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MINIMAL CYCLE BASES OF THE LEXICOGRAPHIC PRODUCT OF GRAPHS

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Abstract

A construction of minimum cycle bases of the lexicographic product of graphs is presented. Moreover, the length of a longest cycle of a minimal cycle basis is determined.

Keywords: cycle space, lexicographic product, cycle basis.

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1. Introduction

Minimum cycle bases (MCBs) of a cycle spaces have a variety of applications in sciences and engineering, for example, in structural flexibility analysis, electrical networks and in chemical structure storage and retrieval systems (see [6, 7] and [14]).

In general, the total length $l(G)$ of a minimum cycle basis and the length of the longest cycle in a minimum cycle basis $\lambda(G)$ are not minor monotone (see [11]). Hence, there does not seem to be a general way of extending minimum cycle bases of a certain collection of partial graphs of G to a minimum cycle basis of G. Global upper bound $l(G) \leq \dim \mathcal{C}(G) + \kappa(T(G))$ where $\kappa(T(G))$ is the connectivity of the tree graph of G is proven in [16].

In this paper, we construct a minimum cycle basis for the lexicographic product of two graphs in term of a minimum cycle basis of the second factor, also, we give its total length and the length of its longest cycle.

2. Definitions and Preliminaries

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [10]. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

2.1. Cycle bases

Given graph G. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \varnothing$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, .)$ spanned by the cycles of G (see [11]). Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that the dimension of the cycle space is the *cyclomatic number* or the *first Betti number* (see [4])

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1. \tag{1}$$

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. The *length,* $|C|$, of the element C of the cycle space $\mathcal{C}(G)$ is the number of its edges. The *length* $l(\mathcal{B})$ *of a cycles basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. $\lambda(G)$ is defined to be the minimum length of the longest element in an arbitrary cycle basis of G. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. With $l(G)$ we denote to the sum of the lengths of the cycles in a minimum cycle basis. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element C has weight $|C|$, the greedy algorithm can be used to extract a MCB (see [19]). A cycle is *relevant* if it is contained in some MCB (see [18]).

Proposition 2.1.1 (Plotkin [17])**.** *A cycle C is relevant if and only if it cannot be written as a linear combinations modulo* 2 *of shorter cycles.*

Chickering, Geiger and Heckerman [5], showed that $\lambda(G)$ is the length of the longest element in a MCB.

2.2. Products

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

(1) The *Cartesian product* $G\Box H$ has the vertex set $V(G\Box H) = V(G) \times V(H)$ and the edge set $E(G\Box H) = \{(u_1, v_1)(u_2, v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$.

(2) The *strong product* $G \boxtimes H$ is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$, or $u_1v_1 \in E(G)$ and $u_2 = v_2\}$.

(3) The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$, or $u_1v_1 \in E(G)\}$.

(4) The *wreath product* $G \ltimes H$ has the vertex set $V(G \ltimes H) = V(G) \times V(H)$ and the edge set $E(G \ltimes H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$.

(5) The *direct product* $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.

The H-fiber $a\Box H$ is the subgraph of a product induced with the vertex set $V(u\Box H) = \{(a, v) | v \in V(H)\}$. Analogue is defined the G-fiber $G\Box v$. Let $e \in E(G)$. Then $e[H]$ is called a *fold* of $G[H]$ (See [1] and [10]).
From (1) and by knowing that $|E(G[H])| = |E(G)||V(H)|^2 + |E(H)||V(G)|$, we have that

$$\text{(2)} \quad \dim \mathcal{C}(G[H]) = |E(G)||V(H)|^2 + |E(H)||V(G)| - |V(G)||V(H)| + 1$$

Imrich and Stadler [11] presented a minimum cycle bases of Cartesian product and strong product. They proved the following results:

Theorem 2.2.1. *If G and H are triangle free, then $l(G\Box H) = l(G) + l(H) + 4[|E(G)|(|V(H)| - 1) + |E(H)|(|V(G)| - 1) - (|V(H)| - 1)(|V(G)| - 1)]$ and $\lambda(G\Box H) = \max\{4, \lambda(G), \lambda(H)\}$.*

Theorem 2.2.2. *For any two graphs G and H, $l(G \boxtimes H) = l(G) + l(H) + 3$ $[\dim C(G \boxtimes H) - \dim C(G) - \dim C(H)]$ and $\lambda(G \boxtimes H) = \max\{3, \lambda(G), \lambda(H)\}$.*

Hammack [8] gave a minimal cycle basis of the direct product of two bipartite graphs in term of the minimum cycle bases of the factors. Also, Hammack [9] and Bradshaw and Jaradat [3] presented a minimal cycle bases of the direct product of complete graphs.

In [12] minimum cycle bases of the wreath product have been constructed for some classes of graphs and determined their length and the length of their longest cycles.

The results cited above trigger off the following question: Can we construct a minimum cycle basis for the lexicographic product of graphs? In this paper we will answer this question in affirmative.

After this manuscript was completed the author learned that a similar statement to our main result was proven by Berger in her PhD dissertation [2] and Kaveh and Mirzaie [15].

