MAXIMAL $k$-INDEPENDENT SETS IN GRAPHS

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Abstract

A subset of vertices of a graph $G$ is $k$-independent if it induces in $G$ a subgraph of maximum degree less than $k$. The minimum and maximum cardinalities of a maximal $k$-independent set are respectively denoted $i_k(G)$ and $\beta_k(G)$. We give some relations between $\beta_k(G)$ and $\beta_j(G)$ and between $i_k(G)$ and $i_j(G)$ for $j \neq k$. We study two families of extremal graphs for the inequality $i_2(G) \leq i(G) + \beta(G)$. Finally we give an upper bound on $i_2(G)$ and a lower bound when $G$ is a cactus.

Keywords: $k$-independent, cactus.

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1. Introduction

For notation and graph theory terminology, we in general follow [6, 7]. In a graph $G = (V, E)$ of order $n(G) = n$, the neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$. If $X$ is a subset of vertices, then $N_G(X) = \cup_{v \in X} N_G(v)$. The closed neighborhoods of $v$ and $X$ are respectively $N_G[v] = N(v) \cup \{v\}$ and $N[X] = N(X) \cup X$. The degree of a vertex $v$ of $G$, denoted by $d_G(v)$, is the order of its neighborhood. For a subset $A$ of $V$, let us
denote by $G[A]$ the subgraph induced in $G$ by $A$. If $x$ is a vertex of $V$, then $d_A(x) = |N(x) \cap A|$ and $\Delta(A) = \max\{d_A(x) \mid x \in A\}$. A vertex of degree one is called a leaf and its neighbor is called a support vertex. We denote the set of leaves of a graph $G$ by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell(G)$, $|S(G)| = s(G)$. If $T = P_2$, then $\ell(P_2) = s(P_2) = 2$. A double star $S_{p,q}$ is obtained by attaching $p$ leaves at an endvertex of a path $P_2$ and $q$ leaves at the second one. A cactus is a graph in which every edge is contained in at most one cycle. A graph is called trivial if its order is $n = 1$.

An independent set is a set of vertices whose induced subgraph has no edge. The independence number $\beta(G)$ is the maximum cardinality of an independent set in $G$. The independence domination number $i(G)$ is the minimum cardinality of a maximal independent set in $G$.

In [5] Fink and Jacobson generalized the concepts of independent and dominating sets. A subset $X$ of $V$ is $k$-independent if the maximum degree of the subgraph induced by the vertices of $X$ is less or equal to $k - 1$. The subset $X$ is $k$-dominating if every vertex of $V - X$ is adjacent to at least $k$ vertices in $X$. The lower $k$-independence number $i_k(G)$ is the minimum cardinality of a maximal $k$-independent set in $G$, the $k$-independence number $\beta_k(G)$ is the maximum cardinality of a maximal $k$-independent set, and the $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. A $k$-independent set with maximum cardinality of a graph $G$ is called a $\beta_k(G)$-set. Similarly we define a $i_k(G)$-set and a $\gamma(G)$-set. For $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets and so $i_1(G) = i(G)$, $\beta_1(G) = \beta(G)$, and $\gamma_1(G) = \gamma(G)$.

Note that Borowiecki and Michalak [2] gave a generalization of the concept of $k$-independence by considering other hereditary-induced properties than the property for a subgraph to have maximum degree at most $k - 1$.

On the same way that the minmax parameter $i$ is more difficult to study than $\beta$, very few results are known on $i_k$ while the literature on $\beta_k$, and even more on $\gamma_k$, is rather copious. The irregularity of the behaviour of $i_k$ is shown for instance by the followings two facts. The well-known inequalities $\gamma(G) \leq i(G) \leq \beta(G)$ only extend to $\gamma_k(G) \leq \beta_k(G)$ [3] but $i_k(G)$ may be smaller than $\gamma_k(G)$. The sequence $(\beta_k(G))$ is always non-decreasing while the sequence $(i_k(G))$ is not necessarily monotone. In this paper we show some properties related to $\beta_k$ and $i_k$.

A matching in a graph $G$ is a collection of pairwise non-adjacent edges.
The matching is called \textit{induced} if no two edges of the matching are joined by an edge in \( G \).

2. Bounds on \( \beta_k \) and \( i_k \).

\textbf{Theorem 1.} For every graph \( G \) and integers \( j, k \) with \( 1 \leq j \leq k \), \( \beta_{k+1}(G) \leq \beta_j(G) + \beta_{k-j+1}(G) \).

