

DIRECTED HYPERGRAPHS: A TOOL FOR RESEARCHING DIGRAPHS AND HYPERGRAPHS

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Abstract

In this paper we introduce the concept of *directed hypergraph*. It is a generalisation of the concept of digraph and is closely related with hypergraphs. The basic idea is to take a hypergraph, partition its edges non-trivially (when possible), and give a total order to such partitions. The elements of these partitions are called *levels*. In order to preserve the structure of the underlying hypergraph, we ask that only vertices which belong to exactly the same edges may be in the same level of any edge they belong to. Some little adjustments are needed to avoid directed walks within a single edge of the underlying hypergraph, and to deal with isolated vertices.

The concepts of independent set, absorbent set, and transversal set are inherited directly from digraphs.

As a consequence of our results on this topic, we have found both a class of kernel-perfect digraphs with odd cycles and a class of hypergraphs which have a strongly independent transversal set.

Keywords: hypergraph, strongly independent set, transversal set, kernel.

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1. PRELIMINARY RESULTS

Kernels in Digraphs

For general concepts about digraphs (resp. hypergraphs) we refer the reader to [1, 4, 6] (resp. [3]).

Transversal sets in hypergraphs have been thoroughly studied (cf. [3, 5]), as well as kernels in digraphs, which have applications in several branches of mathematics. For example, in mathematical logic a kernel may represent a minimal set of axioms for a theory, and in game theory it may represent a minimal (in amount of moves) winning strategy for a game between two players. For examples of results on kernels of digraphs, we refer the reader to [6, 7, 8, 9, 12, 13, 14].

Definition. Given a digraph $D = (V(D), F(D))$ and a set $S \subset V(D)$, we say that:

- (1) S is *independent* iff for every pair of vertices $\{x, y\} \subset S$, none of the ordered pairs (x, y) and (y, x) is in $F(D)$;
- (2) S is *absorbent* iff for every vertex $x \in V \setminus S$ there exists a vertex $y \in S$ such that $(x, y) \in F(D)$, and
- (3) S is a *kernel* of D iff it is independent and absorbent.

It should be noticed that if we order the subsets of $V(D)$ according to containment, any kernel of D is both a maximal independent set and a minimal absorbent set. However, it may be not so according to cardinality.

Definition. Given $x \in V(D)$, the set $N^+(x) = \{y \in V(D) \mid (x, y) \in F(D)\}$ is the *set of out-neighbours* of x , and the set $N^-(x) = \{y \in V(D) \mid (y, x) \in F(D)\}$ is the *set of in-neighbours* of x . Given $S \subset V(D)$, the set $N^+(S) = \{y \in V(D) \mid (x, y) \in F(D) \text{ for some } x \in S\}$ is the *set of out-neighbours* of S , and the set $N^-(S) = \{y \in V(D) \mid (y, x) \in F(D) \text{ for some } x \in S\}$ is the *set of in-neighbours* of S .

Definition [12]. Given a digraph D , a set $S \subset V(D)$ is a *semikernel* of D iff it is independent and satisfies the following statement: Given $x \in V \setminus S$, if there exists $s \in S$ such that $(s, x) \in F(D)$, then there exists $s' \in S$ such that $(x, s') \in F(D)$. In other words, a semikernel is an independent set of vertices which absorbs all of its out-neighbours.

Every kernel of a given digraph D is trivially a semikernel of D , but the converse is not true.

Definition. A digraph D is *kernel-perfect* iff every induced subdigraph of D has a kernel.

Theorem (Richardson) [13]. *Every digraph without directed cycles of odd length is kernel-perfect.*

This is one of the most important theorems regarding kernels on digraphs. There are several well known proofs of it. We mention other classical results without proof:

Proposition. *Let D be a digraph. Every closed directed walk of odd length in D has a directed cycle of odd length as a subsequence.*

Theorem (Neumann-Lara) [12]. *Let D be a digraph such that all of its induced subdigraphs has a non-empty semikernel. Then D is kernel-perfect.*

Hypergraphs

Definition. Given a finite set $V = \{x_1, \dots, x_n\}$, a *hypergraph* on V is a family $H = (E_1, \dots, E_m)$ of subsets of V such that the two following conditions are met:

- (1) $\forall i \in \{1, \dots, m\}, E_i \neq \emptyset$,
- (2) $\bigcup_{i=1}^m E_i = V$.

Notice that every isolated vertex must have a loop for the second condition to hold. Each E_i is called an *edge* of H .

Definition. A hypergraph $H = (E_1, \dots, E_m)$ is *simple* iff $\forall i \in \{1, \dots, m\}, E_i \subset E_j \Rightarrow i = j$.

Definition. Given a hypergraph $H = (E_1, \dots, E_m)$ on a set V , a set $S \subset V$ is *independent* iff $\nexists i \in \{1, \dots, m\}$ such that $E_i \subset S$. The set $S \subset V$ is *strongly independent* iff $\forall i \in \{1, \dots, m\}, |E_i \cap S| \leq 1$. In the case of graphs (hypergraphs in which every edge has two vertices) both concepts coincide.

Definition. Given a hypergraph $H = (E_1, \dots, E_m)$ on a set V , a set $S \subset V$ is *transversal* iff $\forall i \in \{1, \dots, m\}, |E_i \cap S| \geq 1$.

Definition. Given a hypergraph $H = (E_1, \dots, E_m)$ on a set V and an integer $k \geq 2$, a *cycle of length k* is a sequence $C = (x_0, E_0, x_1, \dots, x_{k-1}, E_{k-1}, x_k = x_0)$ such that:

- (1) $\forall \{i, j\} \subset \{0, \dots, k-1\}, E_i \neq E_j$,
- (2) $\forall \{i, j\} \subset \{0, \dots, k-1\}, x_i \neq x_j$,
- (3) $\forall i \in \{0, \dots, k-1\}, x_i, x_{i+1} \in E_i$.

Every hypergraph $H = (E_1, \dots, E_m)$ without cycles of odd length has an independent transversal set. In fact, every hypergraph H such that every cycle of odd length in it has an edge containing at least three vertices of the cycle, has an independent transversal set. This follows directly from [3], Chapter 5, Theorem 7.

The original motivation for this work was the search for families of hypergraphs with strongly independent transversal sets. In Figure 1 we have some examples of hypergraphs with no strongly independent transversal sets. Examples a) and b) may suggest to look for the desired families among simple hypergraphs without cycles of odd length, although example c) shows that not all such hypergraphs have a set with the required properties. Here we present one of such families.