3. Lexicographic Product

In this section, we construct a minimum cycle basis and determine the length of a minimum cycle basis and the minimum length of the longest cycle in an arbitrary cycle basis of the lexicographic product of two graphs. For any edges ab, uv and a vertex w, we set the following cycles:

$$\mathcal{P}_{ab,w}^{uv} = (a,w)(b,u)(b,v)(a,w).$$

Also, for a graph H with $E(H) = \{u_1v_1, u_2v_2, \ldots, u_{|E(H)|}v_{|E(H)|}\}$, we let

$$\mathcal{P}_{ab,w}^{H} = \bigcup_{i=1}^{|E(H)|} \mathcal{P}_{ab,w}^{u_iv_i}.$$

Lemma 3.1. *For any tree T with $w \in V(T)$ and any edge $ab, \mathcal{P}_{ab,w}^{T}$ is linearly independent. Moreover, any linear combination of $\mathcal{P}_{ab,w}^{T}$ contains an edge of the form $(a,w)(b,u)$ for some vertex $u \in V(T)$.*

Proof. Note that $\mathcal{P}_{ab,w}^{T} = \cup_{i=1}^{|E(T)|}\mathcal{P}_{ab,w}^{u_iv_i}$. The first part follows from noting that each cycle $\mathcal{P}_{ab,w}^{u_iv_i}$ contains the edge $b \times u_iv_i$ which occurs in no other cycle of $\mathcal{P}_{ab,w}^{T}$. The second part follows from being that $E(\mathcal{P}_{ab,w}^{T}) = \{(a,w)(b,u) | u \in V(T)\} \cup E(b\Box T)$, and noting that $E(b\Box T)$ is an edge set of a tree and any linear combination of cycles is a cycle or an edge disjoint union of cycles. ∎

The following result from [13] will be needed in the forthcoming results:

Proposition 3.2. *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \varnothing$). Then $A \cup B$ is linearly independent.*

Lemma 3.3. *Let T be a tree of order greater than or equal to 2 and ab be an edge. Then $\mathcal{P}^T_{ab} = (\cup_{w\in V(T)}\mathcal{P}^T_{ab,w}) \cup \mathcal{P}^T_{ba,w_0}$ is a linearly independent set of cycles for some fixed vertex $w_0 \in V(T)$.*

Proof. Since $\mathcal{P}^T_{ab,w}$ is linearly independent for each $w \in V(T)$ by Lemma 3.1 and since $E(\mathcal{P}^T_{ab,w}) \cap E(\mathcal{P}^T_{ab,v}) = E(b \times T)$ whenever $w \neq v$ which is a tree, as a result by Proposition 3.2 $\cup_{w\in V(T)}\mathcal{P}^T_{ab,w}$ is linearly independent. Also, since

$$E\left(\cup_{w\in V(T)}\mathcal{P}^T_{ab,w}\right) \cap E\left(\mathcal{P}^T_{ba,w_0}\right) = \{(a,w)(b,w_0) \,|\, w \in V(T)\}$$

which is an edge set of a star, we have $\mathcal{P}^T_{ab}$ is linearly independent by Proposition 3.2. ■

Now, let G be a graph with $E(G) = \{a_1b_1, a_2b_2, \ldots, a_{|E(G)|}b_{|E(G)|}\}$. Also, let

$$\mathcal{P}^T_G = \bigcup_{i=1}^{|E(G)|} \mathcal{P}^T_{a_ib_i}.$$

Lemma 3.4. *Let G be any graph and T be a tree. Then $\mathcal{P}^T_G$ is a linearly independent set.*

Proof. We use mathematical induction on $|E(G)|$. If G consists only of one edge, say $G = a_1b_1$, then $\mathcal{P}^T_G = \mathcal{P}^T_{a_1b_1}$. And so, the result follows by Lemma 3.3. Assume the statement is true for all graphs with less edges than G. Note that $\mathcal{P}^T_G = \left(\cup_{i=1}^{|E(G)|-1}\mathcal{P}^T_{a_ib_i}\right) \cup \mathcal{P}^T_{a_{|E(G)|}b_{|E(G)|}}$. By the inductive step and Lemma 3.3, each of $\cup_{i=1}^{|E(G)|-1}\mathcal{P}^T_{a_ib_i}$ and $\mathcal{P}^T_{a_{|E(G)|}b_{|E(G)|}}$ is linearly independent. Note that

$$E\left(\bigcup_{i=1}^{|E(G)|-1} \mathcal{P}^T_{a_ib_i}\right) \cap E\left(\mathcal{P}^T_{a_{|E(G)|}b_{|E(G)|}}\right) \subseteq E\left(\{a_{|E(G)|}, b_{|E(G)|}\}\Box T\right)$$

which is an edge set of a forest. Thus, by Proposition 3.2, $\mathcal{P}^T_G$ is linearly independent. ■

Throughout this paper, T_G denotes a spanning tree of the graph G.

Lemma 3.5. *$\mathcal{P}^T_G \cup \mathcal{B}_{G\Box w_0}$ is linearly independent where $\mathcal{B}_{G\Box w_0}$ is a basis for the G-fiber $G\Box w_0$.*

Proof. Note that if G is a tree, then $\mathcal{B}_{G\Box w_0} = \varnothing$ and so the result is obtaind by Lemma 3.4. Therefore, we now consider the case that G is not a tree. By Lemma 3.4, $\mathcal{P}^T_{T_G}$ is linearly independent. Now, since $\mathcal{B}_{G\Box w_0}$ is a basis of the G-fiber $G\Box w_0$, any linear combination of $\mathcal{B}_{G\Box w_0}$ must contain an edge of $E((G - T_G)\Box w_0)$, which is not in any cycle of $\mathcal{P}^T_{T_G}$. Thus, $\mathcal{P}^T_{T_G} \cup \mathcal{B}_{G\Box w_0}$ is linearly independent. Now, we proceed using mathematical induction on the number of edges in $G - T_G$ to show that $\mathcal{P}^T_G \cup \mathcal{B}_{G\Box w_0}$ is linearly independent. Let $E(G - T_G) = \{a_{|E(T_G)|+1}b_{|E(T_G)|+1}, a_{|E(T_G)|+2}b_{|E(T_G)|+2}, \dots, a_{|E(G)|}b_{|E(G)|}\}$. Note that

$$\begin{aligned}
&E\left(\mathcal{P}^T_{a_{|E(T_G)|+1}b_{|E(T_G)|+1}}\right) \cap E\left(\mathcal{P}^T_{T_G} \cup \mathcal{B}_{G\Box w_0}\right)\\
&= E\left(\{a_{|E(T_G)|+1}, b_{|E(T_G)|+1}\}\Box T\right) \cup \{(a_{|E(T_G)|+1}, w_0)(b_{|E(T_G)|+1}, w_0)\}
\end{aligned}$$

