THE BONDAGE NUMBER OF GRAPHS: GOOD AND BAD VERTICES

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Abstract

The domination number \( \gamma(G) \) of a graph \( G \) is the minimum number of vertices in a set \( D \) such that every vertex of the graph is either in \( D \) or is adjacent to a member of \( D \). Any dominating set \( D \) of a graph \( G \) with \( |D| = \gamma(G) \) is called a \( \gamma \)-set of \( G \). A vertex \( x \) of a graph \( G \) is called: (i) \( \gamma \)-good if \( x \) belongs to some \( \gamma \)-set and (ii) \( \gamma \)-bad if \( x \) belongs to no \( \gamma \)-set. The bondage number \( b(G) \) of a nonempty graph \( G \) is the cardinality of a smallest set of edges whose removal from \( G \) results in a graph with domination number greater then \( \gamma(G) \). In this paper we present new sharp upper bounds for \( b(G) \) in terms of \( \gamma \)-good and \( \gamma \)-bad vertices of \( G \).

Keywords: bondage number, \( \gamma \)-bad/good vertex.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [11]. We denote the vertex set and the edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. The subgraph induced by \( S \subseteq V(G) \) is denoted by \( \langle S, G \rangle \). For a vertex \( x \) of \( G \), \( N(x, G) \) denote the set of all neighbors of \( x \) in \( G \), \( N[x, G] = N(x, G) \cup \{x\} \) and the degree of \( x \) is \( \deg(x, G) = |N(x, G)| \). The minimum degree of vertices in \( G \) is denoted by \( \delta(G) \) and the maximum degree by \( \Delta(G) \). If \( x \in V(G) \) and \( \emptyset \neq Y \subseteq V(G) \)
we let \(E(x, Y)\) represents the set of edges of \(G\) of the form \(xy\) where \(y \in Y\), and let \(e(x, Y) = |E(x, Y)|\).

A set \(D \subseteq V(G)\) dominates a vertex \(v \in V(G)\) if either \(v \in D\) or \(N(v, G) \cap D \neq \emptyset\). If \(D\) dominates all vertices in a subset \(T\) of \(V(G)\) we say that \(D\) dominates \(T\). When \(D\) dominates \(V(G)\), \(D\) is called a dominating set of the graph \(G\). The domination number \(\gamma(G)\) of a graph \(G\) is the minimum cardinality taken over all dominating sets of \(G\). Any dominating set of cardinality \(\gamma(G)\) is called a \(\gamma\)-set. A dominating set \(D\) is called an efficient dominating set if the distance between any two vertices in \(D\) is at least three. Not all graphs have efficient dominating sets. A vertex \(v\) of a graph \(G\) is critical if \(\gamma(G - v) < \gamma(G)\), and \(G\) is vertex domination-critical if each its vertex is critical. We refer to graphs with this property as \(vc\)-graphs.

Much has been written about the effects on domination related parameters when a graph is modified by deleting an edge. For surveys see [11, Chapter 5] and [12, Chapter 16]. One measure of the stability of the domination number of \(G\) under edge removal is the bondage number defined in [6] (previously called the domination line-stability in [2]). The bondage number \(b(G)\) of a nonempty graph \(G\) is the cardinality of a smallest set of edges whose removal from \(G\) results in a graph with domination number greater than \(\gamma(G)\). Since the domination number of every spanning subgraph of a nonempty graph \(G\) is at least as great as \(\gamma(G)\) ([11]), the bondage number of a nonempty graph is well defined. First results on bondage number can be found in a 1983 article of Bauer et al. [2].

**Theorem 1.1** (Bauer et al. [2]). If \(G\) is a nontrivial graph, then

(i) \(b(G) \leq \deg(u, G) + \deg(v, G) - 1\) for every pair of adjacent vertices \(u\) and \(v\) of \(G\);

(ii) If there exists a vertex \(v \in V(G)\) for which \(\gamma(G - v) \geq \gamma(G)\), then \(b(G) \leq \deg(v, G) \leq \Delta(G)\).

As a corollary of Theorem 1.1(i) it immediately follows the next theorem.

**Theorem 1.2** (Fink et al. [6]). If \(G\) is a graph with no isolated vertices, then \(b(G) \leq \delta(G) + \Delta(G) - 1\).

