Discussiones Mathematicae
Graph Theory 29 (2009) 119–142

EQUITABLE COLORING OF KNESER GRAPHS

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Abstract

The Kneser graph $K(n,k)$ is the graph whose vertices correspond to k-element subsets of set $\{1,2,\ldots,n\}$ and two vertices are adjacent if and only if they represent disjoint subsets. In this paper we study the problem of equitable coloring of Kneser graphs, namely, we establish the equitable chromatic number for graphs $K(n,2)$ and $K(n,3)$. In addition, for sufficiently large n, a tight upper bound on equitable chromatic number of graph $K(n,k)$ is given. Finally, the cases of $K(2k,k)$ and $K(2k+1,k)$ are discussed.

Keywords: equitable coloring, Kneser graph.

2000 Mathematics Subject Classification: 05C15, 05A18.

1. Introduction

The notion of the equitable colorability was introduced by Meyer [9]. A graph $G=(V,E)$ is said to be *equitably r-colorable* if its vertex set V can be partitioned into r classes $V_1,V_2,\ldots,V_r$ such that each V_i is an independent set — that is, no two vertices from V_i are adjacent, $i=1,\ldots,r$ — and $|\#V_i-\#V_j|\le 1$ for every i,j, where $\#S$ denotes the cardinality of a given set S. Such partition $V_1,V_2,\ldots,V_r$ is called an *equitable partition*. The smallest integer r for which G is equitably r-colorable is known as the

*Supported by The Visby Programme Scholarship 01224/2007 (The Swedish Institute).

equitable chromatic number of G and denoted by $\chi_=(G)$. Since an equitable coloring is a proper coloring, we have:

$$\chi_=(G) \geq \chi(G), \tag{1}$$

where $\chi(G)$ denotes the ordinary chromatic number of graph G. It is worth pointing out that equitable r-colorability of G does not imply that G is equitably $(r+1)$-colorable. A counterexample is the complete bipartite graph $K_{3,3}$ which can be equitably colored with two colors, but not with three.

Applications of equitable coloring can be found in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. We can model this problem as coloring the vertices of the graph G whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. Another applications of equitable coloring can be found in [2].

In this paper we investigate the equitable colorability of Kneser graphs. Given two positive integers n and k, $n \geq 2k$, the *Kneser graph* $K(n,k)$ is the graph whose vertices correspond to k-element subsets of set $\{1,2,\ldots,n\}$ and two vertices are adjacent if and only if they represent disjoint subsets. In 1955, Kneser [5] conjectured that $\chi(K(n,k)) \geq n-2k+2$, $n \geq 2k \geq 2$, which was positively verified by Lovasz in 1978 [8].

Theorem 1.1 [8]. *Let $K(n,k)$ be a Kneser graph and $n \geq 2k \geq 2$. Then*

$$\chi(K(n,k)) = n-2k+2.$$

Since then several types of colorings of Kneser graphs have been considered. For example, the multichromatic number and the circular chromatic number (also known as the "star chromatic number") of Kneser graphs was investigated in [3, 6] and [4, 11, 12, 13], respectively.

Our results. In this paper we color Kneser graphs in the equitable way. Our results are the following theorems.

Theorem 1.2. *Let $K(n,2)$, $n \geq 4$, be a Kneser graph. Then*

$$\chi_=(K(n,2)) = \begin{cases} n-2 & \text{if } n=4,5,6; \\ n-1 & \text{otherwise.} \end{cases}$$

Theorem 1.3. *Let $K(n,3)$, $n \geq 6$, be a Kneser graph. Then*

$$\chi_{=}(K(n,3)) = \begin{cases} n-4 & \text{if } 6 \leq n \leq 13; \\ n-3 & \text{if } n \in \{14,15\}; \\ n-2 & \text{otherwise.} \end{cases}$$

Theorem 1.4. *For any positive integer k, if n is large enough, then*

$$\chi_{=}(K(n,k)) = n-k+1.$$

Theorem 1.5. *For $k \geq 1$, $\chi_{=}(K(2k,k)) = 2$ and $\chi_{=}(K(2k+1,k)) = 3$.*

2. PRELIMINARIES

For simplicity, to avoid some confusion which may arise when considering vertices of Kneser graphs being subsets themselves, we shall consider equitable colorings of set families. We say that a set family $\mathcal{F}$ is *independent* if for any two sets $S_1, S_2 \in \mathcal{F}$, we have $S_1 \cap S_2 \neq \emptyset$. Next, a set family $\mathcal{F}$ is said to be *equitably r-colorable* if it can be partitioned into r subfamilies $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$ such that each $\mathcal{F}_i$, $i = 1, \ldots, r$, is independent and $|\#\mathcal{F}_i - \#\mathcal{F}_j| \leq 1$ for every $1 \leq i, j \leq r$, where $\#\mathcal{S}$ denotes the cardinality of a given family $\mathcal{S}$; such partition $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$ is called an *equitable partition*. And finally, the smallest integer r for which $\mathcal{F}$ is equitably r-colorable is called the *equitable chromatic number* of $\mathcal{F}$ and denoted by $\chi_{=}(\mathcal{F})$.

Equitable coloring of Kneser graphs. Let $[n]^{(k)}$ denotes the family of all k-element subsets of the set $[n] = \{1, \ldots, n\}$. Following the above definitions, it easy to see that the equitable chromatic number $\chi_{=}(K(n,k))$ of Kneser graph $K(n,k)$ is equivalent to the equitable chromatic number $\chi_{=}([n]^{(k)})$ of family $[n]^{(k)}$, and therefore, by estimating bounds on $\chi_{=}([n]^{(k)})$ we estimate bounds on $\chi_{=}(K(n,k))$ as well, and *vice versa*.

Dominator of a family. Let $\mathcal{F}$ be a family of subsets of a given set X, and let D be a subset of X. We say that set D is a *dominator of* $\mathcal{F}$ if for every $S \in \mathcal{F}$ we have $S \cap D \neq \emptyset$. A dominator of $\mathcal{F}$ of the minimum cardinality is called the *minimum dominator of* $\mathcal{F}$. In 1967, Hilton and Milner [1] proved

Theorem 2.1 [1]**.** *Let $\mathcal{F} \subseteq [n]^{(k)}$ be an independent family of size at least*

$$\binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 2.$$

Then

$$\bigcap_{S\in\mathcal{F}} S \neq \emptyset.$$

The above theorem immediately leads to the following corollary which we shall extensively use throughout the paper.

Corollary 2.2 [6]. *Let $\mathcal{F} \subseteq [n]^{(k)}$ be an independent family of size at least*

$$\binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 2.$$

Then

$$\bigcap_{S\in\mathcal{F}} S = \{i\}$$

for some $i \in [n]$, that is, $\{i\}$ is the minimum one-element dominator of family $\mathcal{F}$, and moreover, dominator $\{i\}$ is unique.

Proof. It follows from the fact that if $\bigcap_{S\in\mathcal{F}} S$ has at least two elements, then $\mathcal{F}$ is of size at most $\binom{n-2}{k-2}$, which is smaller than $\binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 2$ — a contradiction. ■

3. THE EQUITABLE CHROMATIC NUMBER $\chi_=([n]^{(2)})$

In this section, we provide the exact bounds on the equitable chromatic number $\chi_=([n]^{(2)})$. For small values of $n = 4, 5, 6$, we explicitly give the minimum equitable colorings, while for larger values of $n \geq 7$, the proof is constructive, and for necessity, we make use of Corollary 2.2.

Theorem 3.1.

(a) *For $n = 4, 5, 6$, we have $\chi_=([n]^{(2)}) = n - 2$.*

(b) *For $n \geq 7$, we have $\chi_=([n]^{(2)}) = n - 1$.*

Proof. (a) Proper equitable partitions of $[n]^{(2)}$, $n = 4, 5, 6$, respectively, are given in Figure 1(a-c). The optimality of the given bounds follows from (1) and the fact that $\chi(K(n, 2)) = n - 2$ by Theorem 1.1.

(b) For necessity, suppose on the contrary that $\chi_=([n]^{(2)}) \leq n - 2$, and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n-l}$, $l \geq 2$, be an equitable partition of family $[n]^{(2)}$ into $(n - l)$ independent subfamilies. As $|\#\mathcal{F}_i - \#\mathcal{F}_j| \leq 1$, $1 \leq i, j \leq n - l$, this forces

$$\#\mathcal{F}_i = \left\lfloor \frac{\binom{n}{2}}{n-l} \right\rfloor \text{ or } \#\mathcal{F}_i = \left\lceil \frac{\binom{n}{2}}{n-l} \right\rceil, \ 1 \le i \le n-l.$$

a)

$\mathcal{F}_1$	$\mathcal{F}_2$
$\{1,2\}$	$\{1,4\}$
$\{1,3\}$	$\{2,4\}$
$\{2,3\}$	$\{3,4\}$

b)

$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
$\{1,2\}$	$\{1,4\}$	$\{1,5\}$
$\{1,3\}$	$\{2,4\}$	$\{2,5\}$
$\{2,3\}$	$\{3,4\}$	$\{3,5\}$
		$\{4,5\}$

c)

$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$
$\{1,2\}$	$\{1,4\}$	$\{1,5\}$	$\{1,6\}$
$\{1,3\}$	$\{2,4\}$	$\{2,5\}$	$\{2,6\}$
$\{2,3\}$	$\{3,4\}$	$\{3,5\}$	$\{3,6\}$
	$\{4,6\}$	$\{4,5\}$	$\{5,6\}$

Figure 1. Equitable partitions of a) $[4]^{(2)}$, b) $[5]^{(2)}$, and c) $[6]^{(2)}$.

Since $n \ge 7$ and $l \ge 2$, we have $\#\mathcal{F}_i \ge 4$, $i = 1, \ldots, n-l$. By Corollary 2.2, there exist two different elements $x, y \in [n]$ such that neither $\{x\}$ nor $\{y\}$ is the (unique) minimum dominator of family $\mathcal{F}_i$, $i = 1, \ldots, n-l$. Consequently, 2-element set $\{x, y\} \notin \bigcup_{i=1}^{n-l} \mathcal{F}_i$, and thus $\bigcup_{i=1}^{n-l} \mathcal{F}_i \ne [n]^{(2)}$ — a contradiction.

