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THE BONDAGE NUMBER OF GRAPHS: GOOD AND BAD VERTICES

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Abstract

The *domination number* $\gamma(G)$ of a graph G is the minimum number of vertices in a set D such that every vertex of the graph is either in D or is adjacent to a member of D. Any dominating set D of a graph G with $|D| = \gamma(G)$ is called a γ-set of G. A vertex x of a graph G is called: (i) γ-good if x belongs to some γ-set and (ii) γ-bad if x belongs to no γ-set. The *bondage number* $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater then $\gamma(G)$. In this paper we present new sharp upper bounds for $b(G)$ in terms of γ-good and γ-bad vertices of G.

Keywords: bondage number, γ-bad/good vertex.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, *et al.* [11]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x, G)$ denote the set of all neighbors of x in G, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of x is $deg(x, G) = |N(x, G)|$. The minimum degree of vertices in G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. If $x \in V(G)$ and $\emptyset \neq Y \subseteq V(G)$

we let $E(x, Y)$ represents the set of edges of G of the form xy where $y \in Y$, and let $e(x, Y) = |E(x, Y)|$.

A set $D \subseteq V(G)$ *dominates* a vertex $v \in V(G)$ if either $v \in D$ or $N(v, G) \cap D \neq \emptyset$. If D dominates all vertices in a subset T of $V(G)$ we say that D *dominates* T. When D dominates $V(G)$, D is called a *dominating set* of the graph G. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. Any dominating set of cardinality $\gamma(G)$ is called a γ-*set*. A dominating set D is called an *efficient dominating set* if the distance between any two vertices in D is at least three. Not all graphs have efficient dominating sets. A vertex v of a graph G is *critical* if $\gamma(G - v) < \gamma(G)$, and G is *vertex domination-critical* if each its vertex is critical. We refer to graphs with this property as *vc-graphs*.

Much has been written about the effects on domination related parameters when a graph is modified by deleting an edge. For surveys see [11, Chapter 5] and [12, Chapter 16]. One measure of the stability of the domination number of G under edge removal is the bondage number defined in [6] (previously called the *domination line-stability* in [2]). The *bondage number* $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$ ([11]), the bondage number of a nonempty graph is well defined. First results on bondage number can be found in a 1983 article of Bauer *et al.* [2].

Theorem 1.1 (Bauer *et al.* [2])**.** *If G is a nontrivial graph, then*

(i) $b(G) \leq deg(u, G) + deg(v, G) - 1$ *for every pair of adjacent vertices u and v of G;*

(ii) *If there exists a vertex $v \in V(G)$ for which $\gamma(G - v) \geq \gamma(G)$, then* $b(G) \leq deg(v, G) \leq \Delta(G)$.

As a corollary of Theorem 1.1(i) it immediately follows the next theorem.

Theorem 1.2 (Fink *et al.* [6])**.** *If G is a graph with no isolated vertices, then* $b(G) \leq \delta(G) + \Delta(G) - 1$.

An extension of a result in Theorem 1.1 which include distance 2 vertices is the next theorem.

Theorem 1.3 (Hartnell and Rall [10] and Teschner [17]). *If u and v are different vertices of G such that the distance between them is at most* 2, *then* $b(G) \leq deg(u,G) + deg(v,G) - 1$.

A generalization of Theorem 1.2 was found independently by Hartnell and Rall [10] and Teschner [17].

Theorem 1.4 (Hartnell and Rall [10] and Teschner [17]). *If G has edge-connectivity $\lambda(G) \geq 1$, then $b(G) \leq \Delta(G) + \lambda(G) - 1$.*

Hartnell and Rall [9] improved the bound of Theorem 1.1(i) for adjacent vertices.

Theorem 1.5 (Hartnell and Rall [9]). *For every pair of u and v of adjacent vertices of G, $b(G) \leq \deg(u,G) + e(v, V(G) - N[u,G]) = deg(u,G) + deg(v,G) - 1 - |N(u,G) \cap N(v,G)|$.*

In [18], Wang, by careful consideration of the nature of the edges from the neighbors of u and v, further refine this bound.