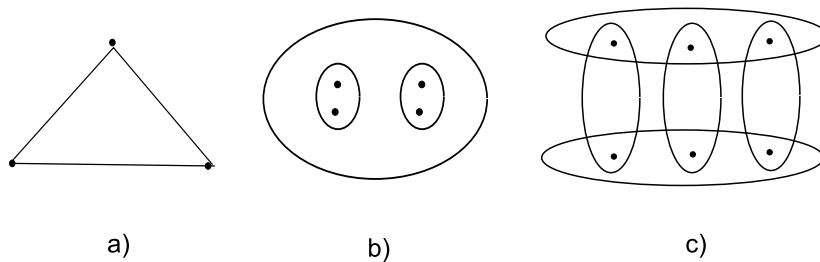


Figure 1. Hypergraphs with no strongly independent transversal sets.

2. DIRECTED HYPERGRAPHS

Definitions

As has already been mentioned, we could intuitively consider a directed hypergraph as a hypergraph with a non-trivial order relation defined on its (non-loop) edges. Formally speaking, we may start defining a directed hypergraph and then consider (or not) its underlying hypergraph, or we

may take a given hypergraph, remove the loops and assign a "direction" to the remaining edges. The first approach is preferred for studying directed hypergraphs as a mathematical object or when using them as a tool for researching digraphs, and the second works better when looking for results on hypergraphs.

To begin with, we define a concept regarding hypergraphs:

Definition. Given a hypergraph $H = (E_1, \dots, E_m)$, we define a partition $P = \{P_1, \dots, P_k\}$ of $V(H)$ in the following way: $\{x, y\} \subset V(H)$ is contained in an element of P iff x and y belong to exactly the same edges of H . We call this the *natural partition* of $V(H)$, and the partition defined over each edge $E \in H$ as $\{P_i \cap E \mid P_i \in P\}$ is the *natural partition* of E .

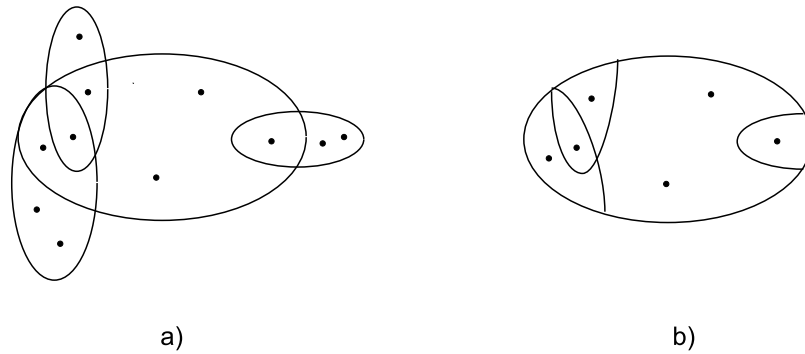


Figure 2. a) Natural partition of a hypergraph.
b) Natural partition of an edge.

Definition. Given a finite set V , a *directed hypergraph* $D = (X, F)$ on V is a subset $X \subset V$ and a set of triples $F = \{A_1, \dots, A_m\}$ such that for every $i \in \{1, \dots, m\}$, $A_i = (E_i, P_i, \leq_i)$, where $E_i \subseteq V$, $|E_i| > 1$, $E_i \cap X = \emptyset$; P_i is a non-trivial partition on E_i , and \leq_i is a total order on P_i . The following conditions must also be met:

$$(1) X \cup \bigcup_{i=1}^m E_i = V.$$

(2) For every $i \in \{1, \dots, m\}$, if $\{x, y\} \subset E_i$ and there exists $j \in \{1, \dots, m\}$ such that $x \in E_j$, $y \notin E_j$, then x and y belong to different elements of P_i . That is, if we consider the underlying hypergraph $H = (E_1, \dots, E_m)$, the partition of each edge is a refinement of its natural partition. If this

condition is not met we may have the following situation, rather unpleasant: Two vertices x, y and three arcs A, B, C such that $x \in V(A)$ but $x \notin V(B)$, $y \in V(B)$ but $y \notin V(A)$, and both x and y are in $V(C)$, but nevertheless there is no directed walk from x to y , nor from y to x (see Figure 3.b). As an additional condition, we could ask the intersection between any two levels of any arcs to be empty, so that the set of all levels in D is a refinement of the natural partition of H . For researching hypergraphs, we restrict ourselves to an even narrower class of directed hypergraphs; when concerned about digraphs, it is probably better to let levels of different arcs overlap (as long as they remain within every arc they belong to).

(3) Given $E \subset V$, if there are arcs A_i and A_j such that $E_i = E = E_j$, then $P_i = P_j$ and either $\leq_i = \leq_j$, or $\forall \{x, y\} \subset E, x \leq_i y \Leftrightarrow y \leq_j x$. This is done to avoid the existence of directed walks of length larger than 2 "supported" by a single edge of the underlying hypergraph. The condition could be changed or omitted altogether when not looking for results on hypergraphs (see Figure 3.c).

When possible, we represent each arc as an "earthworm", whose segments are its levels, following the order given. We draw an arrow from the first level to the second, except in the case of symmetrical arcs, when we use a two headed arrow for both arcs.

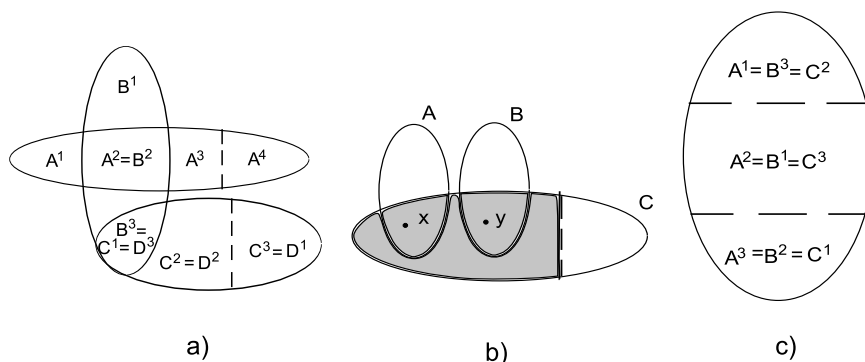


Figure 3. a) Directed hypergraph: $A^2 = B^2, B^3 = C^1 = D^3, C^2 = D^2, C^3 = D^1$.

b) If the shaded area is only one level of A , there is neither a directed walk from x to y nor from y to x .

c) Condition 3 omitted: (x, A, y, B, z, C, x) is a directed cycle of length 3.

We say that $x \in V$ is a *vertex* of D , and that A_i is an *arc* of D . We call X the *set of isolated vertices* of D , and $x \in X$ an *isolated vertex* of D .