An extension of a result in Theorem 1.1 which include distance 2 vertices is the next theorem.
Theorem 1.3 (Hartnell and Rall [10] and Teschner [17]). If \( u \) and \( v \) are different vertices of \( G \) such that the distance between them is at most 2, then \( b(G) \leq \deg(u, G) + \deg(v, G) - 1 \).

A generalization of Theorem 1.2 was found independently by Hartnell and Rall [10] and Teschner [17].

Theorem 1.4 (Hartnell and Rall [10] and Teschner [17]). If \( G \) has edge-connectivity \( \lambda(G) \geq 1 \), then \( b(G) \leq \Delta(G) + \lambda(G) - 1 \).

Hartnell and Rall [9] improved the bound of Theorem 1.1(i) for adjacent vertices.

Theorem 1.5 (Hartnell and Rall [9]). For every pair of \( u \) and \( v \) of adjacent vertices of \( G \),

\[
\begin{align*}
\text{Theorem 1.5:} & \quad \text{For every pair of } u \text{ and } v \text{ of adjacent vertices of } G, \\
& \quad b(G) \leq \deg(u, G) + e(v, V(G) - N[u, G]) = \deg(u, G) + \deg(v, G) - 1 - |N(u, G) \cap N(v, G)|.
\end{align*}
\]

In [18], Wang, by careful consideration of the nature of the edges from the neighbors of \( u \) and \( v \), further refine this bound.

Theorem 1.6 (Wang [18]). For each edge \( uv \) of a graph \( G \), let

\[
\begin{align*}
T_1(u, v) &= N[u, G] \cap N(v, G), \\
T_2(u, v) &= \{ w : w \in N(v, G) \text{ and } N[w, G] \subseteq N[v, G] - \{u\} \}, \\
T_3(u, v) &= \{ w : w \in N(v, G) \text{ and } N[w, G] \subseteq N[x, G] - \{u\}, \text{ where } x \in N(u, G) \cap N(v, G) \}, \text{ and} \\
T_4(u, v) &= \{ w : w \in N(v, G) - (T_1(u, v) \cup T_2(u, v) \cup T_3(u, v)) \}. \\
\end{align*}
\]

Then \( b(G) \leq \min_{u \in V(G), v \in N(u, G)} \{ \deg(u, G) + |T_4(u, v)| \} \).

The concept of \( \gamma \)-bad/good vertices in graphs was introduced by Fricke et al. in [7]. A vertex \( v \) of a graph \( G \) is called:

(i) [7] \( \gamma \)-good, if \( v \) belongs to some \( \gamma \)-set of \( G \) and

(ii) [7] \( \gamma \)-bad, if \( v \) belongs to no \( \gamma \)-set of \( G \).

For a graph \( G \) we define:

\[
\begin{align*}
\mathbf{G}(G) &= \{ x \in V(G) : x \text{ is } \gamma \text{-good} \}; \\
\mathbf{B}(G) &= \{ x \in V(G) : x \text{ is } \gamma \text{-bad} \}; \\
V^{-}(G) &= \{ x \in V(G) : \gamma(G - x) < \gamma(G) \}.
\end{align*}
\]

Clearly, \( \{ \mathbf{G}(G), \mathbf{B}(G) \} \) is a partition of \( V(G) \). In this paper we present new sharp upper bounds for \( b(G) \) in terms of \( \gamma \)-good and \( \gamma \)-bad vertices of \( G \).
2. Good and Bad Vertices

Our main result in this section is the next theorem.

**Theorem 2.1.** Let $G$ be a graph.

(i) If $V(G) \neq V^{-}(G)$, then $b(G) \leq \min \{ \deg(x, G) - (\gamma(G) - x) - \gamma(G) \} : x \in V(G) - V^{-}(G)$.  

(ii) If $G$ has a $\gamma$-bad vertex, then $b(G) \leq \min \{|N(x, G) \cap G(G)| : x \in B(G)\}$.  

(iii) If $V_{-}^{-}(G) = \{ x \in V^{-}(G) : \deg(x, G) \geq 1 \} \neq \emptyset$, then $b(G) \leq \min_{x \in V_{-}^{-}(G), y \in B(G-x)} \{ \deg(x, G) + |N(y, G) \cap G(G-x)| \}$.  

**Proof.** Notice that if $x \in V(G)$ is isolated then $x$ is critical and $\gamma$-good.  

(i) Let $x \in V(G)$ with $\gamma(G - x) = \gamma(G) + p$, $p \geq 0$. If $p = 0$, then $b(G) \leq \deg(x, G)$ by Theorem 1.1 (ii). Now, we need the following lemma.