which is an edge set of a tree. Thus, by Proposition 3.2, $\mathcal{P}^T_{T_G} \cup \mathcal{B}_{G\Box w_0} \cup \mathcal{P}^T_{a_{|E(T_G)|+1}b_{|E(T_G)|+1}}$ is linearly independent. By continuing in this way, using mathematical induction and by Lemma 3.3, we have that both of $\mathcal{P}^T_{T_G} \cup \mathcal{B}_{G\Box w_0} \cup \left(\cup_{i=|E(T_G)|+1}^{|E(G)|-1} \mathcal{P}^T_{a_i b_i}\right)$ and $\mathcal{P}^T_{a_{|E(G)|}b_{|E(G)|}}$ are linearly independent. Now,

$$\begin{aligned}
&E\left(\mathcal{P}^T_{a_{|E(G)|}b_{|E(G)|}}\right) \cap E\left((\mathcal{P}^T_{T_G} \cup \mathcal{B}_{G\Box w_0} \cup \left(\bigcup_{i=|E(T_G)|+1}^{|E(G)|-1} \mathcal{P}^T_{a_i b_i}\right)\right)\\
&= E\left(\{a_{|E(G)|}, b_{|E(G)|}\}\Box T\right) \cup \{(a_{|E(G)|}, w_0)(b_{|E(G)|}, w_0)\}
\end{aligned}$$

which is a tree. Hence, by Proposition 3.2, $\mathcal{P}^T_G \cup \mathcal{B}_{G\Box w_0}$ is an independent set. ■

Let G and H be two graphs, T_G and T_H be any two spanning trees of G and H, respectively. Let a_1 be an end vertex of T_G. Construct a rooted tree, T^*_G, by assuming that a_1 is the root and all the other vertices of T_G are directed a way from a_1. Consider the edges of the rooted tree are the following $e_1 = a_1b_1, e_2 = a_2b_2, \dots, e_{|E(T_G)|} = a_{|E(T_G)|}b_{|E(T_G)|}$. Without loss of generality we can order the edges in such a way that $\delta(a_1, a_i) \le \delta(a_1, a_{i+1}) = \delta(a_1, b_{i+1}) - 1$ where $\delta(x, y)$ denotes the distant between x and y. In this way we guarantee that $V(e_i) \cap V(\cup_{j=1}^{i-1} e_j)$ is exactly one vertex, a_i. For a vertex $w_0 \in V(H)$, set

$$\mathcal{A}_{T_G^*,w_0}^{(H-T_H)} = \left(\mathcal{P}_{b_1a_1,w_0}^{H-T_H}\right) \cup \left(\bigcup_{i=1}^{|E(T_G)|} \mathcal{P}_{a_ib_i,w_0}^{H-T_H}\right).$$

Lemma 3.6. *The set $\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}$ is linearly independent.*

Proof. Note that $E\left(\mathcal{P}_{a_ib_i,w_0}^{H-T_H}\right) \cap E\left(\mathcal{P}_{a_jb_j,w_0}^{H-T_H}\right) = \varnothing$ for each $i \neq j$. Thus, $\cup_{i=1}^{|E(T_G)|}\mathcal{P}_{a_ib_i,w_0}^{H-T_H}$ is linearly independent. Also, note that $E\left(\mathcal{P}_{b_1a_1,w_0}^{H-T_H}\right) \cap E\left(\cup_{i=1}^{|E(T_G)|}\mathcal{P}_{a_ib_i,w_0}^{H-T_H}\right) = \{(a_1,w_0)(b_1,w_0)\}$. Thus, by Proposition 3.2, $\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}$ is linearly independent. ■

We now have the results needed to prove the following result:

Theorem 3.7. *Let G and H be any two graphs. Then $\mathcal{B}(G,H) = \mathcal{P}_G^{T_H} \cup (\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}) \cup \mathcal{B}_{G\Box w_0}$ is a basis for $\mathcal{C}(G[H])$ where $\mathcal{B}_{G\Box w_0}$ is a basis for $\mathcal{C}(G\Box w_0)$.*

Proof. Since

$$E\left(\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}\right) - E(T_G\Box(H-T_H)) = \{(b_i,w_0)(a_i,w)|w \in V(H)\} \cup \left(\bigcup_{i=1}^{|E(T_H)|} \{(a_i,w_0)(b_i,w)|w \in V(H)\}\right),$$

which is an edge set of a tree, as a result each linear combination of cycles of $\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}$ must contain an edge of $a_i\Box(H-T_H)$ for some $a_i \in V(G)$ which is not in any cycle of $\mathcal{P}_G^{T_H} \cup \mathcal{B}_{G\Box w_0}$. Thus, $\mathcal{B}(G,H)$ is linearly independent. Now, for any vertex $w \in V(H)$ and edge ab, we have that

$$|\mathcal{P}_{ab,w}^{T_H}| = \sum_{uv\in E(T_H)} |\mathcal{P}_{ab,w}^{uv}| = \sum_{uv\in E(T_H)} 1 = |E(T_H)|. \tag{3}$$

Similarly, we have that

$$|\mathcal{P}_{ba,w}^{H-T_H}| = |E(H-T_H)|. \tag{4}$$

Thus, by (3) and the defintion of $\mathcal{P}_{ab}^{T_H}$, we have

$$\begin{aligned}|\mathcal{P}_{ab}^{T_H}| &= |\mathcal{P}_{ba,w_0}^{T_H}| + \sum_{w\in V(T_H)} |\mathcal{P}_{ab,w}^{T_H}| \\ &= |E(T_H)| + \sum_{w\in V(T_H)} |E(T_H)| = |E(T_H)| + |V(H)||E(T_H)| \\ &= |V(H)|^2 - 1.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathcal{P}_{G}^{T_H}| &= \sum_{i=1}^{|E(G)|} |\mathcal{P}_{a_ib_i}^{T_H}| = \big(|V(H)|^2 - 1\big)\,|E(G)| \\ &= |E(G)||V(H)|^2 - |E(G)|.\end{aligned}$$