We write $P_i = \{A_i^1, \dots, A_i^{r(i)}\}$, where $r(i) = |P_i|$ and $A_i^1 \leq_i A_i^2 \leq_i \dots \leq_i A_i^{r(i)}$. We say that A_i^j is the j -level of A_i . If $x \in A_i^1$, x is a *minimum* of A_i ; if $x \in A_i^{r(i)}$, x is a *maximum* of A_i . Since trivial partitions are not allowed, there are no arcs with only one level. If $x \in A_i^j$, $y \in A_i^k$, and $j < k$, we say that A_i is an xy -arc and that y *absorbs* x . If $x \in S_1 \subset V$, $y \in S_2 \subset V$, then A_i is an S_1y -arc, an xS_2 -arc, and an S_1S_2 -arc; we say also that S_2 *absorbs* x . If $S \subset V$ absorbs all vertices in $V \setminus S$, S is an *absorbent set* in D .

Definition. Given a directed hypergraph $D = (X, F)$, where $F = \{A_1, \dots, A_m\}$ and $\forall i \in \{1, \dots, m\}$, $A_i = (E_i, P_i, \leq_i)$, the hypergraph $H = (E_1, \dots, E_m) \cup X'$ is the *underlying hypergraph* of D , where $X' = \{\text{loop on } x \mid x \in X\}$.

Definition. Given a hypergraph $H = (E_1, \dots, E_m) \cup \{X'\}$, where $\forall i \in \{1, \dots, m\}$, $|E_i| > 1$ and $X' = \{\text{edges whose cardinality is } 1\}$, we say that $D = (X, F)$ is a directed hypergraph *generated* by H iff $X = \{x \in V(H) \mid x \text{ belongs only to an edge of cardinality } 1\}$ and $F = \{A_1, \dots, A_m, A'_1, \dots, A'_m\}$, where $V(A_i) = V(A'_i) = E_i$, $P_i = P'_i = \text{natural partition of } E_i$, \leq_i is any total order on P_i , and \leq'_i is the "inverse order" of \leq_i (that is, if $M, N \in P_i = P'_i$ and $M \leq_i N$, then $N \leq'_i M$). If D is a directed hypergraph generated by H , we say that H *generates* D .

Notice that given a hypergraph H , there may be several directed hypergraphs generated by H (because there may be several non-equivalent total orders in the natural partition), or there may be not even one. For example, if H is not simple, is not connected, or has but one edge, the natural partition of its edges may not generate a directed hypergraph, for arcs with only one level are not allowed. In most cases, the directed hypergraphs generated by a given hypergraph H are only a small subset of the directed hypergraphs whose underlying hypergraph is H . Since they inherit its structure more faithfully than any other, they are the best choice for studying properties of hypergraphs.

Definition. A *directed walk of length* n is a sequence $C = (x_0, A_0, x_1, \dots, x_{n-1}, A_{n-1}, x_n)$ such that $A_{n-1} \neq A_0$ and for every $i \in \{0, \dots, n-2\}$ we have that $A_i \neq A_{i+1}$, and such that for every $i \in \{0, \dots, n-1\}$ the arc A_i is an $x_i x_{i+1}$ -arc. The *length* of C is $l(C) = n$. Notice that the minimum length of any directed walk is 1, for no vertex may belong to different levels of a given arc. A *closed directed walk of length* n is a directed walk of length n such that $x_0 = x_n$. A *directed cycle* $C = (x_0, A_0, x_1, \dots, x_{n-1}, A_{n-1},$

$x_n = x_0$) is a closed directed walk such that for every $\{i, j\} \subset \{0, \dots, n - 1\}$ we have that $x_i \neq x_j$ and $A_i \neq A_j$.

Definition. Let D be a directed hypergraph. A set $S \subset V(D)$ is *independent* iff there are no SS -arcs (that is, there are no xy -arcs such that $\{x, y\} \subset S$). Equivalently, $S \subset V(D)$ is independent iff for every arc A_i in D we have that $S \cap A_i^j \neq \emptyset \Rightarrow S \cap A_i \subset A_i^j$.

Definition. Given a directed hypergraph $D = (X, F)$ on a set V and $S \subset V$, we may consider the triples $B_i = (E_i \cap S, P'_i, \leq'_i)$, where

$$P'_i = \{A_i^j \cap S \mid 1 \leq j \leq r(i), A_i^j \cap S \neq \emptyset\}$$

and \leq'_i is the order induced by \leq_i in P'_i . The directed subhypergraph of D induced by S is $D[S] = (S', F')$, where $F' = \{B_i \mid |P'_i| \geq 2\}$ and $S' = S \setminus \{x \in E_i \mid B_i \in F'\}$. When considering the induced order of an arc, we start from the minimum and proceed increasingly. This is done to avoid ambiguousness, as shown in Figure 4. We say that B_i is the arc *induced by* A_i in $D[S]$, and that A_i *induces* B_i in $D[S]$. Notice that if $T \subset V$ is independent in D and S is any subset of V , then $T \cap S$ is independent in $D[S]$. It is also important to observe that if H is a hypergraph, $U \subset V(H)$, and D is a directed hypergraph generated by H , then $D[U]$ may not be the directed hypergraph generated by $H[U]$, which does not necessarily exist (see Figure 5).

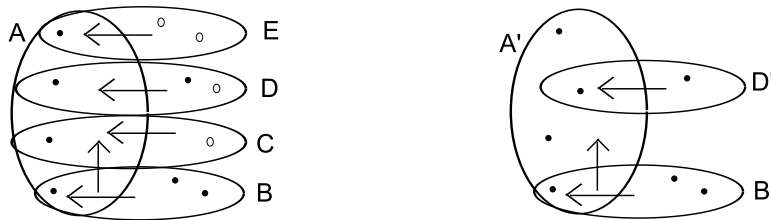


Figure 4. There could have been ambiguousness on defining the order of A' .

Definition. Given a directed hypergraph $D = (X, F)$, where $F = \{A_1, \dots, A_m\}$, an arc A_i in D is *symmetrical* iff there exists an arc A_j in D such that $V(A_i) = V(A_j)$, $P_i = P_j$, and for every two levels $\{N_1, N_2\} \subset V(A_i)$ we

have that $N_1 \leq_i N_2 \Leftrightarrow N_2 \leq_j N_1$. If an arc A is symmetrical, we denote by A' the arc with the same underlying set of vertices (which also has the same partition and the "inverse" order). So (3) on the definition of directed hypergraph states that if there are two arcs with the same underlying set of vertices, they are either equal or symmetrical.

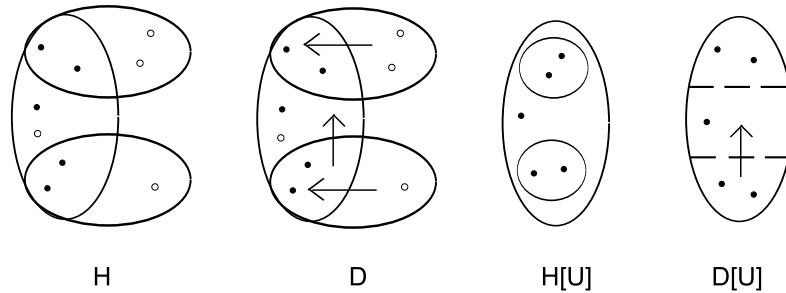


Figure 5. Let U be the set of black vertices. $H[U]$ generates no directed hypergraph.