**Lemma 2.1.1 ([2]).** If $v$ is a vertex of a graph $G$ and $\gamma(G - v) > \gamma(G)$, then $v$ is not an isolate and is in every $\gamma$-set of $G$.

We return to the proof of Theorem 2.1. Assume $p \geq 1$. By the above lemma, it follows that $x$ is in every $\gamma$-set of $G$. Let $M$ be a $\gamma$-set of $G$. Then $Q = (M - \{x\}) \cup N(x, G)$ is a dominating set of $G - x$ which implies $\gamma(G) + p = \gamma(G - x) \leq |Q| = \gamma(G) - 1 + \deg(x, G)$. Hence $1 \leq p \leq \deg(x, G) - 1$. Let $S \subseteq E(x, N(x, G)) = E_x$ and $|S| = \deg(x, G) - p$. Then $\gamma(G - S) \geq \gamma(G - E_x) = \gamma(G - x) + 1 - p = \gamma(G) + 1$ which implies $b(G) \leq |S| = \deg(x, G) + \gamma(G) - \gamma(G - x)$.

(ii) **Fact 1.** Let $x \in B(G), y \in G(G), xy \in E(G)$ and $\gamma(G - xy) = \gamma(G)$. Then $G(G - xy) \subseteq G(G)$ and $B(G - xy) \supseteq B(G)$.  

**Proof.** Every $\gamma$-set of $G - xy$ is a $\gamma$-set of $G$. \hfill $\square$

**Fact 2.** If $x \in B(G)$, then $\gamma(G - E(x, G(G))) > \gamma(G)$.  

**Proof.** Assume to the contrary, that $\gamma(G_{1}) = \gamma(G)$, where $G_{1} = G - E(x, G(G))$. By Fact 1 we have $B(G_{1}) \supseteq B(G)$ which implies $N[x, G_{1}] \subseteq B(G_{1})$. But this is clearly impossible. \hfill $\square$

The result immediately follows by Fact 2.
(iii) Let \( x \in V_1^-(G) \) and \( M \) be a \( \gamma \)-set of \( G - x \). Then clearly no neighbor of \( x \) is in \( M \) which implies \( \emptyset \neq N(x, G) \subseteq B(G - x) \). Since 
\[ \gamma(G - E(x, N(x, G))) = \gamma(G) \]
it follows that \( b(G) \leq \deg(x, G) + b(G - x) \).

By (ii), \( b(G - x) \leq |N(y, G) \cap G(G - x)| \) for any \( y \in B(G - x) \). Hence 
\[ b(G) \leq \deg(x, G) + |N(y, G) \cap G(G - x)|. \]

\[ \square \]

**Lemma 2.2.** Under the notation of Theorem 1.6, if \( u \) is critical, then 
\( (T_1(u, v) - \{u\}) \cup T_2(u, v) \cup T_3(u, v) \subseteq N(v, G) \cap B(G - u). \)

**Proof.** From definitions \( T_1(u, v) \cup T_2(u, v) \cup T_3(u, v) \subseteq N(v, G) \). By the proof of Theorem 2.1 (iii), \( N(u, G) \subseteq B(G - u) \). Since \( T_1(u, v) - \{u\} \subseteq N(u, G) \) we have \( T_1(u, v) - \{u\} \subseteq B(G - u) \).

Observe that if \( H \) is a graph, \( z \in B(H), y \in V(H) \) and \( N[y, H] \subseteq N[z, H] \) then clearly \( y \in B(H) \). From this fact and \( N(u, G) \subseteq B(G - u) \) it immediately follows that \( T_2(u, v) \cup T_3(u, v) \subseteq B(G - u). \)

By Lemma 2.2, if \( u \) is a critical vertex of a graph \( G \), then 
\[ \deg(u, G) + \min_{v \in N(u, G)} \{T_4(u, v)\} \geq \deg(u, G) + \min_{v \in N(u, G)} \{|N(v, G) \cap G(G - u)|\}. \]

Hence Theorem 1.6 (and clearly Theorems 1.1, 1.2 and 1.5) can be seen to follow from Theorem 2.1. Any graph \( G \) with \( b(G) \) achieving the upper bound of some of Theorems 1.1, 1.2, 1.5 and 1.6 can be used to show that the bound of Theorem 2.1 is sharp. For such examples see [5, 6, 9, 14, 18].

