

ON LOCATING AND DIFFERENTIATING-TOTAL DOMINATION IN TREES

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Abstract

A total dominating set of a graph $G = (V, E)$ with no isolated vertex is a set $S \subseteq V$ such that every vertex is adjacent to a vertex in S . A total dominating set S of a graph G is a locating-total dominating set if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and S is a differentiating-total dominating set if for every pair of distinct vertices u and v in V , $N[u] \cap S \neq N[v] \cap S$. Let $\gamma_t^L(G)$ and $\gamma_t^D(G)$ be the minimum cardinality of a locating-total dominating set and a differentiating-total dominating set of G , respectively. We show that for a nontrivial tree T of order n , with ℓ leaves and s support vertices, $\gamma_t^L(T) \geq \max\{2(n + \ell - s + 1)/5, (n + 2 - s)/2\}$, and for a tree of order $n \geq 3$, $\gamma_t^D(T) \geq 3(n + \ell - s + 1)/7$, improving the lower bounds of Haynes, Henning and Howard. Moreover we characterize the trees satisfying $\gamma_t^L(T) = 2(n + \ell - s + 1)/5$ or $\gamma_t^D(T) = 3(n + \ell - s + 1)/7$.

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1. INTRODUCTION

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is the size of its open neighborhood. A *leaf* of a tree T

is a vertex of degree one, while a *support vertex* of T is a vertex of degree at least two adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. We denote the order of a tree T by n , the number of leaves by ℓ , and the number of support vertices by s . A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A *subdivided star* SS_q is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex. A corona of a graph H is the graph G formed from H by adding a new vertex v' for each vertex $v \in V(H)$ and the edge $v'v$. For a subset $S \subseteq V$, we denote by $\langle S \rangle$ the subgraph induced by the vertices of S .

A subset S of vertices of V is a *total dominating set* of G if every vertex in V is adjacent to a vertex in S . The *total domination number*, $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G .

In this paper we are interested in two types of total-dominating sets, namely locating-total dominating sets, and differentiating-total dominating sets defined as follows: A total dominating set S of a graph G is called a *locating-total dominating set* (LTDS) if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and S is called a *differentiating-total dominating set* (DTDS) if for every pair of distinct vertices u and v in V , $N[u] \cap S \neq N[v] \cap S$. The *locating-total domination number*, $\gamma_t^L(G)$ is the minimum cardinality of a LTDS of G , and the *differentiating-total domination number*, $\gamma_t^D(G)$ is the minimum cardinality of a DTDS of G . A LTDS of minimum cardinality is called a $\gamma_t^L(G)$ -set. Likewise we define a $\gamma_t^D(G)$ -set. Note that a tree T of order n admits a LTDS (resp., DTDS) if $n \geq 2$ (resp., $n \geq 3$) since the entire vertex set is such a set. Also for every $\gamma_t^D(G)$ -set D there is no component of size 2 in the subgraph induced by D , for otherwise the two vertices u, v of such a component would satisfy $N[u] \cap D = N[v] \cap D = \{u, v\}$. Locating-total domination and differentiating-total domination were introduced by Haynes, Henning and Howard [4].

In this paper we establish sharp bounds on $\gamma_t^L(T)$, and $\gamma_t^D(T)$ for trees T . More precisely, we show that if T is a tree of order $n \geq 2$, with ℓ leaves and s support vertices, then $\gamma_t^L(T) \geq \max\{2(n + \ell - s + 1)/5, (n + 2 - s)/2\}$ and if T is a tree of order $n \geq 3$, then $\gamma_t^D(T) \geq 3(n + \ell - s + 1)/7$. Then we give a characterization of trees with $\gamma_t^L(T) = 2(n + \ell - s + 1)/5$, or $\gamma_t^D(T) = 3(n + \ell - s + 1)/7$.

We sometimes consider the removing of an edge of a tree T . If uv is an edge of T , then we denote by T_u (resp., T_v) the subtree of T that contains

u (resp., v) obtained by removing uv . The following notation and fact will be used in the proofs. Let n_1, ℓ_1, s_1 be the order, the number of leaves and support vertices of T_u , respectively, and likewise let n_2, ℓ_2, s_2 for T_v . Clearly $n_1 + n_2 = n$, and if n_1 and $n_2 \geq 3$, then $\ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, where q is the number of new support vertices in T_u and T_v with $0 \leq q \leq 2$. Also if D is a $\gamma_t^L(T)$ -set or $\gamma_t^D(T)$ -set, then let $D_u = D \cap V(T_u)$, and $D_v = D \cap V(T_v)$.