Definition. A directed hypergraph D is *symmetrical* iff every arc in D is symmetrical.

Notice that if a given directed hypergraph D is symmetrical, then every directed subhypergraph of D is also symmetrical. Observe also that a directed hypergraph generated by any given hypergraph is always symmetrical.

Transversal Kernels

In this section we focus on results regarding symmetrical directed hypergraphs. Since the directed hypergraph generated by any given hypergraph is always symmetrical (when it exists), we may restrict ourselves to this kind of directed hypergraphs when looking for applications to hypergraphs.

Definition. Let D be a directed hypergraph on V . A set $K \subset V$ is a *kernel* iff it is independent and absorbent.

Definition. Let D be a directed hypergraph on V . A set $S \subset V$ is a *semikernel* iff it is independent and for every $y \in V \setminus S$ such that there exists an Sy -arc, there exists also a yS -arc.

Definition. Let D be a directed hypergraph. $S \subset V(D)$ is a *semitransversal* of D iff S is a semikernel and the following holds: for every $y \in V \setminus S$ such

that there exists an Sy -arc, and for every arc A such that y is a minimum of A , we have that $V(A) \cap S \neq \emptyset$.

Notice that if D is a symmetrical directed hypergraph, $S \subset V(D)$ is a semitransversal of D iff for every $y \in V \setminus S$ such that there exists an Sy -arc, and for every arc A such that y is a minimum of A , we have that $V(A) \cap S \neq \emptyset$.

Definition. Let D be a directed hypergraph. $T \subset V$ is a *transversal kernel* (k -*transversal*) of D iff T is independent, absorbent and transversal ($T \cap V(A) \neq \emptyset$ for every arc A in D). Observe that, according to containment, a transversal kernel is a maximal independent set, as well as a minimal transversal set and a minimal absorbent set, although it is not necessarily so according to cardinality.

Theorem 1. *Let D be a symmetrical directed hypergraph. If every induced directed subhypergraph of D has a non-empty independent semitransversal, then D has a k -transversal.*

Proof. We will proceed by induction on $|V|$. The theorem holds clearly for every directed hypergraph with at most two vertices. Suppose that the result is true for every symmetrical directed hypergraph with less than n vertices. Let D be a symmetrical directed hypergraph such that $|V| = n$.

Let S be a non-empty independent semitransversal of D , $S^- = \{x \in V \setminus S \mid \text{there exists an } xS\text{-arc in } D\}$, and $S_0 = V \setminus (S \cup S^-)$. We consider separately the two possible cases:

Case 1. $S_0 = \emptyset$.

We will prove that, in this case, S is a k -transversal of D .

(i) S is an independent set, for S is a non-empty independent semitransversal.

(ii) S is absorbent: $V \setminus S = S^-$, so that $x \in V \setminus S \Rightarrow$ there exists an xS -arc.

(iii) S is a transversal set: Let $A \in D$ be any arc. Take a vertex $x \in V(A)$ such that x is a minimum of A . Suppose $V(A) \cap S = \emptyset$. Then $V(A) \subset S^-$, which implies the existence of an xS -arc B . As B is a symmetrical arc, there exists an Sx -arc in D . Then $V(A) \cap S \neq \emptyset$, for S is a semitransversal. Therefore, S is a transversal set.

Case 2. $S_0 \neq \emptyset$.

Let $D_0 = D[S_0]$ be the directed subhypergraph of D induced by S_0 . As $S \neq \emptyset$ and $S \cap S_0 = \emptyset$, we have that $|S_0| < n$. Then D_0 has a k -transversal,

from the inductive hypothesis. Let T_0 be a k -transversal of D_0 . We will prove that $T = S \cup T_0$ is a k -transversal of D .

(i) T is independent: Suppose there is an xy -arc A such that $\{x, y\} \subset T$. The set $\{x, y\} \not\subset S$, for S is independent in D . Likewise, $\{x, y\} \not\subset T_0$, for T_0 is independent in D_0 , and D_0 is an induced directed subhypergraph of D . If $x \in T_0$ and $y \in S$, then A is an xS -arc, which implies that $x \in S^-$; this is a contradiction, for $x \in T_0 \subset S_0$ and $S^- \cap S_0 = \emptyset$. If $x \in S$ and $y \in T_0$, we have that A is an Sy -arc, so that there exists a yS -arc B , because S is a semitransversal; then $y \in S^-$, which is a contradiction, for $y \in T_0 \subset S_0$ and $S^- \cap S_0 = \emptyset$. Therefore, $T \subset V$ is an independent set in D .

(ii) T is an absorbent set in D : Remember that $V = S \cup S^- \cup S_0$. Take $x \in V \setminus T$. If $x \in S^-$, then x is absorbed by $S \subset T$, from the definition of S^- . If $x \in S_0$, since T_0 is a k -transversal of $D_0 = D[S_0]$ and $x \notin T_0$, we have that x is absorbed by $T_0 \subset T$.

(iii) T is a transversal set in D :

Claim. $V(A) \cap T \neq \emptyset$ for every arc $A \in D$ such that there exists a vertex $x \in V(A) \cap S^-$ that is a minimum of A .

Proof. Let A be an arc in D , $x \in V(A) \cap S^-$ such that x is a minimum of A . Since $x \in S^-$, there exists an xS -arc B . Since B is symmetrical and there exists $y \in S$ such that B is an xy -arc, we have that there also exists a yx -arc B' (that is, an Sx -arc). Then, as S is a semitransversal of D and x is a minimum of A , it follows that $\emptyset \neq (V(A) \cap S) \subset (V(A) \cap T)$. So our claim is proven.

Let A be any arc of D and consider the set $M_A = \{x \in V(A) \mid x \text{ is a minimum of } A\}$. We have just seen that $M_A \cap S^- \neq \emptyset$ implies $\emptyset \neq (V(A) \cap S) \subset (V(A) \cap T)$. We may then assume $M_A \subset S_0$. Take $x \in M_A$; if $x \in T_0$, then $x \in (V(A) \cap T_0) \subset (V(A) \cap T)$. If $x \notin T_0$, take $y \in V(A)$ such that y is a maximum of A . If $y \in S_0$, there exists an arc $A_0 \in D_0$ such that A_0 is induced by A in D_0 . Since T_0 is a k -transversal of D_0 , $V(A_0) \cap T_0 \neq \emptyset$, and then $V(A) \cap T \neq \emptyset$. If $y \notin S_0$, then $y \in S^-$, for A is an xy -arc and there are no S_0S -arcs. Since A is symmetrical, there exists a yx -arc A' such that $V(A) = V(A')$ and y is a minimum of A' ; from our claim, $V(A') \cap T \neq \emptyset$, so that $V(A) \cap T \neq \emptyset$. Therefore, T is a k -transversal of D , and the proof of Theorem 1 is complete.