**Example 2.3.** Let \( t \geq 2 \) be an integer. Let \( H_1, H_2, \ldots, H_{t+1} \) be mutually vertex-disjoint graphs such that \( H_{t+1} \) is isomorphic to \( K_{t+3} \) and \( H_i \) is isomorphic to \( K_{t+3} - e \) for \( i = 1, 2, \ldots, t \). Let \( x_{t+1} \in V(H_{t+1}) \) and \( x_i, x_{i+2} \in V(H_i) \), \( x_{i+1}x_{i+2} \notin E(H_i) \) for \( i = 1, 2, \ldots, t \). Define a graph \( R_t \) as follows:
\[ V(R_t) = \{u, v\} \cup (\cup_{k=1}^{t+1} V(H_k)) \]
\[ E(R_t) = (\cup_{k=1}^{t+1} E(H_k)) \cup (\cup_{i=1}^{t+1} \{ux_i, ux_{i+2}\}) \cup \{ux_{t+1}, uv\}. \]

Observe that \( \gamma(R_t) = t + 2, G(R_t) = V(R_t), deg(u, R_t) = 2t + 2, deg(x_{t+1}, R_t) = t + 3, deg(v, R_t) = 1 \) and for each \( y \in V(R_t - \{u, v, x_{t+1}\}) \), \( deg(y, R_t) = t + 2 \). Moreover, \( v \) is a critical vertex and if \( y \in V(R_t) - \{v\} \) then \( \gamma(R_t - y) = \gamma(R_t) \). Hence each of the bounds stated in Theorems 1.1–1.6 is greater than or equals \( t + 2 \).

Consider the graph \( R_t - uv \). Clearly \( \gamma(R_t - uv) = \gamma(R_t) \) and \( B(R_t - uv) = B(R_t - v) = \{u\} \cup V(H_{t+1} - x_{t+1}) \cup (\cup_{k=1}^{t} \{x_{k+1}, x_{k+2}\}) \). Therefore \( N(u, R_t) \cap \)
\[ G(R_t - v) = \{x_{t+1}\} \] which implies that the upper bound stated in Theorem 2.1 (iii) is equals to \( \deg(v, R_t) + |\{x_{t+1}\}| = 2 \). Clearly \( b(R_t) = 2 \) and hence this bound is sharp for \( R_t \).

Figure 1. The graph \( R_2 \).

By Example 2.3, it immediately follows:

**Remark 2.4.** For every integer \( t \geq 2 \), the difference between any upper bound stated in Theorems 1.1–1.6 and the upper bound of Theorem 2.1(iii), provided \( G = R_t \), is greater than or equals \( t \).

3. **vc-Graphs**

The concept of vc-graphs plays an important role in the study of the bondage number. For instance, it immediately follows from Theorem 1.1(ii) that if \( b(G) > \Delta(G) \) then \( G \) is vc-graph. The bondage number of a vc-graphs is examined in [15]. If \( G \) is a vc-graph then \( |V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1 \). In this section we give an upper bound for the bondage number of such vc-graphs. We need the following results.

**Theorem 3.1.** Let \( G \) be a vc-graph.

(i) [3] Then \( |V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1 \).
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(ii) [8] If \(|V(G)| = (\Delta(G) + 1)(\gamma(G) - 1) + 1\), then \(G\) is regular.

Theorem 3.2 [1]. Let \(G\) be a graph.

(i) If \(G\) has vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\), then \(G\) has an efficient dominating set if and only if some subcollection of \(\{N[v_1, G], N[v_2, G], \ldots, N[v_n, G]\}\) partitions \(V(G)\).

(ii) If \(G\) has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of \(G\).

Lemma 3.3. Let \(x\) and \(v\) be different critical vertices of a graph \(G\). Let \(v\) belong to some efficient dominating set of \(G - x\) and let \(G - v\) have an efficient dominating set. Then \(x\) belongs to all efficient dominating sets of \(G - v\) and \(v\) belongs to all efficient dominating sets of \(G - x\).