For sufficiency, consider the following set partition $\mathcal{F}_1, \ldots, \mathcal{F}_{n-1}$:

$$\begin{aligned}
\mathcal{F}_{2j+1} := {} & \{\{2k-1, 2j+1\} \colon 1 \le k \le j\} \cup \{\{2j+1, 2k\} : j < k \le \lfloor n/2 \rfloor\} \cup \\
& \{\{2j+1, n\}\} \\
\mathcal{F}_{2j} := {} & \{\{2k, 2j\} : 1 \le k < j\} \cup \{\{2j, 2k+1\} : j \le k < \lfloor n/2 \rfloor\} \cup \\
& \{\{2j, n\}\}
\end{aligned}$$

All we need is to prove that:

(a) each of families $\mathcal{F}_i$ is independent, $i = 1, \ldots, n-1$;

(b) $\bigcup_{i=1}^{n-1} \mathcal{F}_i = [n]^{(2)}$;

(c) $|\#\mathcal{F}_i - \#\mathcal{F}_j| \le 1$, for $i, j = 1, \ldots, n-1$.

(a) It follows from the fact that for every $i = 1, \ldots, n-1$, element $\{i\}$ is the minimum dominator of $\mathcal{F}_i$, that is, $i \in S$ for every $S \in \mathcal{F}_i$.

(b) Let $\{x, y\}$ be a 2-element subset of $[n]$. We have four cases to consider:

- $\{x, y\} = \{2i, 2j\}$. As $i < j$, we have $\{2i, 2j\} \in \mathcal{F}_{2i}$;
- $\{x, y\} = \{2i, 2j+1\}$. As $i \le j$, we have $\{2i, 2j+1\} \in \mathcal{F}_{2i}$;
- $\{x, y\} = \{2i+1, 2j\}$. As $i < j$, we have $\{2i+1, 2j\} \in \mathcal{F}_{2i+1}$;
- $\{x, y\} = \{2i+1, 2j+1\}$. As $i < j$, we have $\{2i+1, 2j+1\} \in \mathcal{F}_{2i+1}$.

(c) We have to consider two cases.

- n is even. Then:

$$\#\mathcal{F}_{2j+1} = j + (\lfloor \tfrac{n}{2} \rfloor - j) + 0 = \lfloor \tfrac{n}{2} \rfloor;$$
$$\#\mathcal{F}_{2j} = (j-1) + (\lfloor \tfrac{n}{2} \rfloor - j) + 1 = \lfloor \tfrac{n}{2} \rfloor.$$

- n is odd. Then:

$$\#\mathcal{F}_{2j+1} = j + (\lfloor \tfrac{n}{2} \rfloor - j) + 1 = \lfloor \tfrac{n}{2} \rfloor + 1;$$
$$\#\mathcal{F}_{2j} = (j-1) + (\lfloor \tfrac{n}{2} \rfloor - j - 1) + 0 = \lfloor \tfrac{n}{2} \rfloor. \qquad \blacksquare$$

Consequently, by the above theorem and the equality $\chi_=(K(n,2)) = \chi_=([n]^{(2)})$, we obtain

Corollary 3.2. *Let $K(n,2)$, $n \geq 4$, be the Kneser graph. Then*

$$\chi_=(K(n,2)) = \chi_=([n]^{(2)}) = \begin{cases} n-2 & \text{if } n = 4,5,6; \\ n-1 & \text{otherwise.} \end{cases}$$

4. The Equitable Chromatic Number $\chi_=([n]^{(3)})$

In this section we consider the problem of determining the equitable chromatic number $\chi_=([n]^{(3)})$. Similarly as in the case of $[n]^{(2)}$, let us first consider small values of n, namely, $6 \leq n \leq 15$.

Proposition 4.1.

(a) *For $6 \leq n \leq 13$, we have $\chi_=([n]^{(3)}) = n-4$.*

(b) *For $n \in \{14, 15\}$, we have $\chi_=([n]^{(3)}) = n-3$.*

Proof. (a) By Theorem 1.1, we have $\chi([n]^{(3)}) = n-4$, and thus by (1) it follows that $\chi_=([n]^{(3)}) \geq n-4$. Hence all we need is to provide proper equitable (independent) partitions of $[n]^{(3)}$, $6 \leq n \leq 13$, into $(n-4)$ subfamilies $\mathcal{F}_1^n, \ldots, \mathcal{F}_{n-4}^n$.

For $\{x_1, \ldots, x_m\} \subseteq [n]$ and $t \leq m \leq k$, let $[n]^{(k)}_{\{x_1,\ldots,x_m\}/t}$ denote the subfamily of $[n]^{(k)}$ consisting of all sets containing at least t elements of $\{x_1, \ldots, x_m\}$; for simplicity, when $m = t = 1$, we shall write $[n]^{(k)}_x$ instead of $[n]^{(k)}_{\{x\}/1}$. And, for k disjoint sets $S_1, \ldots, S_k \subseteq [n]$, let $S_1 S_2 \ldots S_k$ denote the subfamily of $[n]^{(k)}$ consisting of all sets containing $y_1 \in S_1, y_2 \in S_2, \ldots,$ and $y_k \in S_k$.

Then the proper equitable partitions $\mathcal{F}_1^n, \ldots, \mathcal{F}_{n-4}^n$ of $[n]^{(3)}$, $6 \le n \le 13$, respectively, are as follows (for convenience of the reader, they are depicted in Table 1 and 2 in Appendix):

- $n = 6$:

$$\mathcal{F}_1^6 = [6]^{(3)}_{\{1,2,3\}/2}; \qquad (\#\mathcal{F}_1^6 = 10)$$
$$\mathcal{F}_2^6 = [6]^{(3)}_{\{4,5,6\}/2}. \qquad (\#\mathcal{F}_2^6 = 10)$$

- $n = 7$:

$$\mathcal{F}_1^7 = \mathcal{F}_1^6 \cup \{1\}\{2,3\}\{7\}; \qquad (\#\mathcal{F}_1^7 = 12)$$
$$\mathcal{F}_2^7 = \mathcal{F}_2^6 \cup \{4\}\{5,6\}\{7\}; \qquad (\#\mathcal{F}_2^7 = 12)$$
$$\mathcal{F}_3^7 = [7]^{(3)} \setminus \textstyle\bigcup_{i=1}^{2} \mathcal{F}_i^7. \qquad (\#\mathcal{F}_3^7 = 11)$$

- $n = 8$:

$$\mathcal{F}_1^8 = \mathcal{F}_1^7 \cup \big\{\{2,3,7\},\{1,2,8\}\big\}; \qquad (\#\mathcal{F}_1^8 = 14)$$
$$\mathcal{F}_2^8 = \mathcal{F}_2^7 \cup \big\{\{5,6,7\},\{4,5,8\}\big\}; \qquad (\#\mathcal{F}_2^8 = 14)$$
$$\mathcal{F}_3^8 = \big(\mathcal{F}_3^7 \setminus \big\{\{2,3,7\},\{5,6,7\}\big\}\big) \cup [5]\{7\}\{8\}; \qquad (\#\mathcal{F}_3^8 = 14)$$
$$\mathcal{F}_4^8 = [8]^{(3)} \setminus \textstyle\bigcup_{i=1}^{3} \mathcal{F}_i^8. \qquad (\#\mathcal{F}_4^8 = 14)$$

- $n = 9$:

$$\mathcal{F}_1^9 = \mathcal{F}_1^8 \cup \{1,2\}\{3\}\{8\} \cup \{\{1,2,9\}\}; \qquad (\#\mathcal{F}_1^9 = 17)$$
$$\mathcal{F}_2^9 = \mathcal{F}_2^8 \cup \{4,5\}\{6\}\{8\} \cup \{\{4,5,9\}\}; \qquad (\#\mathcal{F}_2^9 = 17)$$
$$\mathcal{F}_3^9 = \mathcal{F}_3^8 \cup [3]\{7\}\{9\}; \qquad (\#\mathcal{F}_3^8 = 17)$$
$$\mathcal{F}_4^9 = \big(\mathcal{F}_4^8 \setminus \big(\{1,2\}\{3\}\{8\} \cup \{4,5\}\{6\}\{8\}\big)\big) \cup [7]\{8\}\{9\}; \qquad (\#\mathcal{F}_3^8 = 17)$$
$$\mathcal{F}_5^9 = [9]^{(3)} \setminus \textstyle\bigcup_{i=1}^{4} \mathcal{F}_i^9. \qquad (\#\mathcal{F}_5^9 = 16)$$

- $n = 10$:

$$\mathcal{F}_1^{10} = \mathcal{F}_1^9 \cup \{1,2\}\{3\}\{9\} \cup \{\{1,2,10\}\}; \qquad (\#\mathcal{F}_1^{10} = 20)$$
$$\mathcal{F}_2^{10} = \mathcal{F}_2^9 \cup \{4,5\}\{6\}\{9\} \cup \{\{4,5,10\}\}; \qquad (\#\mathcal{F}_2^{10} = 20)$$
$$\mathcal{F}_3^{10} = \mathcal{F}_3^9 \cup [3]\{7\}\{10\}; \qquad (\#\mathcal{F}_3^{10} = 20)$$
$$\mathcal{F}_4^{10} = \mathcal{F}_4^9 \cup [3]\{8\}\{10\}; \qquad (\#\mathcal{F}_4^{10} = 20)$$
$$\mathcal{F}_5^{10} = \big(\mathcal{F}_5^9 \setminus \big(\{1,2\}\{3\}\{9\} \cup \{4,5\}\{6\}\{9\}\big)\big) \cup [8]\{9\}\{10\}; \qquad (\#\mathcal{F}_5^{10} = 20)$$
$$\mathcal{F}_6^{10} = [10]^{(3)} \setminus \textstyle\bigcup_{i=1}^{5} \mathcal{F}_i^{10}. \qquad (\#\mathcal{F}_6^{10} = 20)$$