\textit{Proof.} Let \( T \) be a maximum \((k + 1)\)-independent set of \( G \) and \( X \) both a \( j \)-independent and \( j \)-dominating set of \( G[T] \). Such a set \( X \) exists by [3]. Thus \( \beta_j(G) \geq |X| \). Let \( Y = T - X \). Since \( X \) is \( j \)-dominating in \( G[T] \), \( \Delta(G[Y]) \leq k - j \). Hence \( Y \) is a \((k - j + 1)\)-independent set and therefore \( \beta_{k-j+1}(G) \geq |Y| = |T| - |X| \geq \beta_{k+1}(G) - \beta_j(G) \).

\textbf{Corollary 2.} For every graph \( G \) and every integer \( k \geq 1 \),

(a) \( \beta_{k+1}(G) \leq \beta_k(G) + \beta(G) \),

(b) \( \beta_{k+1}(G) \leq 2\beta_{[k+1/2]}(G) \),

(c) \( \beta_{k+1}(G) \leq (k + 1)\beta(G) \).

The next theorem gives a structural property of the graphs satisfying (c).

\textbf{Theorem 3.} Let \( k \geq 2 \) be an integer and \( G \) a graph such that \( \beta_k(G) = k\beta(G) \). Then every \( \beta_k(G) \)-set \( T \) is the disjoint union of \( \beta(G) \) cliques \( U^j \), \( 1 \leq j \leq \beta \), of order \( k \) and every vertex \( v \in V \setminus T \) has at least one clique \( U^j \) entirely contained in its neighborhood.

\textit{Proof.} Since \( T \) is a \( k \)-independent set, \( \Delta(T) \leq k - 1 \). Let \( X_1 \) be a maximal independent set of \( G[T] \). Every vertex of \( T \setminus X_1 \) has at least one neighbor in \( X_1 \) and thus, \( \Delta(T \setminus X_1) \leq k - 2 \). Let \( X_2 \) be a maximal independent set of \( G[T \setminus X_1] \). Every vertex of \( T \setminus (X_1 \cup X_2) \) has at least one neighbor in \( X_1 \) and one in \( X_2 \), and thus \( \Delta(T \setminus (X_1 \cup X_2)) \leq k - 3 \). We continue the process until the choice of a maximal independent set \( X_{k-1} \) of \( G[T \setminus (X_1 \cup \cdots \cup X_{k-2})] \). Then \( \Delta(T \setminus (X_1 \cup \cdots \cup X_{k-1}) \leq 0 \) and thus the set \( X_k = T \setminus (X_1 \cup \cdots \cup X_{k-1}) \) is independent. Therefore every set \( X_i \) is independent in \( G \) and \( |X_i| \leq \beta(G) \) for \( 1 \leq i \leq k \). Hence \( |T| = \sum_{i=1}^k |X_i| \leq k\beta(G) \) and since \( T = \beta_k(G) = k\beta(G) \), \( |X_i| = \beta(G) \) (\( = \beta \) for short) for \( 1 \leq i \leq k \). Let \( X_1 = \{ u_1^1, u_1^2, \ldots, u_1^j \} \). Then \( k\beta = |T| = |N_T[u_1^1] \cup N_T[u_1^2] \cup \cdots \cup N_T[u_1^j]| \leq \sum_{j=1}^\beta |N_T[u_1^j]| \leq k\beta \) since \( d_T(u_1^j) \leq k - 1 \) for \( 1 \leq j \leq \beta \).
Therefore the sets $N_T[u_i^j]$ are disjoint and $|N[u_i^j]| = k$ for $1 \leq j \leq \beta$. If one of the sets $N_T[u_i^j]$, say $N_T[u_i^j]$, does not induce a clique, let $a$ and $b$ two non-adjacent vertices of $N_T(u_i^j)$. Then $\{a, b, u_i^j, \ldots, u_i^\beta\}$ is an independent set of $\beta + 1$ elements of $G$, a contradiction. Hence each $U_i^j = N[u_i^j]$ is a clique and $G$ is the disjoint unions of $\beta$ cliques of order $k$. Let now $v$ be any vertex in $V \setminus T$. If every clique $U_i^j$ contains a vertex which is not adjacent to $v$, say $u_i^j v \notin E(G)$ for $1 \leq j \leq \beta$, then $\{v, u_i^1, u_i^2, \ldots, u_i^\beta\}$ is an independent set of $G$ of $\beta + 1$ elements, a contradiction which completes

Corollary 4. Every connected graph with order $n$ and clique number $\omega < n$ satisfies $\beta_\omega(G) < \omega \beta(G)$.