Theorem 1.6 (Wang [18]). *For each edge uv of a graph G, let*
$T_1(u,v) = N[u,G] \cap N(v,G)$,
$T_2(u,v) = \{w : w \in N(v,G) \text{ and } N[w,G] \subseteq N[v,G] - \{u\}\}$,
$T_3(u,v) = \{w : w \in N(v,G) \text{ and } N[w,G] \subseteq N[x,G] - \{u\}$, *where*
$x \in N(u,G) \cap N(v,G)\}$, *and*
$T_4(u,v) = \{w : w \in N(v,G) - (T_1(u,v) \cup T_2(u,v) \cup T_3(u,v))\}$.
Then $b(G) \leq min_{u \in V(G), v \in N(u,G)}\{deg(u,G) + |T_4(u,v)|\}$.

The concept of γ-bad/good vertices in graphs was introduced by Fricke *et al.* in [7]. A vertex v of a graph G is called:

(i) [7] γ-*good*, if v belongs to some γ-set of G and

(ii) [7] γ-*bad*, if v belongs to no γ-set of G.

For a graph G we define:
$\mathbf{G}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-good}\}$;
$\mathbf{B}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-bad}\}$;
$V^-(G) = \{x \in V(G) : \gamma(G - x) < \gamma(G)\}$.

Clearly, $\{\mathbf{G}(G), \mathbf{B}(G)\}$ is a partition of $V(G)$. In this paper we present new sharp upper bounds for $b(G)$ in terms of γ-good and γ-bad vertices of G.

2. Good and Bad Vertices

Our main result in this section is the next theorem.

Theorem 2.1. *Let G be a graph.*

(i) *If $V(G) \neq V^-(G)$, then $b(G) \leq \min\{\deg(x,G) - (\gamma(G-x) - \gamma(G)) : x \in V(G) - V^-(G)\}$.*

(ii) *If G has a γ-bad vertex, then $b(G) \leq \min\{|N(x,G) \cap \mathbf{G}(G)| : x \in \mathbf{B}(G)\}$.*

(iii) *If $V_1^-(G) = \{x \in V^-(G) : \deg(x,G) \geq 1\} \neq \emptyset$, then $b(G) \leq \min_{x \in V_1^-(G), y \in \mathbf{B}(G-x)}\{\deg(x,G) + |N(y,G) \cap \mathbf{G}(G-x)|\}$.*

Proof. Notice that if $x \in V(G)$ is isolated then x is critical and γ-good.

(i) Let $x \in V(G)$ with $\gamma(G-x) = \gamma(G) + p$, $p \geq 0$. If $p = 0$, then $b(G) \leq deg(x,G)$ by Theorem 1.1 (ii). Now, we need the following lemma.

Lemma 2.1.1 ([2]). *If v is a vertex of a graph G and $\gamma(G-v) > \gamma(G)$, then v is not an isolate and is in every γ-set of G.*

We return to the proof of Theorem 2.1. Assume $p \geq 1$. By the above lemma, it follows that x is in every γ-set of G. Let M be a γ-set of G. Then $Q = (M - \{x\}) \cup N(x,G)$ is a dominating set of $G - x$ which implies $\gamma(G) + p = \gamma(G-x) \leq |Q| = \gamma(G) - 1 + deg(x,G)$. Hence $1 \leq p \leq deg(x,G) - 1$. Let $S \subseteq E(x, N(x,G)) = E_x$ and $|S| = \deg(x,G) - p$. Then $\gamma(G-S) \geq \gamma(G-E_x) - p = \gamma(G-x) + 1 - p = \gamma(G) + 1$ which implies $b(G) \leq |S| = \deg(x,G) + \gamma(G) - \gamma(G-x)$.

(ii) **Fact 1.** *Let $x \in \mathbf{B}(G), y \in \mathbf{G}(G), xy \in E(G)$ and $\gamma(G-xy) = \gamma(G)$. Then $\mathbf{G}(G-xy) \subseteq \mathbf{G}(G)$ and $\mathbf{B}(G-xy) \supseteq \mathbf{B}(G)$.*

Proof. Every γ-set of $G - xy$ is a γ-set of G. □

Fact 2. *If $x \in \mathbf{B}(G)$, then $\gamma(G - E(x, \mathbf{G}(G))) > \gamma(G)$.*

Proof. Assume to the contrary, that $\gamma(G_1) = \gamma(G)$, where $G_1 = G - E(x, \mathbf{G}(G))$. By Fact 1 we have $\mathbf{B}(G_1) \supseteq \mathbf{B}(G)$ which implies $N[x, G_1] \subseteq \mathbf{B}(G_1)$. But this is clearly impossible. □

The result immediately follows by Fact 2.