- $n = 11$:

$$\mathcal{F}_1^{11} = \mathcal{F}_1^{10} \cup \{1,2\}\{3\}\{10\} \cup \{1\}\{2,3\}\{11\}; \qquad (\#\mathcal{F}_1^{10} = 24)$$
$$\mathcal{F}_2^{11} = \mathcal{F}_2^{10} \cup \{4,5\}\{6\}\{10\} \cup \{4\}\{5,6\}\{11\}; \qquad (\#\mathcal{F}_2^{10} = 24)$$
$$\mathcal{F}_3^{11} = \mathcal{F}_3^{10} \cup [4]\{7\}\{11\}; \qquad (\#\mathcal{F}_3^{10} = 24)$$
$$\mathcal{F}_4^{11} = \mathcal{F}_4^{10} \cup [4]\{8\}\{11\}; \qquad (\#\mathcal{F}_4^{10} = 24)$$
$$\mathcal{F}_5^{11} = \mathcal{F}_5^{10} \cup [3]\{9\}\{11\}; \qquad (\#\mathcal{F}_4^{10} = 23)$$
$$\mathcal{F}_6^{11} = \big(\mathcal{F}_6^{10} \setminus \big(\{1,2\}\{3\}\{10\} \cup \{4,5\}\{6\}\{10\}\big)\big) \cup [7]\{10\}\{11\}; \qquad (\#\mathcal{F}_5^{10} = 23)$$
$$\mathcal{F}_7^{11} = [11]^{(3)} \setminus \textstyle\bigcup_{i=1}^{6} \mathcal{F}_i^{11}. \qquad (\#\mathcal{F}_6^{10} = 23)$$

- $n = 12$:

$$\mathcal{F}_1^{12} = \mathcal{F}_1^{11} \cup \{2\}\{3\}\{11,12\} \cup \{1\}\{2,3\}\{12\}; \qquad (\#\mathcal{F}_1^{11} = 28)$$

$\mathcal{F}_2^{12} = \mathcal{F}_2^{11} \cup \{5\}\{6\}\{11,12\} \cup \{4\}\{5,6\}\{12\};$ $(\#\mathcal{F}_2^{11} = 28)$
$\mathcal{F}_3^{12} = \mathcal{F}_3^{11} \cup [4]\{7\}\{12\};$ $(\#\mathcal{F}_3^{11} = 28)$
$\mathcal{F}_4^{12} = \mathcal{F}_4^{11} \cup [4]\{8\}\{12\};$ $(\#\mathcal{F}_4^{11} = 28)$
$\mathcal{F}_5^{12} = \mathcal{F}_5^{11} \cup [4]\{9\}\{12\};$ $(\#\mathcal{F}_5^{11} = 27)$
$\mathcal{F}_6^{12} = \mathcal{F}_6^{11} \cup [4]\{10\}\{12\};$ $(\#\mathcal{F}_6^{11} = 27)$
$\mathcal{F}_7^{12} = \big(\mathcal{F}_7^{11} \setminus \big\{\{2,3,11\},\{5,6,11\}\big\}\big) \cup [6]\{11\}\{12\};$ $(\#\mathcal{F}_7^{11} = 27)$
$\mathcal{F}_8^{12} = [12]^{(3)} \setminus \bigcup_{i=1}^{7} \mathcal{F}_i^{12}.$ $(\#\mathcal{F}_8^{11} = 27)$

- $n = 13$:

$\mathcal{F}_1^{13} = \mathcal{F}_1^{12} \cup \{1\}\{2,3\}\{13\} \cup \big\{\{2,3,13\}\big\};$ $(\#\mathcal{F}_1^{13} = 31)$
$\mathcal{F}_2^{13} = \mathcal{F}_2^{12} \cup \{4\}\{5,6\}\{13\} \cup \big\{\{5,6,13\}\big\};$ $(\#\mathcal{F}_2^{13} = 31)$
$\mathcal{F}_3^{13} = \mathcal{F}_3^{12} \cup [4]\{7\}\{13\};$ $(\#\mathcal{F}_3^{13} = 32)$
$\mathcal{F}_4^{13} = \mathcal{F}_4^{12} \cup [4]\{8\}\{13\};$ $(\#\mathcal{F}_4^{13} = 32)$
$\mathcal{F}_5^{13} = \mathcal{F}_5^{12} \cup [5]\{9\}\{13\};$ $(\#\mathcal{F}_5^{13} = 32)$
$\mathcal{F}_6^{13} = \mathcal{F}_6^{12} \cup [5]\{10\}\{13\}$ $(\#\mathcal{F}_6^{13} = 32)$
$\mathcal{F}_7^{13} = \mathcal{F}_7^{12} \cup [5]\{11\}\{13\};$ $(\#\mathcal{F}_7^{13} = 32)$
$\mathcal{F}_8^{13} = \mathcal{F}_8^{12} \cup [5]\{12\}\{13\};$ $(\#\mathcal{F}_8^{13} = 32)$
$\mathcal{F}_9^{13} = [13]^{(3)} \setminus \bigcup_{i=1}^{8} \mathcal{F}_i^{13}.$ $(\#\mathcal{F}_9^{13} = 32)$

Observe that:

(1) For $n = 6, \ldots, 12$, we have $\mathcal{F}_i^n \subset \mathcal{F}_i^{n+1}$, $i = 1, \ldots, n-5$.

(2) For $n = 6, \ldots, 13$, we have $\mathcal{F}_1^n \subseteq [n]^{(3)}_{\{1,2,3\}/2}$ and $\mathcal{F}_2^n \subseteq [n]^{(3)}_{\{4,5,6\}/2}$, while each of families $\mathcal{F}_i^n$, $i = 3, \ldots, n-4$, is a *star*, that is, $\mathcal{F}_i^n \subseteq [n]^{(3)}_{i+4}$.

(3) $\#[n]^{(3)} = \binom{n}{3} = \left\lceil \frac{\binom{n}{3}}{n-4} \right\rceil + \left\lceil \frac{\binom{n}{3}-1}{n-4} \right\rceil + \cdots + \left\lceil \frac{\binom{n}{3}-n+5}{n-4} \right\rceil$ and:

For $n = 6, \ldots, 12$, we have $\#\mathcal{F}_i^n = \left\lceil \frac{\binom{n}{3}-i+1}{n-4} \right\rceil$, $i = 1, \ldots, n-4$.

For $n = 13$, we have $\#\mathcal{F}_i^{13} = \left\lceil \frac{\binom{n}{3}+i-(n-4)}{n-4} \right\rceil = \left\lceil \frac{\binom{13}{3}+i-9}{9} \right\rceil$, $i = 1, \ldots, 9$.

(b) For necessity, suppose on the contrary that $\chi_=([n]^{(3)}) = n-4$ $(= \chi([n]^{(3)}))$ for $n \in \{14, 15\}$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n-4}$ be an equitable partition of $[n]^{(3)}$ into $(n-4)$ independent subfamilies. As $|\#\mathcal{F}_i - \#\mathcal{F}_j| \leq 1$, $1 \leq i, j \leq n-4$, this forces

$$\#\mathcal{F}_i = \left\lfloor \frac{\binom{n}{3}}{n-4} \right\rfloor \text{ or } \#\mathcal{F}_i = \left\lceil \frac{\binom{n}{3}}{n-4} \right\rceil, \ 1 \leq i \leq n-4.$$

Since $n \in \{14, 15\}$, we have $\#\mathcal{F}_i \geq 3n-7$, $i = 1, \ldots, n-4$. By Corollary 2.2, there exists the unique one-element dominator of $\mathcal{F}_i$, $i = 1, \ldots, n-4$, and

thus there are three different elements $x, y, z \in [n]$ such that none of them is the minimum dominator of family $\mathcal{F}_i$, $i = 1, \ldots, n-4$. Consequently, 3-element set $\{x, y, z\} \notin \bigcup_{i=1}^{n-4} \mathcal{F}_i$, and thus $\bigcup_{i=1}^{n-4} \mathcal{F}_i \neq [n]^{(3)}$ — a contradiction.

For sufficiency, we need to provide proper equitable (independent) partitions of $[n]^{(3)}$, $n \in \{14, 15\}$, into $(n-3)$ subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_{n-3}$. The partitions are given below; observe that $\#\mathcal{F}_i = \left\lceil \frac{\binom{n}{3}+i-(n-3)}{n-3} \right\rceil$, $i = 1, \ldots, n-3$. (Again, for convenience of the reader, they are depicted in Table 3 and 4, respectively.)

- $n = 14$:

$$\mathcal{F}_1 = [14]^{(3)}_{\{1,2,3\}/2} \setminus \{2, 3, 14\}; \qquad (\#\mathcal{F}_1 = 33)$$
$$\mathcal{F}_2 = [14]^{(3)}_{\{4,5,6\}/2} \setminus \{5, 6, 14\}; \qquad (\#\mathcal{F}_2 = 33)$$
$$\mathcal{F}_3 = [14]^{(3)}_{\{7,8,9\}/2} \setminus \{8, 9, 14\}; \qquad (\#\mathcal{F}_3 = 33)$$
$$\mathcal{F}_4 = [3]\{4,5,6\}\{7\} \cup [6]\{7\}\{10,11,12,13\}; \qquad (\#\mathcal{F}_4 = 33)$$
$$\mathcal{F}_5 = [3]\{4,5,6\}\{8\} \cup [6]\{8\}\{10,11,12,13\}; \qquad (\#\mathcal{F}_5 = 33)$$
$$\mathcal{F}_6 = [3]\{4,5,6\}\{9\} \cup [6]\{9\}\{10,11,12,13\}; \qquad (\#\mathcal{F}_6 = 33)$$
$$\mathcal{F}_7 = [3]\{4,5,6\}\{10\} \cup [9]\{10\}\{11,12\} \cup [6]\{10\}\{14\}; \qquad (\#\mathcal{F}_7 = 33)$$
$$\mathcal{F}_8 = [3]\{4,5,6\}\{11\} \cup [9]\{11\}\{12\} \cup [5]\{11\}\{13\} \cup [10]\{11\}\{14\}; \qquad (\#\mathcal{F}_8 = 33)$$
$$\mathcal{F}_9 = [3]\{4,5,6\}\{12\} \cup [11]\{12\}\{13,14\} \cup \big\{\{10,11,12\},\{12,13,14\}\big\}; \qquad (\#\mathcal{F}_9 = 33)$$
$$\mathcal{F}_{10} = [3]\{4,5,6\}\{13\} \cup [9]\{13\}\{10,14\} \cup \{6,7,8,9,10\}\{11\}\{13\} \cup \big\{\{10,13,14\}\big\}; \qquad (\#\mathcal{F}_{10} = 33)$$
$$\mathcal{F}_{11} = [14]^{(3)} \setminus \bigcup_{i=1}^{10} \mathcal{F}_i. \qquad (\#\mathcal{F}_{11} = 34)$$

Observe that $\mathcal{F}_i \subseteq [14]^{(3)}_{i+3}$, $i = 4, \ldots, n-3$.