Proof. Let \(M\) be an arbitrary efficient dominating set of \(G - v\), \(Q\) be an efficient dominating set of \(G - x\) and \(v \in Q\). Hence the closed neighborhoods of each two different vertices of \(Q\) are vertex disjoint and each vertex of \(Q - \{v\}\) dominates a unique vertex of \(M\), by Theorem 3.2. Since \(|M| = \gamma(G - v) = \gamma(G) - 1 = \gamma(G - x) = |Q|\), there exists exactly one vertex in \(M\), say \(w\), which is not dominated by \(Q - v\). If \(w \neq x\) then \(w\) must be dominated by \(v\), which is impossible because \(|M| = \gamma(G - v) < \gamma(G)\) implies that \(M\) does not dominate \(v\) in \(G\). Therefore \(x\) belongs to all efficient dominating sets of \(G - v\). By symmetry, \(v\) belongs to all efficient dominating sets of \(G - x\).

Theorem 3.4. Let \(G\) be a vc-graph with \((\Delta(G) + 1)(\gamma(G) - 1) + 1\) vertices. Then for every vertex \(x \in V(G)\), \(G - x\) has exactly one \(\gamma\)-set and the unique \(\gamma\)-set of \(G - x\) is efficient dominating.

Proof. Let \(x\) be an arbitrary vertex of \(G\) and \(M\) an arbitrary \(\gamma\)-set of \(G - x\). Since \(|M| = \gamma(G) - 1\) and \(\Delta(G - x) \leq \Delta(G)\), it follows that \(|V(G - x)| \leq |M|((\Delta(G - x) + 1) \leq (\gamma(G) - 1)(\Delta(G) + 1) = |V(G - x)|\).

Hence each element of \(M\) dominates exactly \(\Delta(G) + 1\) vertices in \(G - x\) and the closed neighborhoods of all vertices of \(M\) form a partition of \(V(G - x)\). Now Theorem 3.2 implies that \(M\) is an efficient dominating set of \(G - x\). Hence for each \(u \in V(G)\), all \(\gamma\)-sets of \(G - u\) are efficient dominating. Now if \(v\) is a \(\gamma\)-good vertex of \(G - x\) then by Lemma 3.3, \(v\) belongs to all efficient dominating sets of \(G - x\). Hence \(G - x\) has exactly one \(\gamma\)-set.
Lemma 3.5 ([17]). If $G$ is a nontrivial graph with a unique minimum dominating set, then $b(G) = 1$.

Lemma 3.6. Let $G$ be a graph, $x \in V^-(G)$, $\deg(x, G) \geq 1$ and let $G - x$ have exactly one $\gamma$-set. Then $b(G) \leq \deg(x, G) + 1$.

Proof. It follows by Lemma 3.5 that $b(G - x) = 1$. Hence $b(G) \leq e(x, N(x, G)) + b(G - x) = \deg(x, G) + 1$.

We now state and prove the principal result of this section.

Theorem 3.7 (Teschner [15] when $\gamma(G) = 3$). If $G$ is a nontrivial vc-graph with $(\Delta(G) + 1)\gamma(G) - 1 + 1$ vertices, then $b(G) \leq \Delta(G) + 1 = \delta(G) + 1$.

Proof. Let $x \in V(G)$. By Theorem 3.1 (ii), $\deg(x, G) = \Delta(G) = \delta(G)$ and by Theorem 3.4, $G - x$ has exactly one $\gamma$-set. The result immediately follows by Lemma 3.6.

4. Open Problems

Conjecture 4.1 (Teschner [15]). For any vc-graph $G$, $b(G) \leq 1.5\Delta(G)$.

Teschner [15] has shown that Conjecture 4.1 is true when $\gamma(G) \leq 3$. Note that if $G = K_t \times K_t$ for a positive integer $t \geq 2$, then $b(G) = 1.5\Delta(G)$ as was found independently by Hartnell and Rall [9] and Teschner [16].

Conjecture 4.2 (Hailong Liu and Liang Sun [13]). For any positive integer $r$, there exists a vc-graph $G$ such that $b(G) \geq \Delta(G) + k(G) + r$ where $k(G)$ is the vertex connectivity of $G$.

Motivated by Theorem 2.1(iii) and Theorem 3.6 we state the following:

Conjecture 4.3. For every connected nontrivial vc-graph $G$, \[
\min_{x \in V(G), y \in B(G-x)} \{\deg(x, G) + |N(y, G) \cap G(G-x)|\} \leq 1.5\Delta(G).
\]

Conjecture 4.4. If $G$ is a vc-graph with $(\Delta(G) + 1)(\gamma(G) - 1) + 1$ vertices then $b(G) = \Delta(G) + 1$. 
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