Also, by (4) and by noting that $\dim \mathcal{C}(H) = |E(H - T_H)|$, we have that

$$\begin{aligned}|\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}| &= |\mathcal{P}_{b_1a_1,w_0}^{H-T_H}| + \sum_{i=1}^{|E(T_G)|} |\mathcal{P}_{a_ib_i,w_0}^{H-T_H}| \\ &= |E(H - T_H)| + |E(T_G)||E(H - T_H)| \\ &= \dim \mathcal{C}(H) + |E(T_G)| \dim \mathcal{C}(H).\end{aligned}$$

Moreover, we know that

$$|\mathcal{B}_{G\Box w_0}| = \dim \mathcal{C}(G).$$

Therefore,

$$\begin{aligned}&|\mathcal{B}(G,H)| \\ &= |\mathcal{P}_{G}^{T_H}| + |\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}| + |\mathcal{B}_{G\Box w_0}| \\ &= |E(G)||V(H)|^2 - |E(G)| + \dim \mathcal{C}(H) + |E(T_G)| \dim \mathcal{C}(H) + \dim \mathcal{C}(G) \\ &= |E(G)||V(H)|^2 + \dim \mathcal{C}(H)(|E(T_G)| + 1) - |E(G)| + \dim \mathcal{C}(G) \\ &= |E(G)||V(H)|^2 + \dim \mathcal{C}(H)|V(G)| - |E(G)| + \dim \mathcal{C}(G).\end{aligned}$$

But, by (1)

$$-\;|E(G)| + \dim \mathcal{C}(G) = -\;|V(G)| + 1.$$

Thus,

$$\begin{aligned}
|\mathcal{B}(G,H)| &= |E(G)||V(H)|^2 + \dim \mathcal{C}(H)|V(G)| - |V(G)| + 1 \\
&= |E(G)||V(H)|^2 + (|E(H)| - |V(H)| + 1)|V(G)| - |V(G)| + 1 \\
&= |E(G)||V(H)|^2 + |E(H)||V(G)| - |V(G)||V(H)| + 1 \\
&= \dim \mathcal{C}(G[H])
\end{aligned}$$

where the last equality holds by (2). Therefore, $\mathcal{B}(G,H)$ is a basis for $\mathcal{C}(G[H])$. ■

Remark 3.8. (i) By specializing G to be a tree T, we have that $\mathcal{B}_{G\Box w_0} = \varnothing$ and so, $\mathcal{B}(T,H) = \mathcal{P}_T^{T_H} \cup (\mathcal{A}_{T^*,w_0}^{(H-T_H)})$ is a basis for $\mathcal{C}(T[H])$ where T^* is a rooted tree for T.

(ii) By specializing H to be a tree T, we have that $\mathcal{A}_{T^*,w_0}^{(H-T_H)} = \varnothing$ and so, $\mathcal{B}(G,T) = \mathcal{P}_G^T \cup \mathcal{B}_{G\Box w_0}$ is a basis for $\mathcal{C}(G[T])$.

(iii) By specializing G and H to be trees T_1 and T_2, respectively, we have that $(\mathcal{A}_{T_G^*,w_0}^{(H-T_H)}) \cup \mathcal{B}_{G\Box w_0} = \varnothing$ and so, $\mathcal{B}(T_1,T_2) = \mathcal{P}_{T_1}^{T_2}$ is a basis for $\mathcal{C}(T_1[T_2])$. ■

We now turn our attention to construct a minimal cycle basis for $G[H]$.

Lemma 3.9. *Let C be a cycle of $G[H]$ of length greater than or equal to 4 and contains an edge of an H-fiber. Then C is irrelevant.*

Proof. Assume that C contains an edge of the fiber $a\Box H$ for some $a \in V(G)$. Then we consider two cases:

Case 1. All the edges of C are from $a\Box H$, say $C = (a,v_1)(a,v_2)\ldots(a,v_m)(a,v_1)$. Let $ab \in E(G)$. Then

$$C = \left(\bigoplus_{j=1}^{m-1} \mathcal{P}_{ba,v_1}^{v_j v_{j+1}}\right) \oplus \mathcal{P}_{ba,v_1}^{v_m v_1}.$$

Thus, C is irrelevant.

Case 2. C contains at least one edge which is not in $a\Box H$. Hence we may assume that $(b,w)(a,u)(a,v) \subseteq C$ where $vu \in E(H)$ and $ba \in E(G)$. Let

$$C_1 = C \oplus \mathcal{P}^{uv}_{ba,w}.$$

Then,

$$C = C_1 \oplus \mathcal{P}^{uv}_{ba,w}.$$

Note that C_1 is obtained from C by deleting at least two edges $(b,w)(a,u)$, $(a,u)(a,v)$ and adding at most the edge $(b,w)(a,v)$. Thus, $|C_1| < |C|$. Hence, C is irrelevant. ■

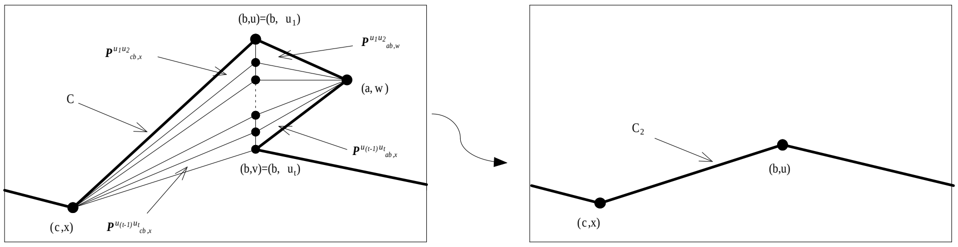

Figure 1. The way of getting C_2 from the ring sum of C and 3-cycles.