- $n = 15$:

$$\mathcal{F}_1 = [14]^{(3)}_{\{1,2,3\}/2}; \qquad (\#\mathcal{F}_1 = 37)$$
$$\mathcal{F}_2 = [3]\{4\}\{5\} \cup [4]\{5\}\{6,7,8,9,10,11,12,13\} \cup [3]\{5\}\{14\}; \qquad (\#\mathcal{F}_2 = 38)$$
$$\mathcal{F}_3 = [3]\{4\}\{6\} \cup [5]\{6\}\{7,8,9,10,11,12,13\}; \qquad (\#\mathcal{F}_3 = 38)$$
$$\mathcal{F}_4 = [3]\{4\}\{7\} \cup [6]\{7\}\{8,10,11,12\} \cup [5]\{7\}\{13\}; \qquad (\#\mathcal{F}_4 = 38)$$
$$\mathcal{F}_5 = [3]\{4\}\{8\} \cup [7]\{8\}\{9,10,11,12,13\}; \qquad (\#\mathcal{F}_5 = 38)$$
$$\mathcal{F}_6 = [3]\{4\}\{9\} \cup [8]\{9\}\{10,11,12\} \cup [6]\{9\}\{13\} \cup [5]\{9\}\{14\}; \qquad (\#\mathcal{F}_6 = 38)$$
$$\mathcal{F}_7 = [3]\{4\}\{10\} \cup [9]\{10\}\{11,12,15\} \cup [8]\{10\}\{14\}; \qquad (\#\mathcal{F}_7 = 38)$$
$$\mathcal{F}_8 = [3]\{4\}\{11\} \cup [10]\{11\}\{12,13,14\} \cup [5]\{11\}\{15\}; \qquad (\#\mathcal{F}_8 = 38)$$
$$\mathcal{F}_9 = [3]\{4\}\{12\} \cup [11]\{12\}\{13,14,15\} \cup \{12\}\{13\}\{14,15\}; \qquad (\#\mathcal{F}_9 = 38)$$
$$\mathcal{F}_{10} = [3]\{4\}\{13\} \cup [9]\{10\}\{13\} \cup [11]\{13\}\{14,15\} \cup \big\{\{6,7,13\},\{7,9,13\},\{8,9,13\},\{13,14,15\}\big\}; \qquad (\#\mathcal{F}_{10} = 38)$$
$$\mathcal{F}_{11} = [3]\{4\}\{14\} \cup [5]\{6\}\{14\} \cup [6]\{7\}\{14\} \cup [7]\{8\}\{14\} \cup \{6,7,8\}\{9\}\{14\} \cup [12]\{14\}\{15\} \cup \big\{\{4,5,14\},\{9,10,14\}\big\}; \qquad (\#\mathcal{F}_{11} = 38)$$
$$\mathcal{F}_{12} = [15]^{(3)} \setminus \bigcup_{i=1}^{11} \mathcal{F}_i. \qquad (\#\mathcal{F}_{12} = 38)$$

Again notice that $\mathcal{F}_i \subseteq [15]^{(3)}_{i+3}$, $i = 2, \ldots, n-3$. ■

For the case $n \geq 16$, let us first provide a simpler upper bound, which we then shall use to derive the tight bound of $(n-2)$ colors.

Proposition 4.2. *For $n \geq 6$, we have $\chi_=([n]^{(3)}) \leq n$.*

Proof. Clearly, by Proposition 4.1, it is sufficient to consider the case $n \geq 16$. Consider triples (i,j,l) such that $i+j+l=n$. We consider three triples (i,j,l), (j,l,i) and (l,i,j) equivalent, that is, two triples are *equivalent* if one is a cyclic shift of the other; for example, triples $(1,2,3)$ and $(3,1,2)$ are equivalent for $n=6$.

First, observe that for every n, we have at most one triple of form (i,i,i), namely, $(n/3,n/3,n/3)$. And, an easy computation shows that if $(n \mod 3)=0$, then the number of non-equivalent triples is equal to $\left(\binom{n-1}{2}-1\right)/3+1=\lceil\binom{n-1}{2}/3\rceil$, otherwise, it is equal to $\binom{n-1}{2}/3$. For example, we have four non-equivalent triples for $n=6$: $(1,1,4)$, $(1,2,3)$, $(1,3,2)$ and $(2,2,2)$.

Following the above definitions, let $\mathcal{T}$ be the set of all non-equivalent triples for a fixed n. Now, for every triple $(i,j,l) \in \mathcal{T}$, define $\mathcal{P}_{(i,j,l)}$ as the set of all 3-vertex sets

$$\{m, ((m+i-1) \bmod n)+1, ((m+i+j-1) \bmod n)+1\},\ m=1,2,\ldots,n.$$

In other words, we consider elements $m, m+i$, and $m+i+j$ on a cycle with integers $1,2,\ldots,n$. For example, if $n=6$ and we consider triple $(1,1,4)$, then

$$\mathcal{P}_{(1,1,4)}=\{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{5,6,1\},\{6,1,2\}\}.$$

It easy to see that if $i,j,l \neq n/3$, then $\#\mathcal{P}_{(i,j,l)}=n$ — otherwise, $\#\mathcal{P}_{(n/3,n/3,n/3)}=n/3$. Moreover, as any 3-element set $S=\{x,y,z\} \in \mathcal{P}_{(x-y,z-y,n+x-z)}$, we have $\bigcup_{(i,j,l)\in\mathcal{T}} \mathcal{P}_{(i,j,l)}=[n]^{(3)}$, that is, any 3-element subset of $[n]$ belongs to $\mathcal{P}_{(i,j,l)}$ for some $(i,j,l) \in \mathcal{T}$.

Now, we shall describe how to divide $[n]^{(3)}$ into equitable independent subfamilies $\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_n$. To obtain such a partition:

- For every $\mathcal{P}_{(i,j,l)}$, where $i,j,l \neq n/3$, we choose exactly one vertex

  $$\{m, ((m+i-1) \bmod n)+1, ((m+i+j-1) \bmod n)+1\}$$

  and add it to $\mathcal{F}_m$ for every $m=1,2,\ldots,n$. Observe that if $n \mod 3 \neq 0$, i.e., there is no triple $(n/3,n/3,n/3)$, then we have just obtained a strongly equitable partition: each family has exactly $\binom{n-1}{2}/3$ sets.

- If $n \mod 3 = 0$, we are left with undistributed sets of family $\mathcal{P}_{(n/3,n/3,n/3)}$. And, to complete our partition, we add them to $n/3$ appropriate subsets, namely, $\mathcal{F}_1, \ldots, \mathcal{F}_{n/3}$. Clearly, we have $|\#\mathcal{F}_x - \#\mathcal{F}_y| \leq 1$, for $x \neq y$.

Finally, observe that in our partition, by the definition, family $\mathcal{F}_m$ consists of sets containing m, $m = 1, 2, \ldots, n$, i.e., $\{m\}$ dominates $\mathcal{F}_m$ ($\mathcal{F}_m$ is a star). Consequently, each family $\mathcal{F}_m$ is independent, which completes the proof. ■

Now, we are ready to prove the main theorem.

Theorem 4.3. *Let $n \geq 16$. Then*

$$\chi_=([n]^{(3)}) = n - 2.$$

Proof. The necessity is established by similar arguments as in the proof of Theorem 3.1 and Proposition 4.1(b). Suppose on the contrary that $\chi_=([n]^{(3)}) \leq n - 3$, and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n-l}$, $l \geq 3$, be an equitable partition of $[n]^{(3)}$ into $(n - l)$ independent subfamilies. As $|\#\mathcal{F}_i - \#\mathcal{F}_j| \leq 1$, $1 \leq i, j \leq n - l$, this forces

$$\#\mathcal{F}_i = \left\lfloor \frac{\binom{n}{3}}{n-l} \right\rfloor \text{ or } \#\mathcal{F}_i = \left\lceil \frac{\binom{n}{3}}{n-l} \right\rceil, \ 1 \leq i \leq n - l.$$

Since $n \geq 16$ and $l \geq 3$, we have $\#\mathcal{F}_i \geq 3n - 7$, $i = 1, \ldots, n - l$. By Corollary 2.2, there exists the unique one-element dominator of $\mathcal{F}_i$, $i = 1, \ldots, l$, and thus there are three different elements $x, y, z \in [n]$ such that none of them is the minimum dominator of family $\mathcal{F}_i$, $i = 1, \ldots, n - l$. Consequently, 3-element set $\{x, y, z\} \notin \bigcup_{i=1}^{n-l} \mathcal{F}_i$, and thus $\bigcup_{i=1}^{n-l} \mathcal{F}_i \neq [n]^{(3)}$ — a contradiction.