2. LOWER BOUNDS ON $\gamma_t^L(T)$

In [4], Haynes, Henning and Howard gave two lower bounds on the locating-total domination number for trees and characterized extremal trees for each lower bound. Let $G = P_n$ be the path on n vertices.

Theorem 1 (Haynes, Henning and Howard [4]).

- (1) If T is a tree of order $n \geq 2$, then $\gamma_t^L(T) \geq 2(n+1)/5$.
- (2) For $n \geq 2$, $\gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Theorem 2 (Haynes, Henning and Howard [4]). *If T is a tree of order $n \geq 3$ with ℓ leaves and s support vertices, then $\gamma_t^L(T) \geq (n + 2(\ell - s) + 1)/3$.*

Our next result improves the lower bound of Theorem 1 for every nontrivial tree T . It also improves Theorem 2 for trees of order $n \geq 4\ell - 4s$. Let \mathcal{F} be the family of trees that can be obtained from r disjoint copies of P_4 and P_3 by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 3. *If T is a tree of order $n \geq 2$, then*

$$\gamma_t^L(T) \geq 2(n + \ell - s + 1)/5,$$

with equality if and only if $T = P_2$ or $T \in \mathcal{F}$.

Proof. We proceed by induction on the order of T . If $n = 2$, then $T = P_2$ and $\gamma_t^L(P_2) = 2(n + \ell - s + 1)/5 = 2$. Every star $K_{1,p}$ ($p \geq 2$) satisfies $\gamma_t^L(K_{1,p}) = p \geq 2(n + \ell - s + 1)/5$ with equality if and only if $p = 2$, that is $T = P_3 \in \mathcal{F}$. This establishes the base cases. Assume that every tree T' of order $2 \leq n' < n$ satisfies $\gamma_t^L(T') \geq 2(n' + \ell' - s' + 1)/5$. Let T be a

tree of order n . Among all $\gamma_t^L(T)$ -sets, let D be one that contains as few leaves as possible. Note that every vertex x of D has at most one private neighbor in $V - D$ for if it had two private neighbors x', x'' , then we would have $N(x') \cap D = N(x'') \cap D = \{x\}$. If $\ell = 2$, then T is a path P_n , and by Theorem 1, $\gamma_t^L(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor \geq 2(n + \ell - s + 1)/5$ with equality if and only if $T = P_4 \in \mathcal{F}$. Thus we may assume that $\ell \geq 3$.

Assume that T contains a strong support vertex y adjacent to at least three leaves. Then D contains y and all its leaves except possibly one. Let $y' \in D$ be any leaf adjacent to y , and let $T' = T - \{y\}$. Clearly $D - \{y'\}$ is a LTDS of T' , $n' = n - 1, \ell' = \ell - 1$, and $s' = s$. By induction on T' , we have $|D - \{y'\}| \geq \gamma_t^L(T') \geq 2(n' + \ell' - s' + 1)/5$, implying that $|D| > 2(n + \ell - s + 1)/5$. Thus every support vertex is adjacent to at most two leaves.

Assume that $\langle D \rangle$ contains a connected component $\langle D_i \rangle$ of diameter at least 3. Then there exists an edge uv , such that $\langle D_i - uv \rangle$ contains no isolated vertices. Clearly each of T_u and T_v has order at least three, D_u and D_v are two LTDS of T_u and T_v , respectively. Recall that $n_1 + n_2 = n, \ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, where $0 \leq q \leq 2$ is defined above. Applying the inductive hypothesis to T_u and T_v , we obtain

$$|D| = |D_u| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.$$

Thus every component of $\langle D \rangle$ has diameter one or two.