Proof. If $\beta_\omega(G) = \omega \beta(G)$, then every $\beta_\omega(G)$-set $T$ consists of disjoint cliques $K_{\omega(G)}$. Since $G$ is connected and different from $K_\omega$, $V \setminus T$ is not empty and every vertex $v \in V \setminus T$ forms with one of these cliques a clique of order $\omega + 1$, a contradiction. ■

Theorem 5. For every graph $G$ and integers $j, k$ with $1 \leq j \leq k$, $i_{k+1}(G) \leq (k - j + 2)i_j(G)$. Equality can occur only when $j = 1$ or $j = k$.

Proof. Let $S$ be a $i_j(G)$-set, $X = \{x \in S|d_S(x) = j - 1\}$ and $Y = S \setminus X = \{y \in S|d_S(y) < j - 1\}$. Since $j < k + 1$, the set $S$ is a $(k + 1)$-independent set of $G$. Let $I$ be a maximal $(k + 1)$-independent set of $G$ containing $S$, $A = N_{I \setminus S}(X)$ and $B = I \setminus (A \cup S)$. Since $I$ is $(k + 1)$-independent, $d_A(x) \leq k - j + 1$ for every $x \in X$, which implies $|A| \leq (k - j + 1)|X|$, and $d_B(y) \leq k$ for every $y \in Y$. Since the $j$-independent set $S$ is maximal in $G$, $d_Y(v) \geq j$ for every $v \in B$. Hence the number $m(Y, B)$ of edges of $G$ between $Y$ and $B$ satisfies $j|B| \leq m(Y, B) \leq k|Y|$, which implies $|B| \leq k|Y|/j \leq (k - j + 1)|Y|$. Therefore

$$i_{k+1}(G) \leq |I| = |A| + |B| + |S| \leq (k - j + 1)(|X| + |Y|) + |S| = (k - j + 2)i_j(G).$$

If $i_{k+1}(G) = (k - j + 2)i_j(G)$, then $i_{k+1}(G) = |I|$, $|A| = (k - j + 1)|X|$, and $|B| = k|Y|/j = (k - j + 1)|Y|$. Equality $|A| = (k - j + 1)|X|$ implies $d_A(x) = k - j + 1$ for every $x \in X$. Equality $|B| = k|Y|/j$ implies $d_B(y) = k$ for every $y \in Y$ and $d_Y(v) = j$ for every $v \in B$. Finally, $k/j = k - j + 1$ if and only if $j = 1$ or $j = k$. 


Case $j = 1$. If $i_{k+1}(G) = (k + 1)i(G)$ for some $k \geq 1$, then $Y = \emptyset$, $X = S$ is an independent set, $A = I \setminus S$, $|A| = k|X|$, the neighborhoods in $A$ of the $i(G)$ vertices of $S$ are disjoint and each of order $k$, and $G[A]$ has maximum degree at most $k - 1$.

Case $j = k$. If $i_{k+1}(G) = 2i_k(G)$ for some $k \geq 1$, then $|A| = |X|$, $|B| = |Y|$, the edges of $G$ between $A$ and $X$ form a perfect matching $M$ and the edges of $G$ between $B$ and $Y$ form a $k$-regular bipartite graph.

**Corollary 6.** For every graph $G$ of order $n$ and maximum degree $\Delta$, $i_{\Delta}(G) \geq n/2$, and this bound is sharp.

**Proof.** Obvious consequence of $i_{k+1}(G) \leq 2i_k(G)$ obtained from Theorem 5 when $j = k$ and $i_{\Delta+1}(G) = n$.

Let $G$ be obtained by attaching one pendant vertex at each vertex of a clique $K_k$. Then $n = 2k$, $\Delta = k$ and $i_k(G) = k$. Hence $i_{\Delta} = n/2$.

**Corollary 7.** If a graph $G$ of order $n \geq 2$ satisfies $i_2(G) = 2i(G)$, then $G$ contains an induced matching of size $i(G)$.

**Proof.** In the equality case $k = j = 1$ in Theorem 5, $S = X$, $G$ contains a perfect matching between $A$ and $X$, and this matching is induced of size $S = i(G)$ since $G[A]$ has maximum degree 0.