For sufficiency, let us partition $[n]^{(3)}$ into subfamilies $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$, where

$$\mathcal{N}_1 = \big\{\{n, x, y\} : 1 \leq x < y \text{ and } 2 \leq y \leq n - 1\big\},$$
$$\mathcal{N}_2 = \big\{\{n - 1, x, y\} : 1 \leq x < y \text{ and } 2 \leq y \leq n - 2\big\},$$
$$\mathcal{N}_3 = [n]^{(3)} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2).$$

Observe that $\mathcal{N}_3$ consists of all 3-element subsets of set $[n - 2] = \{1, 2, \ldots, n - 2\}$, that is, $\mathcal{N}_3 = [n - 2]^{(3)}$. Consequently, by Proposition 4.2, family $\mathcal{N}_3$ can be equitably colored with $n - 2$ colors — let $\mathcal{F}_1, \ldots, \mathcal{F}_{n-2}$ be the relevant partition of $\mathcal{N}_3$; by the constructive proof, each family $\mathcal{F}_m$, $m = 1, \ldots, n - 2$,

is dominated by $\{m\}$. Now, to obtain an equitable partition of $[n]^{(3)}$, all we need is to show how to distribute sets of families $\mathcal{N}_1$ and $\mathcal{N}_2$ into families $\mathcal{F}_1, \ldots, \mathcal{F}_{n-2}$. This can be done as follows.

- Family $\mathcal{N}_1$: each set $\{n, m, y\}$ is assigned to family $\mathcal{F}_m$, $1 \leq m < y$, $2 \leq y \leq n-1$. At this step, the cardinality of family $\mathcal{F}_m$ increases by $n-m-1$, $m = 3, \ldots, n$.
- Family $\mathcal{N}_2$: each set $\{n-1, x, m\}$ is assigned to family $\mathcal{F}_m$, $1 \leq x < m$, $2 \leq m \leq n-2$. At this step, the cardinality of family $\mathcal{F}_m$ increases by $m-1$, $m = 3, \ldots, n$.

Observe that by the construction, each (new) family $\mathcal{F}_m$ is still dominated by $\{m\}$ (and thus it is independent), and we assigned exactly $n-m-1+m-1 = n-2$ sets to each of families $\mathcal{F}_m$, $m = 1, \ldots, n-2$. Consequently, the new partition $\mathcal{F}_1, \ldots, \mathcal{F}_{n-2}$ of $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 = [n]^{(3)}$ remains equitable, which completes the proof. ■

Consequently, by Proposition 4.1 and the above theorem, we get

Corollary 4.4. *Let $K(n,3)$, $n \geq 6$, be the Kneser graph. Then*

$$\chi_=(K(n,3)) = \chi_=([n]^{(3)}) = \begin{cases} n-4 & \text{if } 6 \leq n \leq 13; \\ n-3 & \text{if } n \in \{14, 15\}; \\ n-2 & \text{otherwise.} \end{cases}$$

5. The Equitable Chromatic Number $\chi_=(K(n,k))$

In 1987, Lonc [7] provided an upper bound on the equitable chromatic number of an Kneser graph $K(n,k)$.

Theorem 5.1 [7]**.** *Let $K(n,k)$ be a Kneser graph. Then $\chi_=(K(n,k)) \leq n-k+1$.*

It is easy to see that the similar approach as in the necessity proof of Theorem 4.3 can be adapted to obtain a lower bound on the equitable chromatic number of $K(n,k)$.

Theorem 5.2. *Let k be a positive integer. Then there exists $n_0 \geq 1$ such that*

$$\chi_=([n]^{(k)}) \geq n-k+1$$

holds for every $n \geq n_0$.

Proof. Suppose $[n]^{(k)}$ can be equitably colored with $n-l \le n-k$ colors, $l \ge k$. Then the cardinality $c(n)$ of any color class $\mathcal{F}_i$, $i = 1, \ldots, n-l$, is equal to

$$\left\lfloor \frac{\binom{n}{k}}{n-l} \right\rfloor \text{ or } \left\lceil \frac{\binom{n}{k}}{n-l} \right\rceil .$$

Observe that $c(n) = O(n^{k-1})$ while the expression $w(n) = \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 2$ in Theorem 2.1 is $O(n^{k-2})$, and thus there exists n_0 such that for every $n \ge n_0$ we have $c(n) > w(n)$. Consequently, Corollary 2.2 can be applied, thus resulting in the existence of at most $n-l$ different one-element dominators $\{d_1\}, \ldots, \{d_{n-l}\}$, where $\{d_i\}$ uniquely dominates family $\mathcal{F}_i$. As $l \le k$, there exists a k-element subset $\{x_1, x_2, \ldots, x_k\}$ of $[n]$ such that $\{x_1, x_2, \ldots, x_k\} \cap \{d_1, d_2, \ldots, d_{n-l}\} = \emptyset$, and thus $\{x_1, x_2, \ldots, x_k\} \notin \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{n-l}$, a contradiction. ■

Consequently, we obtain

Corollary 5.3. *For any positive integer k, if n is large enough, then $\chi_=(K(n,k)) = n-k+1$.*

6. $\chi_=(K(2k,k))$ AND $\chi_=(K(2k+1,k))$

We have presented the exact values of $\chi_=(K(n,k))$ for small values of k as well the asymptotic value of $\chi_=(K(n,k))$ for an arbitrary (but fixed) parameter k. It is natural then to consider the case when k is large, i.e., the case $\chi_=(K(2k,k))$ and $\chi_=(K(2k+1,k))$. As graph $K(2k,k)$ consists of $\binom{2k}{k}/2$ copies of the complete 2-vertex graph K_2, we have

Fact 1. $\chi_=(K(2k,k)) = 2$.

Observe next that — by similar arguments as in the proofs of Theorems 3.1, 4.3, and 5.2 — one can show that $\chi_=(K(2k+1,k)) > 2$ (in order to not follow up with the same analysis, we omit details). Consequently, we claim that $\chi_=(K(2k+1,k)) = 3$, and in the following — by generalizing the approach used for constructing the equitable partition of $[7]^{(3)}$ (the proof of Proposition 4.1) — we shall only present the appropriate equitable partitions.

For a given subset $S \subseteq [n]$ and an integer $m \le n$, let $S^{(m)}$ denote the family of all m-element subsets of S. And, for two families $\mathcal{F}'$ and $\mathcal{F}''$, let

$\{\mathcal{F}'\}\{\mathcal{F}''\}$ denote the family of sets (unions) $S' \cup S''$, where $S' \in \mathcal{F}'$ and $S'' \in \mathcal{F}''$. First, we shall consider the case of odd k.

Let $k = 2t+1$, $t \geq 1$. Define families $\mathcal{G}_1^1, \mathcal{G}_1^2, \ldots, \mathcal{G}_1^{t+1}$ as follows:

$$\mathcal{G}_1^1 = [2k]^{(k)}_{\{1,2,\ldots,k\}/t+1};$$

$$\mathcal{G}_1^2 = \{[k]^{(k-1)}\}\{\{\{2k+1\}\}\};$$

$$\mathcal{G}_1^3 = \{[k]^{(k-2)}\}\{\{k+1,\ldots,2k\}^{(1)}\}\{\{\{2k+1\}\}\};$$

$$\ldots$$

$$\mathcal{G}_1^{t+1} = \{[k]^{(k-t)}\}\{\{k+1,\ldots,2k\}^{(t-1)}\}\{\{\{2k+1\}\}\}.$$

Likewise, define families $\mathcal{G}_2^1, \mathcal{G}_2^2, \ldots, \mathcal{G}_2^{t+1}$ as follows:

$$\mathcal{G}_2^1 = [2k]^{(k)}_{\{k+1,\ldots,2k\}/t+1};$$

$$\mathcal{G}_2^2 = \{\{k+1,\ldots,2k\}^{(k-1)}\}\{\{\{2k+1\}\}\};$$

$$\mathcal{G}_2^3 = \{\{k+1,\ldots,2k\}^{(k-2)}\}\{[k]^{(1)}\}\{\{\{2k+1\}\}\};$$

$$\ldots$$

$$\mathcal{G}_2^{t+1} = \{\{k+1,\ldots,2k\}^{(k-t)}\}\{[k]^{(t-1)}\}\{\{\{2k+1\}\}\}.$$

Observe that:

- $\#\mathcal{G}_1^1 = \#\mathcal{G}_2^1 = \sum_{i=0}^{t} \binom{k}{i}^2 = \frac{1}{2} \cdot \binom{2k}{k} < \binom{2k+1}{k}/3.$
- For $i = 2, \ldots, t+1$, $\#\mathcal{G}_1^i = \#\mathcal{G}_2^i = \binom{k}{k-i+1} \cdot \binom{k}{i-2}$, and thus

$$\sum_{i=2}^{t+1} \#\mathcal{G}_1^i = \sum_{i=2}^{t+1} \#\mathcal{G}_2^i = \sum_{i=2}^{t+1} \binom{k}{k-i+1} \cdot \binom{k}{i-2} = \sum_{i=0}^{t-1} \binom{k}{k-i-1} \cdot \binom{k}{i}$$
$$= \frac{1}{2} \cdot \left[\binom{2k}{k-1} - \binom{k}{t}^2 \right].$$

- Consequently (see Appendix):

$$\sum_{i=1}^{t+1} \#\mathcal{G}_1^i = \sum_{i=1}^{t+1} \#\mathcal{G}_2^i = \frac{1}{2} \cdot \left[\binom{2k}{k} + \binom{2k}{k-1} - \binom{k}{t}^2 \right] \geq \binom{2k+1}{k}/3. \quad (*)$$

- By oddness of k and the Pigeon-Hole Principle:

  If $(2k+1) \notin S$ then $S \in \mathcal{G}_1^1 \cup \mathcal{G}_2^1$, and thus, $[2k+1]^{(k)} \setminus (\mathcal{G}_1^1 \cup \mathcal{G}_2^1) = [2k+1]_{2k+1}^{(k)}$.

  Any two sets in $\bigcup_{i=1}^{t+1} \mathcal{G}_1^i$ (in $\bigcup_{i=1}^{t+1} \mathcal{G}_2^i$) have an element in common, while $\bigcup_{i=1}^{t+1} \mathcal{G}_1^i \cap \bigcup_{i=1}^{t+1} \mathcal{G}_2^i = \emptyset$.