If each $w \in V - D$ is a leaf, then D contains for each support vertex all its leaves except possibly one. Hence $|D| \geq n - s$. Since $\ell \geq 3$ and $n - s \geq \ell$, it follows $|D| \geq n - s > 2(n + \ell - s + 1)/5$. Thus there exists a vertex $w \in V - D$ such that w is not a leaf. Assume now that w has a neighbor say $v \in V - D$. Then each of T_w and T_v has order at least three, D_w and D_v are two LTDS of T_w and T_v , respectively. By induction on T_w and T_v , we obtain

$$|D| = |D_w| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.$$

Hence we may assume that $V - D$ is an independent set and so every private neighbor of a vertex of D is a leaf. Suppose now that w has degree at least three and let z be any vertex of $N(w) \cap D$. By removing wz , then T_w has order at least three. If $V(T_z) = \{z, z'\}$ then z' is a leaf of T and so $\{w\} \cup D - \{z'\}$ is a $\gamma_t^L(T)$ -set with less leaves than D , contradicting our assumption on D . Thus T_z has order at least three. Also D_w and D_z are

two LTDS of T_w and T_z , respectively. The rest of the proof is similar to as shown above, which leads to $|D| > 2(n + \ell - s + 1)/5$. Thus every vertex of $V - D$ is either a leaf or has degree two. Note that all cases considered until now do not lead to extremal trees because $|D| > 2(n + \ell - s + 1)/5$. Let W be the set of vertices of $V - D$ having degree two. Since T is a tree, $|W| = k - 1$ where k is the number of connected components of $\langle D \rangle$. Let T' be the forest induced by the vertices of $V(T) - W$ and let T_1, T_2, \dots, T_k the components of T' . Then $n_1 + \dots + n_k = n - |W|$, $\ell_1 + \dots + \ell_k \geq \ell + q$, and $s_1 + \dots + s_k = s + q$, where q is the number of new support vertices. Also $D \cap V(T_i) = D_i$ is a LTDS of T_i , for every $i = 1, \dots, k$. By induction on each T_i , we obtain

$$\begin{aligned} |D| &= \sum_{i=1}^k |D_i| \geq \sum_{i=1}^k 2(n_i + \ell_i - s_i + 1)/5 \geq 2(n - |W| + \ell - s + k)/5 \\ &= 2(n + \ell - s + 1)/5. \end{aligned}$$

Assume now that $\gamma_t^L(T) = 2(n + \ell - s + 1)/5$. Then we have equality throughout the above inequality chain. In particular, $\gamma_t^L(T_i) = 2(n_i + \ell_i - s_i + 1)/5$ for each i , and $\ell_1 + \dots + \ell_k = \ell + q$, and $s_1 + \dots + s_k = s + q$. This means that T' has a new leaf if and only if it has a new support vertex. So each T_i has order at least three. Recall that every component $\langle D_i \rangle$ has diameter one or two. Suppose that for some i , $\langle D_i \rangle$ has diameter two, that is $\langle D_i \rangle$ is a star of center vertex say, x and leaves y_1, y_2, \dots, y_t with $t \geq 2$. We distinguish between three cases. If x is not a support vertex neither in T nor in T_i , then each y_i is support vertex of T_i , and so T_i is a subdivided star but $\gamma_t^L(T_i) > 2(n_i + \ell_i - s_i + 1)/5$, a contradiction. If x is not a support vertex of T but x is a support vertex of T_i , then T_i is a corona of $K_{1,t-1}$, where $|D_i| = t + 1 > 2(n_i + \ell_i - s_i + 1)/5$. Now if x is a support vertex of T with at most two leaves, then every y_j is a support vertex in T_i for either $1 \leq j \leq t$ or $2 \leq j \leq t$, but then $\gamma_t^L(T_i) > 2(n_i + \ell_i - s_i + 1)/5$, a contradiction.

Finally, assume that each connected subgraph $\langle D_i \rangle$ is of diameter one. Then $T_i = P_3$ or P_4 , and the leaves of T_i are leaves in T . Thus every component of T' is either a path P_3 or P_4 where every vertex of W joins two support vertices of any different components T_i, T_j .

Conversely, let $T \in \mathcal{F}$ be a tree obtained from k_1 disjoint copies of P_4 and k_2 disjoint copies of P_3 with $k_1 + k_2 \geq 1$, by adding $k_1 + k_2 - 1$ new vertices, where each new vertex is adjacent to exactly two support vertices.