(iii) Let $x \in V_1^-(G)$ and M be a γ-set of $G - x$. Then clearly no neighbor of x is in M which implies $\emptyset \neq N(x,G) \subseteq \mathbf{B}(G-x)$. Since $\gamma(G - E(x, N(x,G))) = \gamma(G)$ it follows that $b(G) \leq deg(x,G) + b(G-x)$. By (ii), $b(G-x) \leq |N(y,G) \cap \mathbf{G}(G-x)|$ for any $y \in \mathbf{B}(G-x)$. Hence $b(G) \leq deg(x,G) + |N(y,G) \cap \mathbf{G}(G-x)|$. ■

Lemma 2.2. *Under the notation of Theorem* 1.6, *if u is critical, then* $(T_1(u,v) - \{u\}) \cup T_2(u,v) \cup T_3(u,v) \subseteq N(v,G) \cap \mathbf{B}(G-u)$.

Proof. From definitions $T_1(u,v) \cup T_2(u,v) \cup T_3(u,v) \subseteq N(v,G)$. By the proof of Theorem 2.1 (iii), $N(u,G) \subseteq \mathbf{B}(G-u)$. Since $T_1(u,v) - \{u\} \subseteq N(u,G)$ we have $T_1(u,v) - \{u\} \subseteq \mathbf{B}(G-u)$. Observe that if H is a graph, $z \in \mathbf{B}(H), y \in V(H)$ and $N[y,H] \subseteq N[z,H]$ then clearly $y \in \mathbf{B}(H)$. From this fact and $N(u,G) \subseteq \mathbf{B}(G-u)$ it immediately follows that $T_2(u,v) \cup T_3(u,v) \subseteq \mathbf{B}(G-u)$. ■

By Lemma 2.2, if u is a critical vertex of a graph G, then

$\deg(u,G) + \min_{v \in N(u,G)}\{|T_4(u,v)|\} \geq \deg(u,G) + \min_{v \in N(u,G)}\{|N(v,G) \cap \mathbf{G}(G-u)|\} \geq \deg(u,G) + \min_{v \in \mathbf{B}(G-u)}\{|N(v,G) \cap \mathbf{G}(G-u)|\}$.

Hence Theorem 1.6 (and clearly Theorems 1.1, 1.2 and 1.5) can be seen to follow from Theorem 2.1. Any graph G with $b(G)$ achieving the upper bound of some of Theorems 1.1, 1.2, 1.5 and 1.6 can be used to show that the bound of Theorem 2.1 is sharp. For such examples see [5, 6, 9, 14, 18].

Example 2.3. Let $t \geq 2$ be an integer. Let $H_1, H_2, \ldots, H_{t+1}$ be mutually vertex-disjoint graphs such that H_{t+1} is isomorphic to K_{t+3} and H_i is isomorphic to $K_{t+3} - e$ for $i = 1, 2, \ldots, t$. Let $x_{t+1} \in V(H_{t+1})$ and $x_{i1}, x_{i2} \in V(H_i)$, $x_{i1}x_{i2} \notin E(H_i)$ for $i = 1, 2, \ldots, t$. Define a graph R_t as follows:

$V(R_t) = \{u, v\} \cup (\cup_{k=1}^{t+1} V(H_k))$ and

$E(R_t) = (\cup_{k=1}^{t+1} E(H_k)) \cup (\cup_{i=1}^{t}\{ux_{i1}, ux_{i2}\}) \cup \{ux_{t+1}, uv\}$.