The converse of Corollary 7 is not true. For instance the cycle $C_6$ admits an induced matching $M$ of size $i(C_6) = 2$ but $i_2(C_6) = 3 < 2i(C_6)$.

The inequality $i_2(G) \leq 2i(G)$ cannot be improved to $i_2(G) \leq 2\gamma(G)$, even for trees, as shown by the caterpillar obtained by adding $k \geq 5$ pendant vertices at each vertex of a path $P_3$. However the next theorem improves it to $i_2(G) \leq \gamma(G) + i(G)$ in the class of trees and unicyclic graphs.

**Theorem 8.** If the graph $G$ contains at most one cycle, then $i_2(G) \leq \gamma(G) + i(G)$.

**Proof.** Let $S$ be a $i(G)$-set and $I$ a maximal 2-independent set of $G$ containing $S$. With the notation of Theorem 5, $X = S$ is independent, $A = N_I(S) = I \setminus S$, and the edges of $G[I]$ form an induced matching $M$ between $A$ and a subset $A'$ of $S$. Let $Z$ be a $\gamma(G)$-set, $M_1$ the edges of $M$ with no endvertex in $Z$, and $A_1$ ($A'_1$ respectively) the set of the endvertices of the edges of $M_1$ in $A$ ($A'$ respectively). If $\gamma(G) < |M|$, then $M_1 \neq \emptyset$
and since $M$ is induced, the vertices of $A_1 \cup A_1'$ cannot be dominated by vertices in $Z \cap (A \cup A')$. Hence the set $W = Z \setminus (A \cup A')$ is not empty and dominates $A_1 \cup A_1'$. Therefore the induced subgraph $G[W \cup A_1 \cup A_1']$ of order $|W| + 2|M_1|$ contains at least $3|M_1|$ edges. Moreover, since $Z$ contains at least one endvertex of each edge in $M \setminus M_1$, $|W| \leq |Z| - |M \setminus M_1| = (\gamma(G) - |M|) + |M_1| < |M_1|$. Thus $3|M_1| > |W| + 2|M_1|$, which contradicts the assumption that $G$ contains at most one cycle. Therefore $\gamma(G) \geq |M| = |A|$ and $i_2(G) \leq |S| + |A| \leq i(G) + \gamma(G)$.

The result of Theorem 8 is not valid for all graphs as shown by the following example. We consider eight disjoint triangles $x_iy_iy_i$ and identify the vertex $x_i$ with $x_{i+1}$ for $i = 1, 3, 5, 7$. Let $w_1, w_2, w_3, w_4$ denote the resulting new vertices. To complete $G$, we add the edges $w_1w_2, w_1w_3$ and $w_1w_4$. Then $\{w_2, y_3, w_3, y_5, w_4, y_7, y_1, z_1, y_2, z_2\}$ is a $i_2(G)$-set and thus $i_2(G) = 10, \gamma(G) = 4$, and $i(G) = 5$. By attaching $q$ triangles at each vertex $w_i$ instead of 2, $\gamma(G)$ does not change while now $i(G) = 3 + q$ and $i_2(G) = 6 + 2q$. Therefore the difference $i_2(G) - (i(G) + \gamma(G))$ can be done arbitrarily large and the ratio $i_2(G)/(i(G) + \gamma(G))$ arbitrarily close to 2.

The next corollary is another consequence of Theorem 5. A graph $G$ is well-covered if $i(G) = \beta(G)$ and well-k-covered if $i_k(G) = \beta_k(G)$.

**Corollary 9.** For any $k \geq 1$, $i_{k+1}(G) \leq (k + 1)i(G) \leq ki(G) + \beta(G)$ and if $i_{k+1}(G) = ki(G) + \beta(G)$, then $G$ is well-covered and well-(k + 1)-covered.

**Proof.** The inequality comes from Theorem 5 with $j = 1$. If $i_{k+1}(G) = ki(G) + \beta(G)$ then $i(G) = \beta(G)$, that is $G$ is well-covered, and thus $i_{k+1}(G) = (k + 1)\beta(G)$. Therefore $\beta_{k+1}(G) \leq i_{k+1}(G)$ from Corollary 2, which implies $\beta_{k+1}(G) = i_{k+1}(G)$ and proves that $G$ is well-(k + 1)-covered.

### 3. Graphs with $i_2 = i + \beta$

In this section we are interested in graphs $G$ satisfying the equality in Corollary 9 when $k = 1$. We describe two particular classes of them defined by forbidden subgraphs.