Clearly the set of all support vertices plus one leaf from each copy of P_3 forms a minimum LTDS of T of size $2(n + \ell - s + 1)/5$. So extremal trees T achieving $\gamma_t^L(T) = 2(n + \ell - s + 1)/5$ are precisely those of \mathcal{F} . ■

Note that in [2], Chellali and Haynes showed that every nontrivial tree satisfies $\gamma_t(T) \geq (n + 2 - \ell)/2$. Since every LTDS is a total dominating set, $\gamma_t^L(T) \geq (n + 2 - \ell)/2$. Our next result improves this lower bound.

Theorem 4. *If T is a tree of order $n \geq 2$, then $\gamma_t^L(T) \geq (n + 2 - s)/2$.*

Proof. We proceed by induction on the order of T . It is a routine matter to check that the result holds if $\text{diam}(T) \in \{1, 2\}$. Assume that every tree T' of order $2 \leq n' < n$ satisfies $\gamma_t^L(T') \geq (n' + 2 - s')/2$. Let T be a tree of order n and S a $\gamma_t^L(T)$ -set that contains leaves as few as possible.

If the subgraph induced by $V - S$ contains some edge xy , then let T_x and T_y be the trees obtained by removing the edge xy where $x \in T_x$ and $y \in T_y$. Clearly each of T_x and T_y has order at least three, S_x and S_y are two LTDS of T_x and T_y , respectively. Also $n_1 + n_2 = n$, and $s_1 + s_2 = s + q$, where q is the number of new support vertices with $0 \leq q \leq 2$. By induction on T_x and T_y , we have $|S| = |S_x| + |S_y| \geq (n_1 + 2 - s_1)/2 + (n_2 + 2 - s_2)/2 = (n + 4 - s - q) \geq (n + 2 - s)/2$. Thus $V - S$ is independent.

Let w be a vertex of $V - S$ different to a leaf. If w does not exist, then $|S| \geq n - s \geq (n + 2 - s)/2$, since $s \geq 2$. Thus w exists and has at least two neighbors in S . Let z be any neighbor of $N(w) \cap S$, and consider the trees T_w and T_z obtained by removing the edge wz where $w \in T_w$ and $z \in T_z$. If $V(T_z) = \{z, z'\}$, then z' is a leaf of T and $\{w\} \cup S - \{z'\}$ is a $\gamma_t^L(T)$ -set with less leaves than S , a contradiction with our choice of S . Thus T_z has order at least three. Also S_w and S_z are two LTDS of T_w and T_z , respectively. Hence by induction on T_w and T_z and since $n_1 + n_2 = n$, and $s_1 + s_2 = s + q$, where $0 \leq q \leq 2$ is defined as above, we obtain $|S| = |S_w| + |S_z| \geq (n_1 + 2 - s_1)/2 + (n_2 + 2 - s_2)/2 = (n + 4 - s - q) \geq (n + 2 - s)/2$. This achieves the proof. ■

The lower bound of Theorem 4 is sharp for the path P_n with $n \equiv 0 \pmod{4}$ and improves Theorem 3 for nontrivial trees with $n > 4\ell + s - 6$.

3. LOWER BOUND ON $\gamma_t^D(T)$

In [4], Haynes, Henning and Howard gave a lower bound of the differentiating-total domination number of any tree with at least three vertices.

Theorem 5 (Haynes, Henning and Howard [4]).

- (1) If T is a tree of order $n \geq 3$, then $\gamma_t^D(T) \geq 3(n+1)/7$.
- (2) For $n \geq 3$, $\gamma_t^D(P_n) = \lceil 3n/5 \rceil + 1$ if $n \equiv 3 \pmod{5}$ and $\gamma_t^D(P_n) = \lceil 3n/5 \rceil$, otherwise.

Note that as mentioned in [4], since $\gamma_t^D(T) \geq \gamma_t^L(T)$ for trees, the bound of Theorem 2 is also a lower bound for $\gamma_t^D(T)$.

A subset S of vertices of V is an *identifying code* (or a *differentiating domination set*) as defined in [3] if for every pair of distinct vertices u and v in V , $N[u] \cap S \neq N[v] \cap S \neq \emptyset$.