Lemma 3.10. *Let C be a cycle of $G[H]$ of length greater than or equal to 4 such that C contains at least two edges from one fold of $G[H]$. Then C is irrelevant.*

Proof. Let C be a cycle containing at least two edges from the same fold. From Lemma 3.9 we can assume that C contains no edge of $V(G)\Box H$. To this end, we consider the following two cases:

Case 1. $E(C)$ belongs to only one fold, say $ab[H]$ and so belongs to $ab[N_H]$ where N_H is the null graph with vertex set is $V(H)$. By (iii) of Remark 3.8, $\mathcal{P}^{T_H}_{ab}$ is a basis of $ab[T_H]$. Thus, C can be written as a linear combination of some cycles of $\mathcal{P}^{T_H}_{ab}$. Hence, C is irrelevant.

Case 2. $E(C)$ belongs to at least two folds. Then we consider two subcases:

Subcase 2a. There is a fold, say $ab[H]$, contaning two adjacent edges of $E(C)$. Consider $(c, x)(b, u)(a, w)(b, v) \subseteq C$, that is the two adjacent edges in $ab[N_H]$ are $(b, u)(a, w)$ and $(a, w)(b, v)$ and $(c, x)(b, u) \notin E(ab[H])$. Let $u = u_1 u_2 \ldots u_t = v$ be the path of T_H connecting u and v and so $(b, u) = (b, u_1)(b, u_2) \ldots (b, u_t) = (b, v)$ is a path in $G[T_H] \subseteq G[H]$ connecting the two vertices (b, u) and (b, v). Let

$$C_2 = C \oplus \bigoplus_{j=1}^{t-1} \mathcal{P}_{ab,w}^{u_i u_{i+1}} \oplus \bigoplus_{j=1}^{t-1} \mathcal{P}_{cb,x}^{u_i u_{i+1}}.$$

Then

$$C = C_2 \oplus \bigoplus_{j=1}^{t-1} \mathcal{P}_{ab,w}^{u_i u_{i+1}} \oplus \bigoplus_{j=1}^{t-1} \mathcal{P}_{cb,x}^{u_i u_{i+1}}.$$

Note that $|C_2| \leq |C| - 2$ because C_2 is obtained by deleting at least (b, u) (a, w), (a, w) (b, v) and $(c, x)(b, u)$ and adding at most $(c, x)(b, v)$ (see Figure 1). Thus, C is irrelevant.

Subcase 2b. There is no fold containing two adjacent edges of $E(C)$. Consider the fold $ab[H]$ contains two non adjacent edges of $E(C)$, say $(c, x)(b, u)(a, w)$ and $(b, v)(a, y) \subseteq C$. That is the two non adjacent edges of the same fold are $(b, u)(a, w)$ and $(b, v)(a, y)$ and $(c, x)(b, u) \notin ab[H]$. Let $u = u_1 u_2 \ldots u_t = v$ be the path of T_H connecting u and v and so $(b, u) = (b, u_1)(b, u_2) \ldots (b, u_t) = (b, v)$ be a path in $G[T_H] \subseteq G[H]$ connecting the two vertices (b, u) and (b, v). Thus,

$$C_3 = C \oplus \bigoplus_{j=1}^{t-1} \mathcal{P}_{cb,x}^{u_i u_{i+1}} \bigoplus_{j=1}^{t-1} \mathcal{P}_{ab,w}^{u_i u_{i+1}}$$

is a union of at least to edge disjoint cycles (because in this way we unify the vertices (b, u) and (b, v)) each of which is of length less than the length of C, say $C_3 = \cup_{i=1}^{r} C_i^{'}$. Then

$$C_3 = \bigoplus_{j=1}^{r} C_i^{'}.$$

And so,

$$C = \bigoplus_{j=1}^{r} C_i^{'} \oplus \bigoplus_{j=1}^{t} \mathcal{P}_{ab,x}^{u_i u_{i+1}} \bigoplus_{j=1}^{t} \mathcal{P}_{cb,w}^{u_i u_{i+1}}.$$

Thus, C is irrelevant. ■

Lemma 3.11. *Let $C_v = C\Box v$ and $C_u = C\Box u$ be two cycles of the G-fibers $G\Box u$ and $G\Box v$, respectively. Then C_u is a linear combination of C_v with cycles of length* 3.

Proof. Let $C_v = (a_1, v)(a_2, v)\ldots(a_n, v)(a_1, v)$. Let T_H be a spanning tree of H and $v = v_1v_2\ldots v_m = u$ be the path of T_H connecting v and u. Then

$$\left(\bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_ia_{i+1},v}\right)\oplus\left(\bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_{i+1}a_i,u}\right)$$
$$\oplus\left(\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_na_1,v}\right)\oplus\left(\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_1a_n,u}\right) = C_v \oplus C_u.$$

Thus,

$$C_u = C_v \oplus \left(\bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_ia_{i+1},v}\right)\oplus\left(\bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_{i+1}a_i,u}\right)$$
$$\oplus\left(\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_na_1,v}\right)\oplus\left(\bigoplus_{j=1}^{m-1}\mathcal{P}^{v_jv_{j+1}}_{a_1a_n,u}\right).$$

■

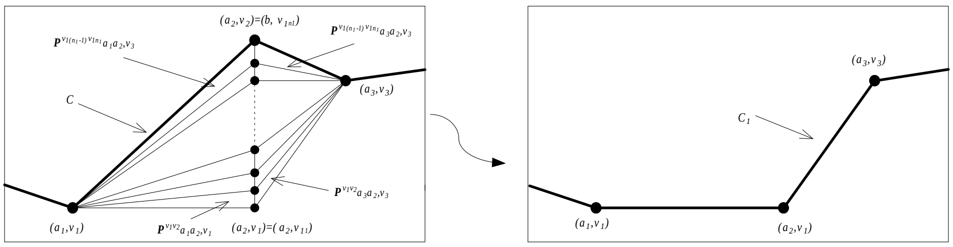

Figure 2. The first step in the procedure of writing the cycle C as a linear combination of cycles of a G-fiber ($G\Box v_1$) and 3-cycles.