Observe that $\gamma(R_t) = t+2, \mathbf{G}(R_t) = V(R_t), \deg(u, R_t) = 2t+2, \deg(x_{t+1}, R_t) = t+3, \deg(v, R_t) = 1, \lambda(R_t) = 1$ and for each $y \in V(R_t - \{v, u, x_{t+1}\})$, $\deg(y, R_t) = t+2$. Moreover, v is a critical vertex and if $y \in V(R_t) - \{v\}$ then $\gamma(R_t - y) = \gamma(R_t)$. Hence each of the bounds stated in Theorems 1.1–1.6 is greater than or equals $t+2$.

Consider the graph $R_t - uv$. Clearly $\gamma(R_t - uv) = \gamma(R_t)$ and $\mathbf{B}(R_t - uv) = \mathbf{B}(R_t - v) = \{u\} \cup V(H_{t+1} - x_{t+1}) \cup (\cup_{k=1}^{t}\{x_{k1}, x_{k2}\})$. Therefore $N(u, R_t) \cap$

$\mathbf{G}(R_t - v) = \{x_{t+1}\}$ which implies that the upper bound stated in Theorem 2.1 (iii) is equals to $\deg(v, R_t) + |\{x_{t+1}\}| = 2$. Clearly $b(R_t) = 2$ and hence this bound is sharp for R_t.

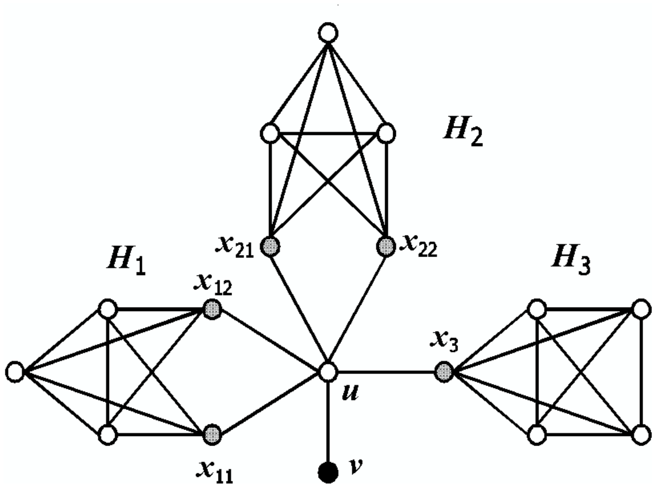

Figure 1. The graph R_2.

By Example 2.3, it immediately follows:

Remark 2.4. For every integer $t \geq 2$, the difference between any upper bound stated in Theorems 1.1–1.6 and the upper bound of Theorem 2.1(iii), provided $G = R_t$, is greater than or equals t.

3. vc-Graphs

The concept of vc-graphs plays an important role in the study of the bondage number. For instance, it immediately follows from Theorem 1.1(ii) that if $b(G) > \Delta(G)$ then G is vc-graph. The bondage number of a vc-graphs is examined in [15]. If G is a vc-graph then $|V(G)| \leq (\Delta(G)+1)(\gamma(G)-1)+1$. In this section we give an upper bound for the bondage number of such vc-graphs. We need the following results.

Theorem 3.1. *Let G be a vc-graph.*

(i) [3] *Then* $|V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$.

(ii) [8] *If* $|V(G)| = (\Delta(G)+1)(\gamma(G)-1)+1$, *then* G *is regular.*

Theorem 3.2 [1]**.** *Let* G *be a graph.*

(i) *If* G *has vertex set* $V(G) = \{v_1, v_2, \ldots, v_n\}$, *then* G *has an efficient dominating set if and only if some subcollection of* $\{N[v_1, G], N[v_2, G], \ldots, N[v_n, G]\}$ *partitions* $V(G)$.

(ii) *If* G *has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of* G.

Lemma 3.3. *Let* x *and* v *be different critical vertices of a graph* G. *Let* v *belong to some efficient dominating set of* $G - x$ *and let* $G - v$ *have an efficient dominating set. Then* x *belongs to all efficient dominating sets of* $G - v$ *and* v *belongs to all efficient dominating sets of* $G - x$.