In [1], Blidia *et al.* showed for trees of order $n \geq 4$ that every identifying code contains at least $3(n + \ell - s + 1)/7$ vertices. Since every differentiating-total dominating set is an identifying code, $3(n + \ell - s + 1)/7$ is a lower bound for $\gamma_t^D(T)$ which improves Theorem 5. Note that the lower bound $3(n + \ell - s + 1)/7$ is better than $(n + 2(\ell - s) + 1)/3$ for trees with $2n > 5\ell - 5s - 2$. For the purpose of characterizing extremal trees we give here a proof of $\gamma_t^D(T) \geq 3(n + \ell - s + 1)/7$, by using a similar argument to that used in the proof of Theorem 3.

Let \mathcal{G} be the family of trees that can be obtained from r disjoint copies of a corona of P_3 , a double star $S_{2,1}$ and a star $K_{1,3}$ by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 6. *If T is a tree of order $n \geq 3$, then*

$$\gamma_t^D(T) \geq 3(n + \ell - s + 1)/7,$$

with equality if and only if $T \in \mathcal{G}$.

Proof. We use an induction on the order of T . If $\text{diam}(T) = 2$, then $T = K_{1,p}$ ($p \geq 2$). Thus $\gamma_t^D(K_{1,2}) = 3 > 3(n + \ell - s + 1)/7$ and for $p \geq 3$, $\gamma_t^D(K_{1,p}) = p \geq 3(n + \ell - s + 1)/7$ with equality if and only if $p = 3$, that is $T = K_{1,3} \in \mathcal{G}$. If $\text{diam}(T) = 3$, then $T = S_{p,q}$. Thus $\gamma_t^D(S_{1,1}) = 3 > 3(n + \ell - s + 1)/7$ and for $\max\{p, q\} \geq 2$, $\gamma_t^D(S_{p,q}) = p + q \geq 3(n + \ell - s + 1)/7$ with equality if and only if $p + q = 3$, that is $T = S_{2,1} \in \mathcal{G}$. This establishes the base cases. Assume that every tree T' of diameter at least 4 and order n' , $5 \leq n' < n$ satisfies $\gamma_t^D(T') \geq 3(n' + \ell' - s' + 1)/7$. Let T be a tree of order n , and D a $\gamma_t^D(T)$ -set. If T is a path P_n with $n \geq 5$, then by Theorem 5(2), $\gamma_t^D(P_n) > 3(n + \ell - s + 1)/7$. Thus we assume that $\ell \geq 3$.

If any strong support vertex y is adjacent to at least four leaves, then let $T' = T - \{y'\}$, where y' is any leaf adjacent to y . Without loss of generality $y' \in D$, and then $D - \{y'\}$ is a DTDS of T' . Hence by induction on T' we have $|D| - 1 \geq \gamma_t^D(T') \geq 3(n' + \ell' - s' + 1)/7$. Since $n' = n - 1, \ell' = \ell - 1$, and $s' = s$, we obtain $|D| > 3(n + \ell - s + 1)/7$. For the next we assume that each support vertex is adjacent to at most three leaves.

Assume, the subgraph $\langle D \rangle$ contains a connected component $\langle D_i \rangle$ of diameter at least 5. Thus there exists an edge uv , such that each connected component of $\langle D_i - uv \rangle$ has diameter at least 2. Then D_u and D_v are two DTDS of T_u and T_v , respectively. Since $n_1 + n_2 = n, \ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, then by induction on T_u and T_v , we obtain

$$|D| = |D_u| + |D_v| \geq 3(n_1 + \ell_1 - s_1 + 1)/7 + 3(n_2 + \ell_2 - s_2 + 1)/7 > 3(n + \ell - s + 1)/7.$$

Thus every component of $\langle D \rangle$ has diameter two, three or four.

Suppose that $\langle V - D \rangle$ contains some edge uv . Then by removing the edge uv , each of T_u and T_v has order at least four, D_u and D_v are two DTDS of T_u and T_v , respectively. By using the induction on T_u and T_v , it follows that $\gamma_t^D(T) > 3(n + \ell - s + 1)/7$. Thus $V - D$ is independent and hence every private neighbor of a vertex of D is a leaf.