Now, the equitable partition $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $[2k+1]^{(k)}$ is a simple consequence of the above observations:

- $\mathcal{F}_1 = \mathcal{G}_1^1 \cup \mathcal{G}'$, where $\mathcal{G}' \subseteq \bigcup_{i=2}^{t+1} \mathcal{G}_1^i$ is chosen arbitrarily, but to satisfy the cardinality constraints of the equitable partition (which is possible by $(*)$).
- Similarly, $\mathcal{F}_2 = \mathcal{G}_2^1 \cup \mathcal{G}''$, where $\mathcal{G}'' \subseteq \bigcup_{i=2}^{t+1} \mathcal{G}_2^i$ is again chosen arbitrarily, but to satisfy the cardinality constraints.
- Finally, $\mathcal{F}_3 = [2k+1]^{(k)} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. (Notice that $\mathcal{F}_3 \subseteq [2k+1]_{2k+1}^{(k)}$.)

Consequently, $\chi_=(K(2k+1,k)) = \chi_=([2k+1]^{(k)}) = 3$ for odd k. The case of even k requires a little bit more effort, however, the idea is the same.

Let $k = 2t$, $t \geq 2$. (If $t = 1$ then $\chi_=([5]^{(2)}) = 3$ — see Figure 1.) Consider first the family $\mathcal{H}$ that consists of sets having t elements of $[k]$ and t elements of $\{k+1, \ldots, 2k\}$; note that $\#\mathcal{H} = \binom{k}{t}^2$. We have the following claim.

Claim 6.1. $\mathcal{H}$ can be partitioned into two (disjoint) subfamilies $\mathcal{H}_1$ and $\mathcal{H}_2$ of the same size such that any two sets in each of subfamilies have an element in common, that is, $\forall_{S_1, S_2 \in H_i}\ S_1 \cap S_2 \neq \emptyset,\ i = 1, 2$.

Proof. The claim immediately follows from the fact that any element (set) $S \in \mathcal{H} \subset [2k]^{(k)}$ has an element in common with all other elements (sets) except its own complement $\overline{S} \in \mathcal{H}$. And thus, $\mathcal{H}$ is a union of complementary pairs whose elements form equitable partitions $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. ■

So let $\mathcal{H}_1 \cup \mathcal{H}_2$ be the equitable partition of $\mathcal{H}$ guaranteed by the above claim. Define now families $\mathcal{G}_1^1, \ldots, \mathcal{G}_1^t$ as follows:

$$\mathcal{G}_1^1 = [2k]_{\{1,2,\ldots,k\}/t+1}^{(k)} \cup \mathcal{H}_1;$$

$$\mathcal{G}_1^2 = \{[k]^{(k-1)}\}\{\{\{2k+1\}\}\};$$

$$\mathcal{G}_1^3 = \{[k]^{(k-2)}\}\{\{k, k+1, \ldots, 2k\}^{(1)}\}\{\{\{2k+1\}\}\};$$

$$\ldots$$

$$\mathcal{G}_1^t = \{[k]^{(k-t+1)}\}\{\{k, k+1, \ldots, 2k\}^{(t-2)}\}\{\{\{2k+1\}\}\}.$$

Likewise, define families $\mathcal{G}_2^1, \ldots, \mathcal{G}_2^t$ as follows:

$$\mathcal{G}_2^1 = [2k]^{(k)}_{\{k,k+1,\ldots,2k\}/t+1} \cup \mathcal{H}_2;$$

$$\mathcal{G}_2^2 = \{\{k, k+1, \ldots, 2k\}^{(k-1)}\}\{\{\{2k+1\}\}\};$$

$$\mathcal{G}_2^3 = \{\{k, k+1, \ldots, 2k\}^{(k-2)}\}\{[k]^{(1)}\}\{\{\{2k+1\}\}\};$$

$$\ldots$$

$$\mathcal{G}_2^t = \{\{k, k+1, \ldots, 2k\}^{(k-t+1)}\}\{[k]^{(t-2)}\}\{\{\{2k+1\}\}\}.$$

Similarly to the case of odd k, observe that:

- $\#\mathcal{G}_1^1 = \#\mathcal{G}_2^1 = \frac{1}{2} \cdot \binom{2k}{k} < \binom{2k+1}{k}/3$.
- For $i = 2, \ldots, t$, $\#\mathcal{G}_1^i = \#\mathcal{G}_2^i = \binom{k}{k-i+1} \cdot \binom{k}{i-2}$, and thus

$$\sum_{i=2}^{t} \#\mathcal{G}_1^i = \sum_{i=2}^{t} \#\mathcal{G}_2^i = \sum_{i=2}^{t} \binom{k}{k-i+1} \cdot \binom{k}{i-2} = \sum_{i=0}^{t-2} \binom{k}{k-i-1} \cdot \binom{k}{i}$$
$$= \frac{1}{2} \cdot \left[\binom{2k}{k-1} - 2 \cdot \binom{k}{t} \cdot \binom{k}{t-1}\right].$$

- Consequently, for $t \geq 4$ (see Appendix):

$$\sum_{i=1}^{t} \#\mathcal{G}_1^i = \sum_{i=1}^{t} \#\mathcal{G}_2^i = \frac{1}{2} \cdot \left[\binom{2k}{k} + \binom{2k}{k-1} - 2 \cdot \binom{k}{t} \cdot \binom{k}{t-1}\right]$$
$$\geq \binom{2k+1}{k}/3. \qquad (**)$$

- By Claim 6.1 and the Pigeon-Hole Principle:
  If $(2k+1) \notin S$ then $S \in \mathcal{G}_1^1 \cup \mathcal{G}_2^1$, and thus, $[2k+1]^{(k)} \setminus (\mathcal{G}_1^1 \cup \mathcal{G}_2^1) = [2k+1]^{(k)}_{2k+1}$.

Any two sets in $\mathcal{G}_1^1$ (in $\mathcal{G}_2^1$) have an element in common.

Any two sets in $\bigcup_{i=1}^t \mathcal{G}_1^i$ (in $\bigcup_{i=1}^t \mathcal{G}_2^i$) have an element in common, while $\bigcup_{i=1}^t \mathcal{G}_1^i \cap \bigcup_{i=1}^t \mathcal{G}_2^i = \emptyset$.

Now — see the construction used for the case of odd k — if $t \geq 4$ then one can easily construct an equitable partitions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $[2k+1]^{(k)}$. And thus, $\chi_=([2k+1]^{(k)}) = 3$ for even $k \geq 8$ as well.

We are only left with the case $k = 4$ and $k = 6$. Both cases may be solved with the same approach, and hence we shall only present the appropriate equitable partitions of $[13]^{(6)}$.†

Let $\mathcal{F}_1$ be the family of all 6-element subsets of $[13]$ having at least 5 elements from $[9]$, that is, $\mathcal{F}_1 = [13]^{(6)}_{[9]/5}$. Next, let $\mathcal{F}_2$ be the family of all 6-element subsets of $[13]$ having at least 2 elements from $\{10, 11, 12\}$, that is, $\mathcal{F}_2 = [13]^{(6)}_{\{10,11,12\}/2}$. Finally, let $\mathcal{F}_3 = [13]^{(6)} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. Observe that:

- $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint.
- By the definition and the Pigeon Hole Principle, any two sets in $\mathcal{F}_1$ (in $\mathcal{F}_2$) have an element in common.
- If $13 \notin S$ then $S \in \mathcal{F}_1 \cup \mathcal{F}_2$, that is, $[13]^{(6)} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \subseteq [13]^{(6)}_{13}$.
- $\#\mathcal{F}_1 = \binom{9}{5} \cdot \binom{4}{1} + \binom{9}{6} = 588$, and $\#(\mathcal{F}_1 \cap [13]^{(6)}_{13}) = \binom{9}{5} = 126$.
- Consequently, by moving 16 sets $S'_1, \ldots, S'_{16}$ of $\mathcal{F}_1$ to $\mathcal{F}_3$ such that $\{13\} \in S'_i$, $i = 1, \ldots, 16$, we obtain the new $\mathcal{F}_1$ with $\#\mathcal{F}_1 = 588 - 16 = 572$.
- $\#\mathcal{F}_2 = \binom{3}{2} \cdot \binom{10}{4} + \binom{3}{3} \cdot \binom{10}{3} = 750$, and $\#(\mathcal{F}_2 \cap [13]^{(6)}_{13}) = \binom{3}{2} \cdot \binom{9}{3} + \binom{3}{3} \cdot \binom{9}{2} = 288$.
- Consequently, by moving 178 sets $S''_1, \ldots, S''_{178}$ of $\mathcal{F}_2$ to $\mathcal{F}_3$ such that $\{13\} \in S''_i$, $i = 1, \ldots, 178$, we obtain the new $\mathcal{F}_2$ with $\#\mathcal{F}_2 = 750 - 178 = 572$.
- By the construction, the new family $\mathcal{F}_3 \subseteq [13]^{(6)}_{13}$ is of size $378 + 16 + 188 = 572$.
- And hence $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ forms the desired equitable partition of $[13]^{(6)}$.

It is worth pointing out that the above approach may be applied for other arbitrary but small values of k, i.e., it fails for $k \geq 10$.

†For $k = 4$, we have $\mathcal{F}_1 \subset [9]^{(4)}_{\{1,2,3,4,5\}/3}$, $\mathcal{F}_2 \subset [9]^{(4)}_{\{6,7,8\}/2}$, and $\mathcal{F}_3 \subset [9]^{(4)}_9$.

Summarizing up, we obtain the following theorem.

Theorem 6.2. *For* $k \geq 1$, $\chi_{=}(K(2k,k)) = 2$ *and* $\chi_{=}(K(2k+1,k)) = 3$.