**Definition 10**

- The graphs $F$ of the family $\mathcal{F}$ are formed by five disjoint cliques $X_i$ of cardinality at least 2 together with all the edges between $X_i$ and $X_{i+1}$ for $1 \leq i \leq 5 \pmod{5}$.
The graphs $C_4$ and $g$ are shown in Figure 1.

![Figure 1]

Clearly every non-trivial clique and every graph of $\mathcal{F}$ satisfies $i_2(G) = i(G) + \beta(G)$ with $i(G) = \beta(G) = 1$ for a clique, $i(G) = \beta(G) = 2$ for a graph of $\mathcal{F}$.

**Theorem 11.** Let $G$ be a graph such that $i_2(G) = i(G) + \beta(G)$. Then

1. If $G$ is $g$-free, the component of $G$ are $\lambda_1$ non-trivial cliques with $\lambda_1 = i(G)$.
2. If $G$ is $C_4$-free, the components of $G$ are $\lambda_1 \geq 0$ non-trivial cliques and $\lambda_2 \geq 0$ graphs of $\mathcal{F}$ with $\lambda_1 + 2\lambda_2 = i(G)$.

**Proof.** From Corollary 9, $i(G) = \beta(G)$ and $i_2(G) = \beta_2(G)$. The maximal independent sets of $G$ have all the same cardinality and each maximal 2-independent set induces a matching $M$ of size $i(G)$. In particular, $G$ has no isolated vertex. We make an induction on the common value $\lambda$ of $i(G)$ and $\beta(G)$. If $\lambda = 1$, then $G$ is a clique of cardinality at least 2. For $\lambda > 1$, suppose the property true when $i_2(G) = i(G) + \beta(G) < 2\lambda$ and let $G$ be a $g$-free or $C_4$-free graph such that $i_2(G) = i(G) + \beta(G) = \beta_2(G) = 2\lambda$. Let $a$ be a vertex of $G$ such that $\beta(N[a])$ is maximum and let $G' = G - N[a]$. Every maximal independent set of $G'$ can be completed to a maximal independent set of $G$ by adding $a$. Hence $i(G') = \beta(G') = \lambda - 1$. If $S'$ is a $i_2(G')$-set then, by Theorem 5, $|S'| = i_2(G') \leq 2i(G') = 2\lambda - 2$. The 2-independent set $S' \cup \{a\}$ of $G$ can be completed to a maximal 2-independent set of $G$ by adding at most one vertex of $N(a)$. Hence $i_2(G) \leq |S'| + 2 \leq 2\lambda$. Since $i_2(G) = 2\lambda$, $i_2(G') = 2\lambda - 2 = i(G') + \beta(G')$. By the induction hypothesis applied to $G'$, which is $g$-free or $C_4$-free as $G$, the components of $G'$ are $\mu_1$ non-trivial cliques if $G$ is $g$-free, $\mu_1$ non-trivial cliques and $\mu_2$ graphs of $\mathcal{F}$ if $G$ is $C_4$-free, with $\mu_1 + 2\mu_2 = \lambda - 1$. To continue, we distinguish two cases.
Case 1. \(N[a]\) is a clique of \(G\). By the choice of \(a\), \(N[x]\) is a clique for every vertex \(x\) of \(G\). Therefore the components of \(G\) are \(\lambda\) non-trivial cliques.

Case 2. \(N[a]\) is not a clique. Let \(b\) and \(c\) be two non-adjacent vertices of \(N(a)\). If for each component \(H\) of \(G'\), \(V(H) \setminus (N(b) \cup N(c)) \neq \emptyset\) if \(H\) is a clique and \(\beta(V(H) \setminus (N(b) \cup N(c))) = 2\) if \(H \in \mathcal{F}\), then \(\beta(G) \geq \beta(G') + 2 = \lambda + 1\) which is impossible. Therefore there exists a component \(H\) of \(G'\) such that either \(H\) is a non-trivial clique and \(V(H) \subseteq N(b) \cup N(c)\) or \(H \in \mathcal{F}\) and \(\beta(V(H) \setminus (N(b) \cup N(c))) < 2\).