Let w be any vertex of $V - D$ different to a leaf. If w does not exist, then $|D| \geq n - s \geq 3(n + \ell - s + 1)/7$ with equality only if T is a corona of a path P_3 or a double star $S_{2,1}$. Thus $T \in \mathcal{G}$. Now if w has degree at least three, then let z be any vertex of $N(w) \cap D$. Then by removing wz , T_w has order at least seven and T_z has order at least three, D_w and D_z are two DTDS of T_w and T_z , respectively. The rest of the proof is similar to as shown above and so $|D| > 3(n + \ell - s + 1)/7$. Thus every vertex of $V - D$ is either a leaf or has degree two.

Let W be the set of vertices of $V - D$ having degree two. Since T is a tree, $|W| = k - 1$ where k is the number of connected components of $\langle D \rangle$. Let T' be the forest induced by the vertices of $V(T) - W$ and let T_1, T_2, \dots, T_k the components of T' . Then $n_1 + \dots + n_k = n - |W|, \ell_1 + \dots + \ell_k \geq \ell + q$, and $s_1 + \dots + s_k = s + q$, where q is the number of new support vertices. Also $D \cap V(T_i) = D_i$ is a DTDS of T_i , for every $i = 1, \dots, k$. By induction on each T_i , we obtain

$$\begin{aligned} |D| &= \sum_{i=1}^k |D_k| \geq \sum_{i=1}^k 3(n_i + \ell_i - s_i + 1)/7 \geq 3(n - |W| + \ell - s + k)/7 \\ &= 3(n + \ell - s + 1)/7. \end{aligned}$$

Assume now that $\gamma_t^D(T) = 3(n + \ell - s + 1)/7$. Then we have equality throughout this inequality chain. In particular, $\gamma_t^D(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ for each i , and $\ell_1 + \dots + \ell_k = \ell + q$, and $s_1 + \dots + s_k = s + q$. Thus T' contains a new leaf if and only if it has a new support vertex. So each T_i has order at least four. Recall that each component of $\langle D \rangle$ has diameter two, three or four. We first assume that the subgraph $\langle D_i \rangle$ has diameter three or four. We will show that no leaf of T_i is contained in $\langle D_i \rangle$. Assume to the contrary that a leaf $y \in V(T_i) \cap D_i$ and let $z \in D_i$ be its support vertex. Note that y may be a new leaf in T_i . Consider the tree $T'_i = T_i - \{y\}$. Then $\langle D_i - \{y\} \rangle$ has diameter at least two and $D_i - \{y\}$ is a DTDS of T'_i , with $n'_i = n_i - 1$, $\ell'_i \geq \ell_i - 1$, and $s'_i \leq s_i$. It follows that $|D_i - \{y\}| \geq \gamma_t^D(T'_i) \geq 3(n'_i + \ell'_i - s'_i + 1)/7$ and so $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction since $|D_i| = 3(n_i + \ell_i - s_i + 1)/7$. Thus $\langle D_i \rangle$ contains no leaf of T_i and hence every support vertex of T_i is adjacent to exactly one leaf. Now let k_1 be the number of support vertices of T_i . Thus T_i has k_1 leaves. Let $k_2 = n_i - 2k_1$. Clearly $k_1 + k_2 \geq 4$ since $\langle D_i \rangle$ is a component of diameter three or four, but then $|D_i| = k_1 + k_2 > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction.

Thus for each $i = 1, \dots, k$, the subgraph $\langle D_i \rangle$ has diameter two, and so $\langle D_i \rangle$ is a star of center vertex x and leaves y_1, y_2, \dots, y_t with $t \geq 2$. Note that $|D_i| = t + 1$. If x is not a support vertex neither in T nor in T_i , then each y_i is support vertex of T_i . Hence T_i is a subdivided star with $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction. If x is not a support vertex of T but it is a vertex support of T_i , then T_i is a corona of $K_{1,t-1}$, where $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$. Now if x is a support vertex of T with at most three leaves, then every y_j is a support vertex in T_i for either $1 \leq j \leq t$, $2 \leq j \leq t$, or $3 \leq j \leq t$, but then $\gamma_t^L(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ if and only if $T_i = K_{1,3}, S_{2,1}$ or T_i is a corona of a path P_3 . Thus every component of T' is either a path $K_{1,3}, S_{2,1}$ or corona of P_3 where every vertex of W joins two support vertices. Therefore extremal trees T achieving $\gamma_t^L(T) = 3(n + \ell - s + 1)/7$ are precisely those of \mathcal{G} .

The converse is easy to show. ■

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