APPENDIX

Table 1. Equitable partition of $[n]^{(3)}$, $6 \leq n \leq 9$; for simplicity, $\{i,j,l\}$ is written as ijl.

n	$\mathcal{F}_1^n$	$\mathcal{F}_2^n$	$\mathcal{F}_3^n$	$\mathcal{F}_4^n$	$\mathcal{F}_5^n$
6	123, 124, 134, 234, 125, 135, 235, 126, 136, 236 $\#\mathcal{F}_1^6 = 10$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456 $\#\mathcal{F}_2^6 = 10$			
7	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137 $\#\mathcal{F}_1^7 = 12$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467 $\#\mathcal{F}_2^7 = 12$	147, 157, 167, 247, 257, 267, 347, 357, 367, 237, 567 $\#\mathcal{F}_3^7 = 11$		
8	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128 $\#\mathcal{F}_1^8 = 14$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458 $\#\mathcal{F}_2^8 = 14$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578 $\#\mathcal{F}_3^8 = 14$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 138, 238, 468, 568 $\#\mathcal{F}_4^8 = 14$	
9	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128, 138, 238, 129 $\#\mathcal{F}_1^9 = 17$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458, 468, 568, 459 $\#\mathcal{F}_2^9 = 17$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578, 179, 279, 379 $\#\mathcal{F}_3^9 = 17$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 189, ..., 789 $\#\mathcal{F}_4^9 = 17$	149, 159, 169, 249, 259, 269, 349, 359, 369, 479, 579, 679, 139, 239, 469, 569 $\#\mathcal{F}_5^9 = 16$

Table 2. Equitable partition of $[n]^{(3)}$, $10 \leq n \leq 13$.

n	$\mathcal{F}_1^n$	$\mathcal{F}_2^n$	$\mathcal{F}_3^n$	$\mathcal{F}_4^n$	$\mathcal{F}_5^n$	$\mathcal{F}_6^n$	$\mathcal{F}_7^n$	$\mathcal{F}_8^n$	$\mathcal{F}_9^n$
10=A	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128, 138, 238, 129, 139, 239, 12A $\#\mathcal{F}_1^{10} = 20$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458, 468, 568, 459, 469, 569, 45A $\#\mathcal{F}_2^{10} = 20$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578, 179, 279, 379 17A, 27A, 37A $\#\mathcal{F}_3^{10} = 20$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 189, ..., 789 18A, 28A, 38A $\#\mathcal{F}_4^{10} = 20$	149, 159, 169, 249, 259, 269, 349, 359, 369, 479, 579, 679, 19A, ..., 89A $\#\mathcal{F}_5^{10} = 20$	14A, 15A, 16A, 24A, 25A, 26A, 34A, 35A, 36A, 47A, 57A, 67A, 48A, ..., 78A 13A, 23A, 46A, 56A $\#\mathcal{F}_6^{10} = 20$			
11=B	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128, 138, 238, 129, 139, 239, 12A, 13A, 23A, 12B, 13B $\#\mathcal{F}_1^{11} = 24$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458, 468, 568, 459, 469, 569, 45A, 46A, 56A, 45B, 46B $\#\mathcal{F}_2^{11} = 24$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578, 179, 279, 379 17A, 27A, 37A 17B, ..., 47B $\#\mathcal{F}_3^{11} = 24$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 189, ..., 789 18A, 28A, 38A 18B, ..., 48B $\#\mathcal{F}_4^{11} = 24$	149, 159, 169, 249, 259, 269, 349, 359, 369, 479, 579, 679, 19A, ..., 89A 19B, 29B, 39B $\#\mathcal{F}_5^{11} = 23$	14A, 15A, 16A, 24A, 25A, 26A, 34A, 35A, 36A, 47A, 57A, 67A, 48A, ..., 78A 1AB, ..., 7AB $\#\mathcal{F}_6^{11} = 23$	14B, 15B, 16B, 24B, 25B, 26B, 34B, 35B, 36B, 57B, 67B, 58B, 68B, 78B, 49B, ..., 89B 8AB, 9AB, 23B, 56B $\#\mathcal{F}_7^{11} = 23$		
12=C	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128, 138, 238, 129, 139, 239, 12A, 13A, 23A, 12B, 13B, 23B, 12C, 13C, 23C $\#\mathcal{F}_1^{12} = 28$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458, 468, 568, 459, 469, 569, 45A, 46A, 56A, 45B, 46B, 56B, 45C, 46C, 56C $\#\mathcal{F}_2^{12} = 28$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578, 179, 279, 379 17A, 27A, 37A 17B, ..., 47B 17C, ..., 47C $\#\mathcal{F}_3^{12} = 28$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 189, ..., 789 18A, 28A, 38A 18B, ..., 48B 18C, ..., 48C $\#\mathcal{F}_4^{12} = 28$	149, 159, 169, 249, 259, 269, 349, 359, 369, 479, 579, 679, 19A, ..., 89A 19B, 29B, 39B 19C, ..., 49C $\#\mathcal{F}_5^{12} = 27$	14A, 15A, 16A, 24A, 25A, 26A, 34A, 35A, 36A, 47A, 57A, 67A, 48A, ..., 78A 1AB, ..., 7AB 1AC, ..., 4AC $\#\mathcal{F}_6^{12} = 27$	14B, 15B, 16B, 24B, 25B, 26B, 34B, 35B, 36B, 57B, 67B, 58B, 68B, 78B, 49B, ..., 89B 8AB, 9AB, 1BC, ..., 6BC $\#\mathcal{F}_7^{12} = 27$	14C, 15C, 16C, 24C, 25C, 26C, 34C, 35C, 36C, 57C, 67C, 58C, 68C, 78C, 59C, ..., 89C 5AC, ..., 9AC 7BC, ..., ABC $\#\mathcal{F}_8^{12} = 27$	
13=D	123, 124, 134, 234, 125, 135, 235, 126, 136, 236, 127, 137, 237, 128, 138, 238, 129, 139, 239, 12A, 13A, 23A, 12B, 13B, 23B, 12C, 13C, 23C, 12D, 13D, 23D $\#\mathcal{F}_1^{13} = 31$	145, 146, 156, 245, 246, 256, 345, 346, 356, 456, 457, 467, 567, 458, 468, 568, 459, 469, 569, 45A, 46A, 56A, 45B, 46B, 56B, 45C, 46C, 56C, 45D, 46D, 56D $\#\mathcal{F}_2^{13} = 31$	147, 157, 167, 247, 257, 267, 347, 357, 367, 178, ..., 578, 179, 279, 379 17A, 27A, 37A 17B, ..., 47B 17C, ..., 47C 17D, ..., 47D $\#\mathcal{F}_3^{13} = 32$	148, 158, 168, 248, 258, 268, 348, 358, 368, 678, 189, ..., 789 18A, 28A, 38A 18B, ..., 48B 18C, ..., 48C 18D, ..., 48D $\#\mathcal{F}_4^{13} = 32$	149, 159, 169, 249, 259, 269, 349, 359, 369, 479, 579, 679, 19A, ..., 89A 19B, 29B, 39B 19C, ..., 49C 19D, ..., 59D $\#\mathcal{F}_5^{13} = 32$	14A, 15A, 16A, 24A, 25A, 26A, 34A, 35A, 36A, 47A, 57A, 67A, 48A, ..., 78A 1AB, ..., 7AB 1AC, ..., 4AC 1AD, ..., 5AD $\#\mathcal{F}_6^{13} = 32$	14B, 15B, 16B, 24B, 25B, 26B, 34B, 35B, 36B, 57B, 67B, 58B, 68B, 78B, 49B, ..., 89B 8AB, 9AB, 1BC, ..., 6BC 1BD, ..., 5BD $\#\mathcal{F}_7^{13} = 32$	14C, 15C, 16C, 24C, 25C, 26C, 34C, 35C, 36C, 57C, 67C, 58C, 68C, 78C, 59C, ..., 89C, 5AC, ..., 9AC, 7BC, ..., ABC, 1CD, ..., 5CD $\#\mathcal{F}_8^{13} = 32$	14D, 15D, 16D, 24D, 25D, 26D, 34D, 35D, 36D, 57D, 67D, 58D, 68D, 78D, 69D, 79D, 89D, 6AD, ..., 9AD, 6BD, ..., ABD, 6CD, ..., BCD $\#\mathcal{F}_9^{13} = 32$

Table 3. Equitable partition of $[14]^{(3)}$, $[14] = \{1, 2, \ldots, 9, \mathrm{A}, \ldots, \mathrm{E}\}$.

$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_6$
123, ..., 12E, 134, ..., 13E, 234, ..., 23D $\#\mathcal{F}_1 = 33$	145, 245, 345, 457, ..., 45E, 146, 246, 346, 467, ..., 46E, 156, ..., 456, 567, ..., 56D $\#\mathcal{F}_2 = 33$	178, ..., 678, 789, ..., 78E, 179, ..., 679, 79A, ..., 79E, 189, ..., 689, 89A, ..., 89D $\#\mathcal{F}_3 = 33$	147, 157, 167, 247, 257, 267, 347, 357, 367, 17A, ..., 67A, 17B, ..., 67B, 17C, ..., 67C, 17D, ..., 67D $\#\mathcal{F}_4 = 33$	148, 158, 168, 248, 258, 268, 348, 358, 368, 18A, ..., 68A, 18B, ..., 68B, 18C, ..., 68C, 18D, ..., 68D $\#\mathcal{F}_5 = 33$	149, 159, 169, 249, 259, 269, 349, 359, 369, 19A, ..., 69A, 19B, ..., 69B, 19C, ..., 69C, 19D, ..., 69D $\#\mathcal{F}_6 = 33$
$\mathcal{F}_7$	$\mathcal{F}_8$	$\mathcal{F}_9$	$\mathcal{F}_{10}$	$\mathcal{F}_{11}$	
14A, 15A, 16A, 24A, 25A, 26A, 34A, 35A, 36A, 1AB, ..., 9AB, 1AC, ..., 9AC, 1AE, ..., 6AE $\#\mathcal{F}_7 = 33$	14B, 15B, 16B, 24B, 25B, 26B, 34B, 35B, 36B, 1BC, ..., 9BC, 1BD, ..., 5BD, 1BE, ..., ABE $\#\mathcal{F}_8 = 33$	14C, 15C, 16C, 24C, 25C, 26C, 34C, 35C, 36C, 1CD, ..., BCD, 1CE, ..., BCE, ABC, CDE $\#\mathcal{F}_9 = 33$	14D, 15D, 16D, 24D, 25D, 26D, 34D, 35D, 36D, 1AD, ..., 9AD, 6BD, ..., ABD, 1DE, ..., ADE $\#\mathcal{F}_{10} = 33$	14E, 15E, 16E, 24E, 25E, 26E, 34E, 35E, 36E, 17E, ..., 67E, 18E, ..., 68E, 19E, ..., 69E, 7AE, 8AE, 9AE, 23E, 56E, 89E, BDE $\#\mathcal{F}_{11} = 34$	

Table 4. Equitable partition of $[15]^{(3)}$, $[15] = \{1, 2, \ldots, 9, \mathrm{A}, \ldots, \mathrm{F}\}$.

