. Then $(u_0, x_1, x_2, z_{k-1})$ is a $C_4$ that is not quasi-monochromatic, a contradiction.

We conclude that there is no $u_2u_0$-monochromatic path in $D$.

4. $\ell(\gamma) \geq 4$. It follows from (1) and (3).

5. There is no $u_0u_2$-monochromatic path in $D$.

Assume that there exists a $u_0u_2$-monochromatic path in $D$. Then $\gamma_1 = (u_0, u_2, u_3, \ldots, u_n, u_0)$ would be a $\gamma$-cycle such that $\ell(\gamma_1) < \ell(\gamma)$ contradicting the choice of $\gamma$.

6. If $T_i$ and $T_{i+1}$ have different colours then there is no $u_{i+2}u_i$-monochromatic path and there is no $u_iu_{i+2}$-monochromatic path either.

This follows the same way as (3) and (5).

7. If $T_i$ and $T_{i+1}$ have different colours and $\ell(T_i) = 1$, for some $i \in \{0, \ldots, n\}$, then $\ell(T_{i+1}) = 1$.

W.l.o.g. suppose that $\ell(T_0) = 1$. Suppose by contradiction that $\ell(T_1) \geq 2$. If $\ell(T_1) = 2$, let $T_1 = (u_1, z, u_2)$. In this case $(u_0, u_1, z, u_2)$ is a $u_0u_2$-path of length 3. Since $D$ is a 3-quasitransitive digraph then $(u_0, u_2) \in A(D)$ or $(u_2, u_0) \in A(D)$, contradicting (5) or (3) respectively. We may assume that $\ell(T_1) > 2$. Let $T_1 = (u_1 = z_0, z_1, \ldots, z_k = u_2)$. Then $(u_0, u_1, z_1, z_2)$ is a $u_0z_2$-path of length 3. Since $D$ is a 3-quasitransitive digraph $(u_0, z_2) \in A(D)$ or $(z_2, u_0) \in A(D)$. If $(u_0, z_2) \in A(D)$ then it is coloured 1 and $D[\{u_0, u_1, z_1, z_2\}]$ contains a $T_4$ that is not quasi-monochromatic, a contradiction. If $(z_2, u_0) \in A(D)$ then it is coloured 2 and $(u_1, z_1, z_2, u_0)$ is a $u_1u_2$-monochromatic path contradicting that $\gamma$ is a $\gamma$-cycle. We conclude that $\ell(T_1) = 1$.

8. If $T_i$ and $T_{i+1}$ have different colours and $\ell(T_i) = 1$ then $T_{i+2}$ is coloured with the same colour of $T_i$.

W.l.o.g. suppose that $i = 0$, $T_0$ is coloured 1 and $T_1$ is coloured 2. $\ell(T_0) = 1$ and assertion (7) imply that $\ell(T_1) = 1$. Let $T_2 = (u_2, x_1, \ldots, x_t = u_3)$. Then $C = (u_0, u_1, u_2, x_1)$ is a $u_0x_1$-walk of length 3. The definition of $\gamma$-cycle implies that $x_1 \neq u_1$ and from assertion (3) we obtain that $x_1 \neq u_0$.

So $C$ is a $u_0x_1$-path of length 3. Since $D$ is a 3-quasitransitive digraph
\[(u_0, x_1) \in A(D)\] or \[(x_1, u_0) \in A(D)\]. From the hypothesis that every \(C_4\) and \(T_4\) in \(D\) is quasi-monochromatic, then the arc between \(x_1\) and \(u_0\) and \((u_2, x_1)\) have the same colour. If \((x_1, u_0) \in A(D)\) then \((u_2, x_1, u_0)\) is a \(w_3u_0\)-monochromatic path contradicting assertion (3). We may assume that \((u_0, x_1) \in A(D)\). Then \((u_0, x_1)\) and \((u_2, x_1)\) are coloured 1. Hence \(T_2\) is coloured 1.

To conclude the proof of the theorem we will analyze 5 possible cases.

Case 1. Suppose that \(\ell(T_0) = 1\).

Applying assertions (7) and (8) repeatedly we have that \(\ell(T_i) = 1\) for every \(i \in \{0, \ldots, n\}\). \(T_i\) is coloured 1 if \(i\) is even and \(T_i\) is coloured 2 if \(i\) is odd. This implies that \(\gamma = (u_0, u_1, \ldots, u_n, u_0)\) is a 2-coloured cycle in \(D\) such that the colours of its arcs are alternated, so \(n\) is odd.