Lemma 3.12. *Let C be a cycle of $G[H]$ of length greater than or equal to* 3 *such that C neither contains an edge of an H-fiber nor contains two edges of the same fold. Then C is a ring sum of cycles of length* 3 *with cycles of a G-fiber.*

Proof. Let $C = (a_1, v_1)(a_2, v_2) \dots (a_m, v_m)(a_1, v_1)$. Since C neither contains an edge of an H-fiber nor contains two edges of the same fold, as a result C contains no edge of $\cup_{a \in v(G)}(a \Box H)$ and contains at most one edge of $e[N_H]$ for each $e \in E(G)$. And so, $a_1 a_2 \dots a_m a_1$ is a cycle or edge disjoint union of cycles of G. Now, we show that C is a linear combination of 3-cycles with cycles of the fiber $G \Box v_1$. Let $v_1 = v_{1_1} v_{1_2} \dots v_{1_{n_1}} = v_2$ be the path of T_H connecting the two vertices v_1 and v_2. Let

$$C_1 = C \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_1 a_2, v_1}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_3 a_2, v_3}^{v_{1_j} v_{1_{j+1}}} \right).$$

Note that C_1 is obtained from C by pulling the vertex (a_2, v_2) to (a_2, v_1). Thus, C_1 is a cycle or an edge disjoint union of cycles each of which has the same properties as C (see Figure 2). Now, Let $v_1 = v_{2_1} v_{2_2} \dots v_{2_{n_2}} = v_3$ be the path of T_H joining the two vertices v_1 and v_3. Let

$$C_2 = C_1 \oplus \left(\bigoplus_{j=1}^{n_2 - 1} \mathcal{P}_{a_2 a_3, v_2}^{v_{2_j} v_{2_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_2 - 1} \mathcal{P}_{a_4 a_3, v_4}^{v_{2_j} v_{2_{j+1}}} \right).$$

Similarly, note that C_2 is obtained from C_1 by pulling the vertex (a_3, v_3) to (a_3, v_1). Thus, C_2 is a cycle or an edge disjoint union of cycles each of which has the same properties as C_1. By continuing in this process, we get a cycle or edge disjoint union of cycles C_{m-1} which obtained from cycles of length 3 and C_{m-2} each of which has the same properties as C_{m-2}. Moreover, each vertex of C_{m-1} lies on the fiber $G \Box v_1$ except possibly the vertex (a_m, v_m). To this end, let $v_1 = v_{(m-1)_1} v_{(m-1)_2} \cdots v_{(m-1)_{n_{(m-1)}}} = v_m$ be the path of T_H joining the two vertices v_1 and v_m. Let

$$C_m = C_{m-1} \oplus \left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_{m-1} a_m, v_{m-1}}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_1 a_m, v_1}^{v_{1_j} v_{1_{j+1}}} \right).$$

Then $C_m = (a_1, v_1)(a_2, v_1) \dots (a_m, v_1)(a_1, v_1)$ which is a cycle or edge disjoint union of cycles of the fiber $G \Box v_1$. Hence,

$$C_m = C \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_1 a_2, v_1}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_3 a_2, v_3}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \cdots \oplus$$
$$\left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_{m-1} a_m, v_{m-1}}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_1 a_m, v_1}^{v_{1_j} v_{1_{j+1}}} \right).$$

Thus,

$$C = C_m \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_1 a_2, v_1}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_1 - 1} \mathcal{P}_{a_3 a_2, v_3}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \cdots \oplus$$
$$\left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_{m-1} a_m, v_{m-1}}^{v_{1_j} v_{1_{j+1}}} \right) \oplus \left(\bigoplus_{j=1}^{n_{(m-1)} - 1} \mathcal{P}_{a_1 a_m, v_1}^{v_{1_j} v_{1_{j+1}}} \right).$$

■

Lemma 3.13. *Every cycle of length three of $G[H]$ which contains at least one edge of an H-fiber can be written as a linear combination of $\mathcal{P}_G^{T_H} \cup (\mathcal{A}_{T_G^*, w_0}^{(H-T_H)})$.*

Proof. Let C be a 3-cycle which contains at least one edge of an H-fiber. We consider two cases:

Case 1. C is subgraph of an H-fiber. Then C is a subgraph of $T_G[H]$. By (i) of Remark 3.8 $\mathcal{P}_{T_G}^{T_H} \cup (\mathcal{A}_{T_G^*, w_0}^{(H-T_H)})$ is a basis of $T_G[H]$. Thus, C can be written as a linear combination of $\mathcal{P}_{T_G}^{T_H} \cup (\mathcal{A}_{T_G^*, w_0}^{(H-T_H)})$.

Case 2. C is not a subgraph of an H-fiber. Since C contains at least one edge of an H-fiber, C belongs to a fold of H. Note that, by (i) of Remark 3.8,

$$\mathcal{P}_{ab,w_0}^{H-T_H} \cup \mathcal{P}_{ba,w_0}^{H-T_H} \cup \left(\cup_{w \in V(T)} \mathcal{P}_{ab,w}^{T_H} \right) \cup \mathcal{P}_{ba,w_0}^{T_H}$$

is a basis for $ab[H]$ for any $ab \in E(G)$. Thus, to prove the lemma it is enough to show that each cycle of $\mathcal{P}_{ab,w_0}^{H-T_H} \cup \mathcal{P}_{ba,w_0}^{H-T_H}$ can be written as a linear combination of $\mathcal{P}_G^{T_H} \cup (\mathcal{A}_{T_G^*,w_0}^{(H-T_H)})$ for any $ab \in E(G)$. Let $\{e_1 = a_1b_1, e_2 = a_2b_2 \ldots e_{|E(T_G)|} = a_{|E(T_G)|}b_{|E(T_G)|}\}$ be the edge set of the rooted tree T_G^*. Note that

$$\mathcal{P}_{b_1a_1,w_0}^{H-T_H} \cup \mathcal{P}_{a_1b_1,w_0}^{H-T_H} \subseteq \mathcal{P}_G^{T_H} \cup \mathcal{A}_{T_G^*,w_0}^{(H-T_H)}.$$