$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_6$
123, ..., 12F, 134, ..., 13F, 234, ..., 23F $\#\mathcal{F}_1 = 37$	145, 245, 345, 156, ..., 15E, 256, ..., 25E, 356, ..., 35E, 456, ..., 45D $\#\mathcal{F}_2 = 38$	146, 246, 346, 167, ..., 16D, 267, ..., 26D, 367, ..., 36D, 467, ..., 46D, 567, ..., 56D $\#\mathcal{F}_3 = 38$	147, 247, 347, 178, ..., 17D, 278, ..., 27D, 378, ..., 37D, 478, ..., 47D, 578, ..., 57D, 678, ..., 67C $\#\mathcal{F}_4 = 38$	148, 248, 348, 189, ..., 18D, 289, ..., 28D, 389, ..., 38D, 489, ..., 48D, 589, ..., 58D, 689, ..., 68D, 789, ..., 78D $\#\mathcal{F}_5 = 38$	149, 249, 349, 19A, ..., 19E, 29A, ..., 29E, 39A, ..., 39E, 49A, ..., 49E, 59A, ..., 59E, 69A, ..., 69D, 79A, 79B, 79C, 89A, 89B, 89C $\#\mathcal{F}_6 = 38$
$\mathcal{F}_7$	$\mathcal{F}_8$	$\mathcal{F}_9$	$\mathcal{F}_{10}$	$\mathcal{F}_{11}$	$\mathcal{F}_{12}$
14A, 24A, 34A, 1AB, ..., 9AB, 1AC, ..., 9AC, 1AE, ..., 8AE, 1AF, ..., 9AF $\#\mathcal{F}_7 = 38$	14B, 24B, 34B, 1BC, ..., ABC, 1BD, ..., ABD, 1BE, ..., ABE, 1BF, ..., 5BF $\#\mathcal{F}_8 = 38$	14C, 24C, 34C, 1CD, ..., BCD, 1CE, ..., BCE, 1CF, ..., BCF, CDE, CDF $\#\mathcal{F}_9 = 38$	14D, 24D, 34D, 1AD, ..., 9AD, 1DE, ..., BDE, 1DF, ..., BDF, 67D, 79D, 89D, DEF $\#\mathcal{F}_{10} = 38$	14E, 24E, 34E, 16E, ..., 56E, 17E, ..., 67E, 18E, ..., 78E, 69E, 79E, 89E, 1EF, ..., CEF, 9AE,45E $\#\mathcal{F}_{11} = 38$	14F, 24F, 34F, 15F, ..., 45F, 16F, ..., 56F, 17F, ..., 67F, 18F, ..., 78F, 19F, ..., 89F, 6BF, ..., ABF $\#\mathcal{F}_{12} = 38$

6.1. Proof of inequality $(*)$

$$\binom{2k+1}{k}/3 \leq \frac{1}{2} \cdot \left[\binom{2k}{k} + \binom{2k}{k-1} - \binom{k}{t}^2\right]$$

is replaced with

$$\binom{k}{t}^2 \leq \left[\binom{2k}{k} + \binom{2k}{k-1}\right] - \frac{2}{3} \cdot \binom{2k+1}{k}$$

$$= \binom{2k+1}{k} - \frac{2}{3} \cdot \binom{2k+1}{k} = \frac{1}{3} \cdot \binom{2k+1}{k}.$$

Therefore, as $k = 2t+1$, all we need is to prove the following inequality:

$$\binom{2t+1}{t}^2 \leq \frac{1}{3} \cdot \binom{4t+3}{2t+1}.$$

Proof. The proof is by induction on t.

- $t = 1$: $\binom{3}{1}^2 = 9 \leq 35/3 = \frac{1}{3} \cdot \binom{7}{3}$,
- t: $\binom{2t+1}{t}^2 \leq \frac{1}{3} \cdot \binom{4t+3}{2t+1}$,
- $t+1$: $\binom{2t+3}{t+1}^2 \leq^? \frac{1}{3} \cdot \binom{4t+7}{2t+3}$,

$$\binom{2t+3}{t+1}^2 =$$

$$= \frac{(2t+3)!}{(t+1)! \cdot (t+2)!} \cdot \frac{(2t+3)!}{(t+1)! \cdot (t+2)!}$$

$$= \frac{(2t+1)! \cdot (2t+2) \cdot (2t+3)}{t! \cdot (t+1) \cdot (t+1)! \cdot (t+2)} \cdot \frac{(2t+1)! \cdot (2t+2) \cdot (2t+3)}{t! \cdot (t+1) \cdot (t+1)! \cdot (t+2)}$$

$$= \binom{2t+1}{t}^2 \cdot \frac{(2t+2) \cdot (2t+3) \cdot (2t+2) \cdot (2t+3)}{(t+1) \cdot (t+2) \cdot (t+1) \cdot (t+2)}$$

$$= 4 \cdot \binom{2t+1}{t}^2 \cdot \frac{(2t+3)^2}{(t+2)^2}$$

$$\leq_{\text{by i.h.}} 4 \cdot \frac{1}{3} \cdot \binom{4t+3}{2t+1} \cdot \frac{(2t+3)^2}{(t+2)^2}$$

$$= 4 \cdot \frac{1}{3} \cdot \binom{4t+7}{2t+3} \cdot \frac{(2t+2) \cdot (2t+3) \cdot (2t+3) \cdot (2t+4)}{(4t+4) \cdot (4t+5) \cdot (4t+6) \cdot (4t+7)} \cdot \frac{(2t+3)^2}{(t+2)^2}$$

$$= \frac{1}{3} \cdot \binom{4t+7}{2t+3} \cdot \frac{2 \cdot (2t+3)^3}{(4t+5) \cdot (4t+7) \cdot (t+2)}$$

$$\leq \frac{1}{3} \cdot \binom{4t+7}{2t+3}, \quad \text{as } \frac{2 \cdot (2t+3)^3}{(4t+5) \cdot (4t+7) \cdot (t+2)} \leq 1 \text{ for } t \geq 0.$$

■

6.2. Proof of inequality $(**)$

$$\binom{2k+1}{k}/3 \leq \frac{1}{2} \cdot \left[\binom{2k}{k} + \binom{2k}{k-1} - 2 \cdot \binom{k}{t} \cdot \binom{k}{t-1} \right]$$

is replaced with

$$2 \cdot \binom{k}{t} \cdot \binom{k}{t-1} \leq \left[\binom{2k}{k} + \binom{2k}{k-1} \right] - \frac{2}{3} \cdot \binom{2k+1}{k} = \frac{1}{3} \cdot \binom{2k+1}{k}.$$

Therefore, as $k = 2t$, all we need is to prove the following inequality:

$$2 \cdot \binom{2t}{t} \cdot \binom{2t}{t-1} \leq \frac{1}{3} \cdot \binom{4t+1}{2t}.$$

Proof. The proof is by induction on t.

- $t = 4$: $2 \cdot \binom{8}{4} \cdot \binom{8}{3} = 7840 \leq 8103 < 24310/3 = \frac{1}{3} \cdot \binom{17}{8}$,
- t: $2 \cdot \binom{2t}{t} \cdot \binom{2t}{t-1} \leq \frac{1}{3} \cdot \binom{4t+1}{2t}$,
- $t+1$: $2 \cdot \binom{2t+2}{t+1} \cdot \binom{2t+2}{t} \leq^{?} \frac{1}{3} \cdot \binom{4t+5}{2t+2}$,

$$2 \cdot \binom{2t+2}{t+1} \cdot \binom{2t+2}{t} =$$

$$= 2 \cdot \frac{(2t+2)!}{(t+1)! \cdot (t+1)!} \cdot \frac{(2t+2)!}{t! \cdot (t+2)!}$$

$$= 2 \cdot \frac{(2t)! \cdot (2t+1) \cdot (2t+2)}{t! \cdot (t+1) \cdot t! \cdot (t+1)} \cdot \frac{(2t)! \cdot (2t+1) \cdot (2t+2)}{(t-1)! \cdot t \cdot (t+1)! \cdot (t+2)}$$

$$
\begin{aligned}
&= 2 \cdot \binom{2t}{t} \cdot \binom{2t}{t-1} \cdot \frac{(2t+1)\cdot(2t+2)\cdot(2t+1)\cdot(2t+2)}{(t+1)\cdot(t+1)\cdot t\cdot(t+2)} \\
&= 2 \cdot 4 \cdot \binom{2t}{t} \cdot \binom{2t}{t-1} \cdot \frac{(2t+1)^2}{t\cdot(t+2)} \\
&\leq_{\text{by i.h.}} 4 \cdot \frac{1}{3} \cdot \binom{4t+1}{2t} \cdot \frac{(2t+1)^2}{t\cdot(t+2)} \\
&= 4 \cdot \frac{1}{3} \cdot \binom{4t+5}{2t+2} \cdot \frac{(2t+1)\cdot(2t+2)\cdot(2t+2)\cdot(2t+3)}{(4t+2)\cdot(4t+3)\cdot(4t+4)\cdot(4t+5)} \cdot \frac{(2t+1)^2}{t\cdot(t+2)} \\
&= \frac{1}{3} \cdot \binom{4t+5}{2t+2} \cdot \frac{(2t+1)^2\cdot(2t+2)\cdot(2t+3)}{(4t+3)\cdot(4t+5)\cdot t\cdot(t+2)} \\
&\leq \frac{1}{3} \cdot \binom{4t+5}{2t+2}, \quad \text{as } \frac{(2t+1)^2\cdot(2t+2)\cdot(2t+3)}{(4t+3)\cdot(4t+5)\cdot t\cdot(t+2)} \leq 1 \text{ for } t \geq 1.
\end{aligned}
$$

■

Acknowledgment

The authors gratefully acknowledge the suggestions and advice of the anonymous referee which allowed the authors to extend the results of the paper.

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Received 7 December 2007
Revised 9 May 2008
Accepted 9 May 2008