Notice that we needed D to be symmetrical only to prove transversality, so that by omitting (iii) in both cases we have a proof of the following:

Theorem 1'. *Let D be a directed hypergraph. If every induced directed subhypergraph of D has a non-empty semikernel, then D has a kernel.*

Definition. A directed hypergraph D is *kernel-perfect* iff every induced directed subhypergraph of D has a kernel.

Theorem 1''. *Let D be a directed hypergraph. If every induced directed subhypergraph of D has a non-empty semikernel, then D is kernel-perfect.*

Proof. The result follows directly from Theorem 1', because every induced directed subhypergraph of an induced directed subhypergraph of D is itself an induced directed subhypergraph of D .

Theorem 1'' is a generalisation of Neumann-Lara's, for every digraph is a directed hypergraph.

Definition. A directed hypergraph D is *bipartite* iff there is a partition of $V(D)$ in two non-empty independent sets.

Theorem 2. Let D be a symmetrical directed hypergraph. If D is bipartite then D has a k -transversal.

Proof. Since every induced directed subhypergraph of a bipartite directed hypergraph is itself bipartite, and considering Theorem 1, it is enough to show that every bipartite directed hypergraph has a non-empty semitransversal.

Let D be a bipartite directed hypergraph and let $\{V_1, V_2\}$ be a partition of $V(D)$ in two independent sets. Notice that both V_1 and V_2 are independent and transversal sets. If there is a vertex $x \in V_2$ that is a maximum of every arc A of D such that $x \in V(A)$, then $\{x\}$ is a semitransversal of D . If the last statement is not true, we have that for every $x \in V_2$ there is an arc A_x such that x is not a maximum of A_x . Since V_2 is independent, A_x is an xV_1 -arc for every x in V_2 . Then V_1 is a k -transversal of D .

By considering Theorem 1'' instead of Theorem 1, we obtain:

Theorem 2'. *Every bipartite directed hypergraph is kernel-perfect.*

We will now prove that every symmetrical directed hypergraph that has no closed directed walks of odd length has a k -transversal. To achieve this, some preliminary results are needed.

Lemma 1. *Let D be a directed hypergraph. Every closed directed walk of odd length $C = (x_0, A_0, x_1, A_1, x_2, A_2, \dots, x_{2k}, A_{2k}, x_{2k+1} = x_0)$ in D , such that for every $\{i, j\} \subset \{0, \dots, 2k\}$ we have $A_i \neq A_j$, has a directed cycle of odd length as a subsequence.*

Proof. By induction on the length of the closed directed walk.

Let $C = (x_0, A_0, x_1, A_1, x_2, A_2, x_0)$ be a closed directed walk of length 3. From the definition of directed walk we have that $x_0 \neq x_1$, $x_1 \neq x_2$, $x_2 \neq x_0$, $A_0 \neq A_1$, $A_1 \neq A_2$, $A_2 \neq A_0$. Then C is a directed cycle of length 3.

Now suppose that every closed directed walk of length at most $2k - 1$ in which all arcs are different has a directed cycle of odd length as a subsequence, and let $C = (x_0, A_0, x_1, A_1, x_2, A_2, \dots, x_{2k}, A_{2k}, x_0)$ be a closed directed walk of length $2k + 1$ such that for every $\{i, j\} \subset \{0, \dots, 2k\}$ we have $A_i \neq A_j$. If $x_i \neq x_j$ for every $\{i, j\} \subset \{1, \dots, 2k\}$, then C is a directed cycle. If there are $\{i, j\} \subset \{1, \dots, 2k\}$ such that $i < j$ and $x_i = x_j$, then we have two closed directed walks: $C_1 = (x_0, A_0, \dots, x_i = x_j, A_j, x_{j+1}, \dots, x_{2k}, A_{2k}, x_0)$ and $C_2 = (x_j = x_i, A_i, x_{i+1}, \dots, x_{j-1}, A_{j-1}, x_j)$. Notice that $F(C_1) \cap F(C_2) = \emptyset$, $V(C_1) \cap V(C_2) = \{x_j\}$, so that C_1, C_2 are directed walks (since none "uses" any arc but once). Moreover, we have that $l(C_1) \neq 0$, $l(C_2) \neq 0$, and $l(C_1) + l(C_2) = l(C)$, which implies that $l(C_1)$ is odd and $l(C_2)$ is even, or the other way round. In any case, there is a closed directed walk of odd length at most $2k - 1$ which is a subsequence of C . From the inductive hypothesis, such a directed walk has a directed cycle of odd length as a subsequence, and that cycle is also a subsequence of C .

It is important to notice that if there exist $\{p, q\} \subset \{1, \dots, 2k\}$ such that $p < q$, $A_p = A_q$, there may be closed directed walks of odd length with no directed cycle of odd length as a subsequence. In fact, there are directed hypergraphs with closed directed walks of odd length and without directed cycles of odd length at all (Figure 6). However, we can guarantee the existence of a directed cycle of odd length under certain conditions:

Lemma 2. *Let D be a directed hypergraph such that for every closed directed walk of odd length $C = (x_0, A_0, x_1, A_1, x_2, A_2, \dots, x_{2k}, A_{2k}, x_0)$, and for every arc A such that for $\{p, q\} \subset \{1, \dots, 2k\}$ with $p < q$ and $A_p = A = A_q$, we have that x_p belongs to the same level of A_p as x_q , or that x_{p+1} belongs to the same level of A_p as x_{q+1} . Then the following statement holds: If D has a closed directed walk of odd length, D has a directed cycle of odd length.*

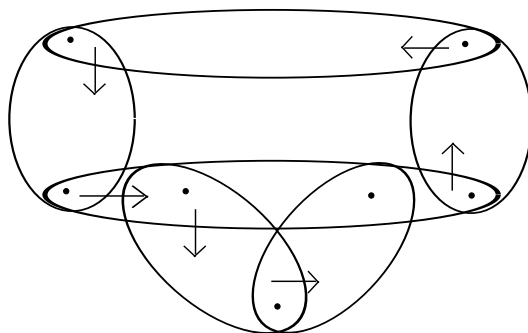


Figure 6. Hypergraph with closed directed walks of odd length and without directed cycles of odd length.