Thus, the result is obtained if $ab = a_1b_1$. Now, we show that the result is true for each edge $e = ab \in E(G)$ different from a_1b_1. To this end, we consider two subcases:

Subcase 1. $e = ab \in T_G^*$. With out loss of generality, we can assume that $e_1e_2\ldots e_l$ be the path of T_G^* joining a_1b_1 with ab, say $e_1 = a_1b_1 = a_{1_1}b_{1_1}$ and $e_2 = a_{1_2}b_{1_2}, \ldots, e_l = a_{1_l}b_{1_l} = ab$. Now, for each $uv \in E(H - T_H)$, let $u = u_1u_2\ldots u_t = v$ be the path of T_H connecting u and v, Hence, $u_1u_2\ldots u_tu_1$ is the cycle of H containing $uv = u_tu_1$. Note that

$$\mathcal{P}_{b_{1_2}a_{1_2},w_0}^{uv} = C_{b_{1_1}}^* \oplus \bigoplus_{i=1}^{t-1} \mathcal{P}_{b_{1_2}a_{1_2},w_0}^{u_iu_{i+1}}$$

where $C_{b_{1_1}}^* : (b_{1_1}, u_1)(b_{1_1}, u_2)\ldots(b_{1_1}, u_t)(b_{1_1}, u_1)$. $C_{b_{1_1}}^*$ can be written as a linear combinations of cycle of $\mathcal{P}_{b_{1_1}a_{1_1},w_0}^{H-T_H} \cup \mathcal{P}_{a_{1_1}b_{1_1},w_0}^{H-T_H} \cup \left(\cup_{w\in V(T)} \mathcal{P}_{a_{1_1}b_{1_1},w}^{T_H}\right) \cup \mathcal{P}_{b_{1_1}a_{1_1},w_0}^{T_H}$ because $C_{b_{1_1}}^*$ is a cycle of $a_{1_1}b_{1_1}[H] = a_1b_1[H]$ and $\mathcal{P}_{b_{1_1}a_{1_1},w_0}^{H-T_H} \cup \mathcal{P}_{a_{1_1}b_{1_1},w_0}^{H-T_H} \cup \left(\cup_{w\in V(T)} \mathcal{P}_{a_{1_1}b_{1_1},w}^{T_H}\right) \cup \mathcal{P}_{b_{1_1}a_{1_1},w_0}^{T_H}$ is a basis of $\mathcal{C}(a_{1_1}b_{1_1}[H])$. Moreover, by (iii) of Remark 3.8, $\mathcal{P}_{b_{1_2}a_{1_2},w_0}^{u_iu_{i+1}}$ can be written as a linear combinations of cycles of $\mathcal{P}_{e_2}^{T_H}$ because $\mathcal{P}_{e_2}^{T_H}$ is a basis of $\mathcal{C}(e_2[T_H])$. Thus, $\mathcal{P}_{b_{1_2}a_{1_2},w_0}^{uv}$ can be written as a linear combination of cycles of $\mathcal{P}_G^{T_H} \cup \mathcal{A}_{T_G^*,w_0}^{(H-T_H)}$. Now, by a similar argument

$$\mathcal{P}_{b_{1_3}a_{1_3},w_0}^{uv} = C_{b_{1_2}}^* \oplus \bigoplus_{i=1}^{t-1} \mathcal{P}_{b_{1_3}a_{1_3},w_0}^{u_iu_{i+1}}$$

where $C_{b_{1_2}}^* : (b_{1_2}, u_1)\ (b_{1_2}, u_2)\ldots(b_{1_2},\ u_t)\ (b_{1_2},\ u_1)$ which can be written as a linear combinations of cycle of $\mathcal{P}_{b_{1_1}a_{1_1},w_0}^{H-T_H} \cup (\cup_{i=1}^{2} \mathcal{P}_{a_{1_i}b_{1_i},w_0}^{H-T_H}) \cup \left(\cup_{w\in V(T)} \cup_{i=1}^{2} \mathcal{P}_{a_{1_i}b_{1_i},w}^{T_H}\right) \cup \mathcal{P}_{b_{1_1}a_{1_1},w_0}^{T_H}$ because $\mathcal{P}_{b_{1_1}a_{1_1},w_0}^{H-T_H} \cup (\cup_{i=1}^{2} \mathcal{P}_{a_{1_i}b_{1_i},w_0}^{H-T_H}) \cup \left(\cup_{w\in V(T)} \cup_{i=1}^{2} \mathcal{P}_{a_{1_i}b_{1_i},w}^{T_H}\right) \cup \mathcal{P}_{b_{1_1}a_{1_1},w_0}^{T_H}$ is a basis of $\mathcal{C}(e_1 \cup e_2[H])$ by (i) of

Remark 3.8. Moreover, by (iii) of Remark 3.8, $\mathcal{P}^{u_iu_{i+1}}_{b_{1_3}a_{1_3},w_0}$ can be written as a linear combinations of cycles of $\mathcal{P}^{T_H}_{e_3}$ because $\mathcal{P}^{T_H}_{e_3}$ is a basis of $\mathcal{C}(e_3[T_H])$. Thus, $\mathcal{P}^{uv}_{b_{1_3}a_{1_3},w_0}$ can be written as a linear combination of cycles of $\mathcal{P}^{T_H}_G \cup \mathcal{A}^{(H-T_H)}_{T^*_G,w_0}$. By continuing in this procedure we show that for each $uv \in E(H - T_H)$,

$$\mathcal{P}^{uv}_{b_{1_l}a_{1_l},w_0} = C^*_{b_{1_l}} \oplus \bigoplus_{i=1}^{t-1} \mathcal{P}^{u_iu_{i+1}}_{b_{1_l}a_{1_l},w_0}$$