We will prove by induction that \((u_0, u_i) \in A(D)\) for every odd \(i, i \in \{1, \ldots, n\}\). For \(i = 1\), \((u_0, u_1) \in A(D)\), since \(\gamma\) is a cycle. Suppose that \((u_0, u_{2k-1}) \in A(D)\) for \(i = 2k - 1\), where \(k \geq 1\). Now, we will prove that \((u_0, u_{2k+1}) \in A(D)\). We have that \(((u_0, u_1), (u_0, u_{2k-1}), (u_{2k}, u_{2k+1}))\) are coloured 1 and \((u_{2k-1}, u_{2k})\) is coloured 2. Let \(T = (u_0, u_{2k-1}, u_{2k}, u_{2k+1})\).

Then \(T\) is a \(u_0u_{2k+1}\)-path of length 3. Since \(D\) is a 3-quasitransitive digraph \((u_0, u_{2k+1}) \in A(D)\) or \((u_{2k+1}, u_0) \in A(D)\). Hence \(D[V(T)]\) contains a \(T_4\) or a \(C_4\). Since every \(T_4\) and \(C_4\) contained in \(D\) is quasi-monochromatic then the arc between \(u_0\) and \(u_{2k+1}\) is coloured 1. If \((u_{2k+1}, u_0) \in A(D)\) then \((u_{2k}, u_{2k+1}, u_0, u_{2k-1})\) is a \(u_{2k}u_{2k-1}\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle, so \((u_0, u_{2k+1}) \in A(D)\). We conclude that \((u_0, u_i) \in A(D)\) for every odd \(i \in \{1, \ldots, n\}\). Since \(n\) is odd \((u_0, u_n) \in A(D)\), but this contradicts the definition of \(\gamma\)-cycle.

Case 2. Suppose that \(\ell(T_0) = 2\) and \(\ell(T_1) = 1\).

Let \(T_0 = (u_0, x, u_1)\). Then \(C = T_0 \cup T_1\) is a walk of length 3. Assertion (5) implies that \(x \neq u_2\). \(C\) is a path of length 3. Since \(D\) is a 3-quasitransitive digraph \((u_0, u_2) \in A(D)\) or \((u_2, u_0) \in A(D)\). In any case we obtain a contradiction to assertion (5) or (3) respectively.

Case 3. \(\ell(T_0) = 2\) and \(\ell(T_1) \geq 2\).

Let \(T_0 = (u_0, x, u_1)\) and \(T_1 = (u_1, y_1, y_2, \ldots, y_t = u_2)\) where, \(t \geq 2\). Then \(C = T_0 \cup (u_1, y_1)\) is a path of length 3. Since \(D\) is a 3-quasitransitive digraph then \((u_0, y_1) \in A(D)\) or \((y_1, u_0) \in A(D)\). So, \(D[V(C)]\) contains a \(C_4\) or a \(T_4\), by the hypothesis it should be quasi-monochromatic.
Theorem 3.2. Proof: a symmetrical arc. Let $u_0$ be a path of length 3. Then $(u_0, y_1) \in A(D)$. Also $C' = (x, u_1, y_1, y_2)$ is a path of length 3, $(y_2, x) \in A(D)$ and it is coloured 2. Now, $D(\{u_0, y_1, y_2, x\})$ contains a $T_4$ that is not quasi-monochromatic, a contradiction.

Case 4. $\ell(T_0) \geq 4$ and $\ell(T_1) = 1$. Let $T_0 = (u_0, x_1, x_2, \ldots, x_{t-1}, x_t = u_1)$ with $t \geq 4$. We have $C = (x_{t-2}, x_{t-1}, x_t = u_1, u_2)$ is a path of length 3 (the definition of $\gamma$-cycle implies that there is no $u_2 u_1$-monochromatic path). Since $D$ is a 3-quasitransitive digraph $(x_{t-2}, u_2) \in A(D)$ or $(u_2, x_{t-2}) \in A(D)$. So, $D[V(C)]$ contains a $T_4$ or a $C_4$, by hypothesis it should be quasi-monochromatic. Then the arc between $x_{t-2}$ and $u_2$ is coloured 1. If $(u_2, x_{t-2}) \in A(D)$ then $(u_2, x_{t-2}, x_{t-1}, u_1)$ is a $u_2 u_1$-monochromatic path contradicting the definition of $\gamma$-cycle. So $(x_{t-2}, u_2) \in A(D)$. Hence $(u_0, x_1, \ldots, x_{t-2}, u_2)$ is a $u_0 u_2$-monochromatic path contradicting assertion (5).