Proof. Let D be a directed hypergraph and let $C = (x_0, A_0, x_1, A_1, x_2, A_2, \dots, x_{2k}, A_{2k}, x_0)$ be a closed directed walk of length $2k + 1$ such that for $\{p, q\} \subset \{1, \dots, 2k\}$ we have $A_p = A = A_q$. If x_p belongs to the same level of A_p as x_q , or x_{p+1} belongs to the same level of A_p as x_{q+1} , we have that A_p is both a $x_p x_{q+1}$ -arc and an $x_q x_{p+1}$ -arc. Then there are two closed directed walks of length at most $2k - 1$, $C_1 = (x_0, A_0, \dots, x_p, A_p, x_{q+1}, A_{q+1}, \dots, x_{2k}, A_{2k}, x_0)$ and $C_2 = (x_q, A_p, x_{p+1}, \dots, x_{q-1}, A_{q-1}, x_q)$, such that $l(C_1) \neq 0$, $l(C_2) \neq 0$, and $l(C_1) + l(C_2) = l(C)$. Observe that both C_1 and C_2 use the arc $A_p = A_q$ once less than C . By repeating this procedure, we will eventually find a closed directed walk of odd length in which all arcs are different. According to Lemma 1, such a walk has a directed cycle of odd length as a subsequence. It should be noticed that the cycle so found is not necessarily a subsequence of C .

Observe that if $A_p = A_q$ and x_p belongs to the same level of A_p as x_{q+1} , then x_q belongs to a different level of A_p than x_{p+1} , for A_p is both an $x_p x_{p+1}$ -arc and an $x_q x_{q+1}$ -arc. That is, there are at least three different levels in A_p : x_q belongs to one of them, x_{q+1} and x_p belong to another, and x_{p+1} belongs to a third one. Analogously, if x_q belongs to the same level of A_p as x_{p+1} , there is a level to which x_p belongs, a second one which contains x_{p+1} and x_q , and yet another with x_{q+1} . Therefore, if the arc $A_p = A_q$ has only two levels, it must satisfy the conditions asked in Lemma 2. The following result is then proven:

Lemma 3. *Let D be a directed hypergraph which has a closed directed walk of odd length but has no directed cycles of odd length, then at least one arc in D has more than two levels.*

An additional result, which will not be used later, is the following:

Lemma 4. *Let D be a symmetrical directed hypergraph such that in every closed directed walk of odd length $C = (x_0, A_0, x_1, A_1, x_2, A_2, \dots, x_{2k}, A_{2k}, x_0)$, and in every arc A such that for $\{p, q\} \subset \{1, \dots, 2k\}$, with $p < q$ and $A_p = A = A_q$, we have that x_p belongs to a different level of A_p than x_{q+1} and x_q belongs to a different level of A_p than x_{p+1} . Then D has a directed cycle of odd length.*

Proof. In this case we have also two closed directed walks of length at most $2k - 1$: $C_1 = (x_0, A_0, \dots, x_p, [A_p], x_{q+1}, A_{q+1}, \dots, x_{2k}, A_{2k}, x_0)$ and $C_2 = (x_q, [A_p], x_{p+1}, \dots, x_{q-1}, A_{q-1}, x_q)$, such that $l(C_1) \neq 0$, $l(C_2) \neq 0$ and $l(C_1) + l(C_2) = l(C)$. The symbol $[A_p]$ means A_p or A'_p , whichever applies. Since C_1 and C_2 are subsequences of C (except the possible change from A_p to A'_p), we have two closed directed walks, one of which is of odd length, and both using arcs whose underlying set of vertices is $V(A_p)$ (that is, whether A_p or A'_p) once less than C . By means of a reasoning similar to the one used in the proof of Lemma 2, we conclude that D has a directed cycle of odd length (which is not necessarily a subsequence of C).

Lemma 5. *Let D be a directed hypergraph. If D is bipartite, then every arc in D has exactly two levels.*

Proof. Let D be a directed hypergraph, and let A be an arc in D with more than two levels. Let N_1, N_2, N_3 be different levels of A , and take $\{x_1, x_2, x_3\} \subset V(A)$ such that for every $i \in \{1, 2, 3\}$, $x_i \in N_i$. Suppose D is bipartite, and let $\{V_0, V_1\}$ be a partition of $V(D)$ in two independent sets. Without loss of generality, we may assume $x_1 \in V_1$. Since x_1 and x_2 belong to different levels of A , and V_1 is an independent set, it follows that x_2 does not belong to V_1 ; then it belongs to V_0 . In a similar way, since x_1 and x_3 belong to different levels of A and V_1 is an independent set, x_3 does not belong to V_1 ; since x_2 and x_3 belong to different levels of A and V_0 is an independent set, x_3 does not belong to V_0 . Therefore D is not bipartite. This is a contradiction, so Lemma 5 is proven.

Definition. A directed hypergraph D is *strong* iff for every $\{x, y\} \subset V(D)$ there are both an xy -directed walk and a yx -directed walk.

Definition. A directed hypergraph D is *connected* iff for every $\{x, y\} \subset V(D)$ there is either an xy -directed walk or a yx -directed walk.

Definition. Let D be a directed hypergraph, and let D' be a strong directed subhypergraph of D which is not contained in any other strong directed subhypergraph of D . Then D' is a *strong component* of D .

Definition. Let D be a directed hypergraph, and let D' be a connected directed subhypergraph of D which is not contained in any other connected directed subhypergraph of D . Then D' is a *connected component* of D .

Remark. A symmetrical directed hypergraph is connected iff it is strong. Moreover, given any directed hypergraph D , D has a k -transversal T iff all of its connected components have a k -transversal; in that case, T is the union of the k -transversals of every connected component of D . Therefore, a symmetrical directed hypergraph has a k -transversal iff all of its strong components have one.

Theorem 3. *A strong directed hypergraph D such that $|V(D)| \geq 2$ is bipartite iff it has no closed directed walks of odd length.*

Proof. To begin with, we will prove that a bipartite directed hypergraph has no directed cycles of odd length: Let D be a bipartite directed hypergraph, let $\{V_0, V_1\}$ be a partition of $V(D)$ in two independent sets, and let $C = (x_0, A_0, \dots, x_{k-1}, A_{k-1}, x_k = x_0)$ be a directed cycle of length k . We may assume $x_0 \in V_0$. Since C is a cycle, and both V_0 and V_1 are independent sets, it follows that $x_1 \in V_1$ and, in general, for every $i \in \{1, \dots, k-1\}$ and $j \in \{0, 1\}$, we have that $x_i \in V_j$ iff $i \equiv j \pmod{2}$. Therefore, $k \equiv 0 \pmod{2}$ for $x_k = x_0 \in V_0$.

Let D be a directed hypergraph, and let C be a closed directed walk of odd length. As we have seen, if D has a cycle of odd length, it is not bipartite. If D has no cycle of odd length, Lemma 3 states that there is at least an arc in $F(D)$ with more than two levels, so D is not bipartite, according to Lemma 5.