Proof. Let M be an arbitrary efficient dominating set of $G - v$, Q be an efficient dominating set of $G-x$ and $v \in Q$. Hence the closed neighborhoods of each two different vertices of Q are vertex disjoint and each vertex of $Q - \{v\}$ dominates a unique vertex of M, by Theorem 3.2. Since $|M| = \gamma(G-v) = \gamma(G)-1 = \gamma(G-x) = |Q|$, there exists exactly one vertex in M, say w, which is not dominated by $Q-v$. If $w \neq x$ then w must be dominated by v, which is impossible because $|M| = \gamma(G-v) < \gamma(G)$ implies that M does not dominate v in G. Therefore x belongs to all efficient dominating sets of $G - v$. By symmetry, v belongs to all efficient dominating sets of $G - x$. ■

Theorem 3.4. *Let* G *be a vc-graph with* $(\Delta(G)+1)(\gamma(G)-1)+1$ *vertices. Then for every vertex* $x \in V(G)$, $G-x$ *has exactly one* γ*-set and the unique* γ*-set of* $G - x$ *is efficient dominating.*

Proof. Let x be an arbitrary vertex of G and M an arbitrary γ-set of $G - x$. Since $|M| = \gamma(G) - 1$ and $\Delta(G - x) \leq \Delta(G)$, it follows that $|V(G - x)| \leq |M|(\Delta(G - x) + 1) \leq (\gamma(G) - 1)(\Delta(G) + 1) = |V(G - x)|$. Hence each element of M dominates exactly $\Delta(G)+1$ vertices in $G-x$ and the closed neighborhoods of all vertices of M form a partition of $V(G - x)$. Now Theorem 3.2 implies that M is an efficient dominating set of $G - x$. Hence for each $u \in V(G)$, all γ-sets of $G - u$ are efficient dominating. Now if v is a γ-good vertex of $G-x$ then by Lemma 3.3, v belongs to all efficient dominating sets of $G - x$. Hence $G - x$ has exactly one γ-set. ■

Lemma 3.5 ([17])**.** *If G is a nontrivial graph with a unique minimum dominating set, then $b(G) = 1$.*

Lemma 3.6. *Let G be a graph, $x \in V^-(G)$,* $\deg(x, G) \geq 1$ *and let $G - x$ have exactly one γ-set. Then $b(G) \leq deg(x, G) + 1$.*

Proof. It follows by Lemma 3.5 that $b(G - x) = 1$. Hence $b(G) \leq e(x, N(x, G)) + b(G - x) = \deg(x, G) + 1$. ■

We now state and prove the principal result of this section.

Theorem 3.7 (Teschner [15] when $\gamma(G) = 3$)**.** *If G is a nontrivial vc-graph with $(\Delta(G) + 1)(\gamma(G) - 1) + 1$ vertices, then $b(G) \leq \Delta(G) + 1 = \delta(G) + 1$.*

Proof. Let $x \in V(G)$. By Theorem 3.1 (ii), $deg(x, G) = \Delta(G) = \delta(G)$ and by Theorem 3.4, $G - x$ has exactly one γ-set. The result immediately follows by Lemma 3.6. ■

4. Open Problems

Conjecture 4.1 (Teschner [15])**.** For any vc-graph G, $b(G) \leq 1.5\Delta(G)$.

Teschner [15] has shown that Conjecture 4.1 is true when $\gamma(G) \leq 3$. Note that if $G = K_t \times K_t$ for a positive integer $t \geq 2$, then $b(G) = 1.5\Delta(G)$ as was found independently by Hartnell and Rall [9] and Teschner [16].

Conjecture 4.2 (Hailong Liu and Liang Sun [13])**.** For any positive integer r, there exists a vc-graph G such that $b(G) \geq \Delta(G) + k(G) + r$ where $k(G)$ is the vertex connectivity of G.

Motivated by Theorem 2.1(iii) and Theorem 3.6 we state the following:

Conjecture 4.3. For every connected nontrivial vc-graph G,
$\min_{x \in V(G), y \in \mathbf{B}(G-x)}\{\deg(x, G) + |N(y, G) \cap \mathbf{G}(G - x)|\} \leq 1.5\Delta(G)$.

Conjecture 4.4. If G is a vc-graph with $(\Delta(G)+1)(\gamma(G)-1)+1$ vertices then $b(G) = \Delta(G) + 1$.

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