Case 5. $\ell(T_0) \geq 4$ and $\ell(T_1) \geq 2$. Let $T_0 = (u_0, x_1, x_2, \ldots, x_{t-1}, x_t = u_1)$ and $T_1 = (u_1, y_1, y_2, \ldots, y_t = u_2)$. Then $C = (x_{t-2}, x_{t-1}, x_t = u_1, y_1)$ is an $x_{t-2} y_1$-path of length 3 (Remark 2.1). Since $D$ is a 3-quasitransitive digraph then $(x_{t-2}, y_1) \in A(D)$ or $(y_1, x_{t-2}) \in A(D)$. Then $D[V(C)]$ contains a $T_4$ or a $C_4$, by hypothesis it should be quasi-monochromatic. Then the arc between $x_{t-2}$ and $y_1$ is coloured 1. Hence $(y_1, x_{t-2}) \notin A(D)$, $A^+(y_1)$ is coloured 2, $(x_{t-2}, y_1) \in A(D)$ and it is coloured 1. Also, $C' = (x_{t-1}, u_1, y_1, y_2)$ is a $x_{t-1} y_2$-path of length 3. Then $(x_{t-1}, y_2) \in A(D)$ or $(y_2, x_{t-1}) \in A(D)$. Since every $T_4$ and $C_4$ is quasi-monochromatic, we have that $(y_2, x_{t-1}) \in A(D)$ and it is coloured 2. Then $D(\{x_{t-2}, y_1, y_2, x_{t-1}\})$ contains a $T_4$ that is not quasi-monochromatic, a contradiction.

We conclude that $D$ contains no $\gamma$-cycles.

Theorem 3.3. Let $D$ be an $m$-coloured 3-quasitransitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every $C_3$, $C_4$ and $T_4$ contained in $D$ is quasi-monochromatic, then $\mathcal{E}(D)$ is a kernel-perfect digraph.

Proof. By Theorem 1.3 we will prove that every cycle in $\mathcal{E}(D)$ contains a symmetrical arc. Let $C$ a cycle in $\mathcal{E}(D)$. Assume for a contradiction, that $C$ has no symmetrical arcs. Then $C$ is a $\gamma$-cycle in $D$ contradicting Theorem 3.2.
Corollary 3.4. Let $D$ be an $m$-coloured 3-quasitransitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every $C_3, C_4$ and $\tilde{T}_4$ contained in $D$ is quasi-monochromatic, then $D$ has a kernel by monochromatic paths.

Corollary 3.5. Let $T$ be an $m$-coloured tournament such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every $C_3, C_4$ and $e \in T_4$ contained in $D$ is quasi-monochromatic, then $T$ has a kernel by monochromatic paths.

Corollary 3.6. Let $D$ be an $m$-coloured bipartite tournament such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If every $C_4$ and $e \in T_4$ contained in $D$ is quasi-monochromatic, then $D$ has a kernel by monochromatic paths.

Remark 3.1. The condition that $D$ contains no $C_3$ 3-coloured in Theorem 3.3 cannot be dropped. Let $D_n$ be the digraph obtained from $D_{n-1}$ ($D_0$ is a 3-coloured $C_3$) by adding the vertex $v_n$ and arcs $(v_n,v)$ for every $v \in V(D_{n-1})$, all arcs coloured with some colour $j$. $D_n$ is an $m$-coloured 3-quasitransitive digraph with $A^+(z)$ monochromatic for every $z \in V(D_n)$, every $C_3$ and $\tilde{T}_4$ are quasi-monochromatic, $D_n$ contains a $\gamma$-cycle ($C_3$) and $D_n$ has no kernel by monochromatic paths.

Remark 3.2. The condition that every $C_4$ of $D$ is quasi-monochromatic in Theorem 3.2 is tight. Let $D$ be a 3-quasitransitive digraph 2-coloured with $V(D) = \{u,v,w,x\}$ and $A(D) = \{(u,v),(v,w),(w,x),(x,u)\}$ such that $(u,v),(w,x)$ are coloured 1 and $(v,w),(x,u)$ are coloured 2. In $D A^+(z)$ is monochromatic for every $z \in V(D)$, $D$ has a $\gamma$-cycle. Moreover, for each $n$ we give a digraph $D_n$, obtained from $D_0 = D$, that satisfies all the conditions of Theorem 3.2 except the one over $C_4$ and has a $\gamma$-cycle. $D_n$ is obtained from $D_{n-1}$ by adding the vertex $v_n$ and the arcs $(v_n,x)$ and $(v,v_n)$ with colours $j$ (for some $j$) and 2 respectively.

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References


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