Conversely, let D be a strong directed hypergraph without closed directed walks of odd length and such that $|V(D)| \geq 2$. Take $x \in V(D)$ and define $V_0 = \{x\} \cup \{y \in V(D) \mid \text{there exists an } xy\text{-directed walk of even length in } D\}$, $V_1 = \{y \in V(D) \mid \text{there exists an } xy\text{-directed walk of odd length in } D\}$. Then $\{V_0, V_1\}$ is a partition of $V(D)$ in two independent sets:

- (i) $V(D) = V_0 \cup V_1$, for D is strong.

(ii) $V_0 \neq \emptyset \neq V_1$: Since $x \in V_0$, we have that $V_0 \neq \emptyset$. There exists at least a vertex $y \in V(D)$ such that $y \neq x$. If $y \notin V_1$ then $y \in V_0$, for D is strong, so there is an xy -directed walk C such that $l(C)$ is even, that is, such that $l(C) \geq 2$. Then $C = (x = x_0, A_0, x_1, \dots, A_{k-1}, x_k = y)$ contains the subsequence $C' = (x = x_0, A_0, x_1)$, which is an xx_1 -directed walk of length 1. Therefore, $x_1 \in V_1$.

(iii) $V_0 \cap V_1 = \emptyset$: Suppose there exists $y \in V_0 \cap V_1$. Then there are xy -directed walks C_0 and C_1 such that $l(C_0)$ is even and $l(C_1)$ is odd. Since D is strong, there is a yx -directed walk C' . If $l(C')$ is odd, then $C' \cup C_0$ is a closed directed walk of odd length. If $l(C')$ is even, then $C' \cup C_1$ is a closed directed walk of odd length. Therefore, $\nexists y \in V_0 \cap V_1$.

(iv) V_0 and V_1 are independent sets: Suppose there exists $\{y, z\} \subset V_0$ such that there is a yz -arc A in D . Since $y \in V_0$, there is an xy -directed walk of even length $C = (x = x_0, A_0, x_1, \dots, A_{k-1}, x_k = y)$, so that the length of the xz -directed walk $C' = (x = x_0, A_0, x_1, \dots, A_{k-1}, y, A, z)$ is odd. Then $z \in V_0 \cap V_1$, which is impossible. It follows that V_0 is an independent set. We may see that V_1 is independent by means of a similar reasoning.

Theorem 4. *Every symmetrical directed hypergraph D with no closed directed walks of odd length has a k -transversal.*

Proof. As stated in the remark preceding Theorem 3, a symmetrical directed hypergraph has a k -transversal iff all of its strong components have one, so we may assume D to be strong. Since D is strong, by Theorem 3 D is bipartite, and then Theorem 2 states that D has a k -transversal.

Corollary 1. *Let H be a hypergraph such that there exists a directed hypergraph D generated by H with no closed directed walks of odd length. Then H has a strongly independent transversal set.*

Proof. Let H be a hypergraph satisfying the conditions of the corollary, and let D be a directed hypergraph generated by H without closed directed walks of odd length. Theorem 4 states that D has a k -transversal T . Since the partition of every arc is the natural partition of the edges of H , the intersection between any two levels of D is empty. Let N_1, \dots, N_k be the levels of arcs in D such that $\forall i \in \{1, \dots, k\}$, $N_i \cap T \neq \emptyset$. Consider a set $N = \{x_1, \dots, x_k\}$, where $\forall i \in \{1, \dots, k\}$, $x_i \in N_i$. On the other hand, notice that the set of isolated vertices X of D is contained in T . Then $L = X \cup N$ is a strongly independent transversal set in H : Since every level

intersected by T is as well intersected by L , we have that L is a transversal set of H ; since T is independent (that is, there are not two levels N_p and N_q such that $N_p \cap T \neq \emptyset \neq N_q \cap T$ and $\{N_p, N_q\} \subset V(A)$ for some arc A in D) and we take only one vertex from each level, it follows that L is strongly independent in H .

Corollary 2. *Let $H = \{E_1, \dots, E_m\}$ be a hypergraph with no cycles of odd length and such that the natural partition of all its edges has two elements. Then H has a strongly independent transversal set. In particular, every multigraph with no loops nor cycles of odd length has an independent transversal set.*

Proof. Let $H = \{E_1, \dots, E_m\}$ be a hypergraph satisfying the conditions of the corollary. Since the natural partition of every edge of H has two levels, H generates a directed hypergraph D . Since H has no cycles of odd length, D has no directed cycles of odd length (for every arc has but two levels), so that according to Lemma 3 D has no closed directed walks of odd length. Then Corollary 1 states that H has a strongly independent transversal set.

Notice that in such a case, H is always simple, because the following holds for every hypergraph G : If $\{E_i, E_j\} \in G$, $E_i \subset E_j$, $E_i \neq E_j$, then the natural partition of E_j has at least one more element than that of E_i . Also, for every $E \in H$ there is a partition $B = (B_1, B_2)$ of the set $I_E = \{F \in H \mid E \cap F \neq \emptyset\}$, possibly with empty elements, such that $\forall i \in \{1, 2\}$, $\forall F_a, F_b \in B_i$, $E \cap F_a = E \cap F_b$; if $B_1 \neq \emptyset \neq B_2$, then $E \subset I_E$. Moreover, Lemma 5 implies that every hypergraph H such that there exists a bipartite directed hypergraph generated by it, does have a strongly independent transversal set.

Corollary 3. *Let $H = \{E_1, \dots, E_m\}$ be a simple hypergraph with no cycles of odd length and such that the natural partition of all its edges has at most two elements. Then H has a strongly independent transversal set.*

Proof. Let $H = \{E_1, \dots, E_m\}$ be a hypergraph satisfying the conditions of the corollary. Since H is simple, every edge whose natural partition has only one element intersects no other edge. Then we may take one vertex from each of such edges and consider the remaining hypergraph, all of whose edges have exactly two levels: without loss of generality, let $H' = \{E_1, \dots, E_k\}$ be the set of edges of H whose natural partition has one element, and consider the set $T' = \{x_1, \dots, x_k\}$, where for every $i \in \{1, \dots, k\}$, $x_i \in E_i$. The natural partition of every edge of $H'' = H \setminus H'$ has two elements, so we may

apply Corollary 2, obtaining a strongly independent transversal set T'' of H'' . Then $T' \cup T''$ is a strongly independent transversal set of H .

As a consequence, every (multi)graph with loops only in "isolated" vertices and without odd cycles has an independent transversal set.

Open Problem. Characterise all hypergraphs H such that there exists a directed hypergraph D generated by H with no closed directed walks of odd length. Figure 7 shows a hypergraph with cycles of odd length such that the only directed hypergraph generated by it has no directed cycles altogether.