where $C^*_{b_{1_l}} : (b_{1_{l-1}}, u_1)(b_{1_{l-1}}, u_2)\dots(b_{1_{l-1}}, u_t)(b_{1_{l-1}}, u_1)$ which can be written as a linear combinations of cycle of $\mathcal{P}^{T_H}_G \cup (\mathcal{A}^{(H-T_G)}_{T_G,w_0})$. Moreover, by (iii) of Remark 3.8, $\mathcal{P}^{u_iu_{i+1}}_{e_l,w_0}$ can be written as a linear combinations of cycles of $\mathcal{P}^{T_H}_{e_l}$ because it is a basis of $\mathcal{C}(e_l[T_H])$. Thus, each cycle of $\mathcal{P}^{H-T_H}_{ba,w_0}$ can be written as a linear combination of cycles of $\mathcal{P}^{T_H}_G \cup \mathcal{A}^{(H-T_H)}_{T^*_G,w_0}$. Since $\mathcal{P}^{H-T_H}_{ab,w_0} \subseteq \mathcal{P}^{T_H}_G \cup \mathcal{A}^{(H-T_H)}_{T^*_G,w_0}$, as a result each cycle of $\mathcal{P}^{H-T_H}_{ab,w_0} \cup \mathcal{P}^{H-T_H}_{ba,w_0}$ can be written as a linear combinations of cycles of $\mathcal{P}^{T_H}_G \cup \mathcal{A}^{(H-T_H)}_{T^*_G,w_0}$.

Subcase 2. $e = ab \notin T^*_G$. Assume that $P = e_1e_2\dots e_l$ be a path of T^*_G joining $a_1b_1$and a, and $P^* = e^*_1e^*_2\dots e^*_k$ be a path of T^*_G joining a_1b_1 and b. By applying the same argument as in Subcase 1 on $P \cup ab$ and $P^* \cup ba$, we get $\mathcal{P}^{uv}_{ba,w_0}$ and $\mathcal{P}^{uv}_{ab,w_0}$, respectively, for any $uv \in E(H - T_H)$. Thus, each cycle of $\mathcal{P}^{H-T_H}_{ab,w_0} \cup \mathcal{P}^{H-T_H}_{ba,w_0}$ can be written as a linear combination of cycles of $\mathcal{P}^{T_H}_G \cup \mathcal{A}^{(H-T_H)}_{T^*_G,w_0}$. ∎

Note that if each vertex of the cycle basis $\mathcal{B}$ is relevant, then $\mathcal{B}$ is minimal.

Theorem 3.14. *Let G and H be any two graphs. If $\mathcal{B}_{G\Box w_0}$ is a minimal cycle basis of $G\Box w_0$, then $\mathcal{B}(G, H) = \mathcal{P}^{T_H}_G \cup (\mathcal{A}^{(H-T_H)}_{T_G}) \cup \mathcal{B}_{G\Box w_0}$ is a minimal cycle basis of $G[H]$.*

Proof. Let $\mathcal{B}$ be a minimal cycle basis of $G[H]$ obtained by applying the Greedy algorithm. Since $\mathcal{P}^{T_H}_G \cup (\mathcal{A}^{(H-T_H)}_{T^*_G,w_0})$ is a linearly independent set consisting of 3-cycles, as a result we may assume that $\mathcal{P}^{T_H}_G \cup (\mathcal{A}^{(H-T_H)}_{T^*_G,w_0}) \subseteq \mathcal{B}$. Now, let $\mathcal{S} = \mathcal{B} - \left(\mathcal{P}^{T_H}_G \cup (\mathcal{A}^{(H-T_H)}_{T^*_G,w_0})\right)$. By Lemmas 3.9, 3.11 and 3.13 each cycle of $\mathcal{S}$ neither contains an edge of any H-fiber nor contains two edges of the same fold of H. Thus, the cycles of $\mathcal{S}$ must be as in Lemma 3.12. Since $\mathcal{B}$ is a minimal cycle basis, then each cycle of $\mathcal{S}$ is relevant. Now, by Lemmas 3.12 and 3.11, each cycle of $\mathcal{S}$ is a linear combination of cycles of length three (of the form $\mathcal{P}^{uv}_{ab,w}$) with cycle corresponding to a cycle of $G\Box w_0$ of the same length. By Lemma 3.12 and 3.13, those corresponding cycles of $G\Box w_0$ must be linearly independent. Since $|\mathcal{S}| = \dim \mathcal{C}(G\Box w_0)$, as a result the set of corresponding cycles of cycles of $\mathcal{S}$ is a basis for $G\Box w_0$. Thus, $l(\mathcal{S}) \geq l(\mathcal{B}_{G\Box w_0})$. Hence,

$$\begin{aligned} l(\mathcal{B}) &= l\left(\mathcal{P}_G^{T_H} \cup \left(\mathcal{A}_{T_G}^{(H-T_H)}\right)\right) + l(\mathcal{S}) \\ &\geq l\left(\mathcal{P}_G^{T_H} \cup \left(\mathcal{A}_{T_G}^{(H-T_H)}\right)\right) + l\left(\mathcal{B}_{G\Box w_0}\right) \\ &= l(\mathcal{B}(G,H)). \end{aligned}$$

On the other hand, since $\mathcal{B}$ is minimal, we have that

$$l(\mathcal{B}) \leq l(\mathcal{B}(G,H)).$$

Thus,

$$l(\mathcal{B}) = l(\mathcal{B}(G,H)).$$

Therefore, $\mathcal{B}(G,H)$ is minimal. ∎

The following two corollaries are straightforward from theorem 3.14.

Corollary 3.15. $l(G[H]) = 3(|E(G)||V(H)|^2 + \dim \mathcal{C}(H)|V(G)| - |E(G)|) + l(G)$.

Corollary 3.16. $\lambda(G[H]) = \max\{3, \lambda(G)\}$.

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