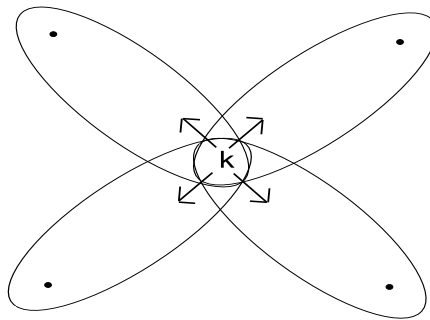


Figure 7. There are k arcs, k vertices in the level which is the minimum of all arcs, and one vertex in all other levels. D has no closed directed walks, while the underlying hypergraph has cycles of every length no greater than k .

We will now see that every directed hypergraph without closed directed walks of odd length is kernel-perfect:

Definition. A strong component T of a directed hypergraph D on a set V is *terminal* iff $\forall x \in V \setminus V(T)$, there are no $V(T)x$ -arcs.

Observe that every directed hypergraph has a terminal strong component.

Definition. A directed hypergraph which is not kernel-perfect is called *kernel-imperfect*. A kernel-imperfect directed hypergraph such that all of its proper induced directed subhypergraphs are kernel-perfect is called *critical-kernel-imperfect (CKI)*.

Notice that given a kernel-imperfect directed hypergraph D , there is always an induced directed subhypergraph of D which is CKI, for every directed hypergraph on a set of one or two vertices is kernel-perfect.

Lemma 6. *Every CKI directed hypergraph is strong.*

Proof. Let D be a CKI directed hypergraph on a set V . Suppose D is not strong, and consider a strong terminal component T of D . Since D is not strong, T is a proper induced directed subhypergraph of D , so it has a kernel K_1 . Since D has no kernel, $\emptyset \neq U = V \setminus (K_1 \cup N^-(K_1)) \neq V$, which implies that $D[U]$ has a kernel K_2 .

We will see that $K_1 \cup K_2$ is independent: There are no K_1K_2 -arcs, for T is terminal; there are no K_2K_1 -arcs, from the definition of U . We also have that $K_2 \cup N^-(K_2) = U$ and $K_1 \cup N^-(K_1) = V \setminus U$, so that $K_1 \cup K_2$ is a kernel of D . This is a contradiction. Then D must be strong.

Theorem 5. *Every directed hypergraph D without closed directed walks of odd length is kernel-perfect.*

Proof. Let D be a kernel-imperfect directed hypergraph without closed directed walks of odd length, and let D' be an induced directed subhypergraph of D which is CKI. Since D' is strong and has no directed walks of odd length, Theorem 3 states D' is bipartite. Then, from Theorem 2', D' is kernel-perfect. This is a contradiction, so Theorem 5 is proven.

Corollary. *Let D be a directed hypergraph without directed cycles of odd length, and such that none of its arcs has more than two levels. Then D is kernel-perfect.*

Proof. This follows from Lemma 3 and Theorem 5.

The Corollary is a generalisation of Richardson's Theorem, for every arc on a (multi)digraph has at most 2 levels.

Associated Digraphs

We may associate a digraph Q to any directed hypergraph D in the following way: $V(Q) = V(D)$, and for every $\{x, y\} \subset V(Q) = V(D)$ there is an xy -arrow in Q iff there is an xy -arc in D . This resembles the 2-section graph of a hypergraph, defined by Berge in [3] and studied by Borowiecki (cf. [5]). Given a directed hypergraph D , to each arc A_i of D corresponds an $r(i)$ -partite tournament. If we ask the intersection of any two levels of D to be empty and k is the total number of levels in D , then Q is a k -partite digraph.

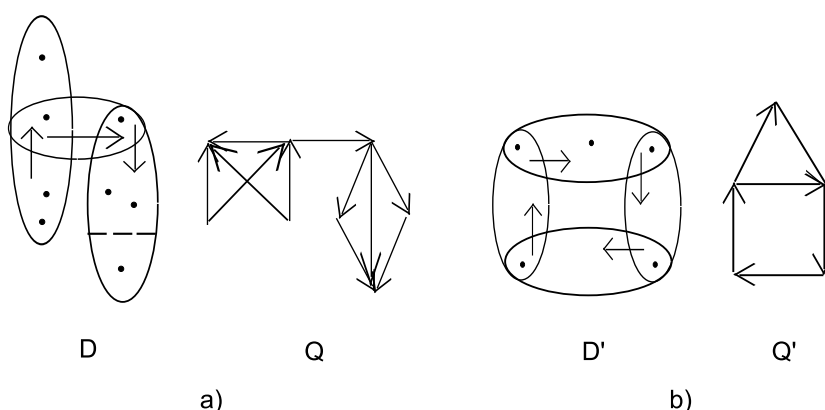


Figure 8. Associated digraph.

Notice that a given digraph may be associated to several directed hypergraphs (to begin with, to itself). From the definition of Q , we have that $S \subset V(D) = V(Q)$ is independent in D iff it is independent in Q , and that S is absorbent en D iff it is absorbent in Q . So, any result on (not necessarily transversal) kernels of directed hypergraphs corresponds to a result on kernels of digraphs. If Q has no closed directed walks of odd length, D has none either. However, there are directed hypergraphs without closed directed walks of odd length whose associated digraphs do have closed directed walks of odd length (that is, directed cycles of odd length), as shown on Figure 8.b. Theorem 5 implies that the associated digraph of every directed hypergraph without closed directed walks of odd length is kernel-perfect.

Open problem. Characterise all digraphs with directed cycles (that is, closed directed walks) of odd length which are associated to directed hypergraphs without closed directed walks of odd length. It is easy to show that every directed cycle of odd length C of such a digraph Q has at least one "jump" of length 2: If $C = (x_0, x_1, x_2, \dots, x_{2k+1} = x_0)$ is a directed cycle of odd length in Q , whether (x_{2k}, x_1) is an arrow of Q , or there exists $i \in \{1, \dots, 2k - 1\}$ such that (x_i, x_{i+2}) is an arrow of Q .

Remark. We may say that a directed hypergraph D is k -transversal-perfect iff every induced directed subhypergraph of D has a k -transversal. Since every directed subhypergraph of a symmetrical directed hypergraph is itself symmetrical, theorems 1, 2, and 4 can be easily extended to results resembling theorems 1'', 2', and 5. However, we already mentioned that induced

directed subhypergraphs of a directed hypergraph D generated by a hypergraph H are not necessarily generated by induced subhypergraphs of H , so the concept is not useful for researching hypergraphs.

Up to now, our efforts have been focused on the study of transversal kernels (that is, sets which are independent, absorbent, and transversal) in directed hypergraphs. However, we think that the concept of directed hypergraph may be useful for studying other aspects of digraphs and hypergraphs.

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