Subcase 2.1. Suppose first that \(H \in \mathcal{F}\) and \(\beta(V(H) \setminus (N(b) \cup N(c))) \leq 1\). Then \(G\) is not \(g\)-free and thus is \(C_4\)-free. We will prove that this subcase is impossible. Let \(X_i\), \(1 \leq i \leq 5\) be the five cliques of \(H\) as described in the definition of \(\mathcal{F}\). Then \(V(H) \setminus (N(b) \cup N(c))\) is a (possibly empty) clique \(U\) and since \(G\) is \(C_4\)-free, \(N(b) \cap N(c) \cap V(H) = \emptyset\). If \(V(H) \subseteq N(b)\), then a maximal 2-independent set \(I\) of \(G\) containing \(\{a, b\}\) contains no other vertex in \(N[a] \cup V(H)\) and at most \(2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6\) vertices in \(G' - V(H)\), that is \(|I| \leq 2\lambda - 4\) which is impossible. Hence \(V(H)\) is not contained in \(N(b)\), neither in \(N(c)\) by symmetry. If \(V(H) \cap N(b)\) and \(V(H) \cap N(c)\) are cliques, then \(U \neq \emptyset\) and \(H\) contains an edge \(uv\) with \(u \in U\) and \(v \in N(c)\). The set \(S = \{u, v, a, b\}\) dominates \(V(H) \cup N[a]\). If \(N(b) \cap V(H)\) is not a clique, say \(b\) is adjacent to \(x_1 \in X_1\) and to \(x_3 \in X_3\), then by \(C_4\)-freeness, \(b\) is adjacent to every vertex \(x_2\) of \(X_2\) and to no vertex of \(X_4 \cup X_5\) for otherwise \(V(H) \subseteq N(b)\). Similarly, if \(V(H) \cap N(c)\) is neither a clique then \(N(c)\) entirely contains \(X_4\) or \(X_5\), say \(N(c)\) contains vertices in \(X_3\) and in \(X_5\) and entirely contains \(X_4\). The set \(U\) is contained in \(X_1 \cup X_5\). If \(U \neq \emptyset\), then \(U\) contains a vertex \(u\) adjacent to some vertex \(v\) in \(X_2\) or in \(X_4\), say \(v \in X_4\). The set \(S = \{u, v, a, b\}\) dominates \(V(H) \cup N[a]\). If \(U = \emptyset\), then \(b\) (respectively \(c\)) entirely dominates \(X_1\) (respectively \(X_3\)). Let \(u\) and \(v\) be vertices in \(X_4\). Again \(S = \{u, v, a, b\}\) dominates \(V(H) \cup N[a]\). Finally, if \(V(H) \cap N(b)\) is not a clique and \(V(H) \cap N(c)\) is a clique \(C\), we can find two adjacent vertices \(u\) and \(v\) with \(u\) in \(C\) or in \(U\), depending on whether \(U\) is or not equal to \(\emptyset\), and \(v\) in \(C\). The set \(S = \{u, v, a, b\}\) dominates \(V(H) \cup N[a]\). In any case, the set \(S\) is 2-independent and a maximal 2-independent set of \(G\) containing \(S\) contains no other vertex in \(V(H) \cup N[a]\) and at most \(2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6\) vertices in \(G' - V(H)\). Hence \(i_2(G) \leq 2\lambda - 2\), a contradiction. Therefore Subcase 2.1 is impossible.
Subcase 2.2. $H$ is a non-trivial clique contained in $N(b) \cup N(c)$. Since every maximal 2-independent set $S$ of $G$ contains at most two vertices in each clique-component and four vertices in each $\mathcal{F}$-component of $G'$, 

$$2\lambda = i_2(G) \leq 2(\mu_1 - 1) + 4\mu_2 + i_2(V(H) \cup N[a]) = 2\lambda - 4 + 4\mu_2,$$

which gives $i_2(V(H) \cup N[a]) \geq 4$. Hence if $S$ is a maximal 2-independent set of $G[V(H) \cup N[a]]$ containing $\{a, c\}$, then $|S \cap (V(H) \setminus N(c))| \geq 2$. Let $x$ and $x'$ be two vertices in $V(H) \setminus N(c) \subseteq V(H) \cap N(b)$. Similarly, $V(H) \setminus N(b)$ contains at least two vertices $y$ and $y'$ which are adjacent to $c$. The induced subgraph $G[\{a, b, c, x, x', y, y'\}]$ is equal to $g$. Hence $G$ is $C_4$-free and $V(H)$ is partitioned into $V(H) \cap N(b)$ and $V(H) \cap N(c)$.

If $\beta(N[a]) > 2$, let $\{a_1, a_2, \ldots, a_t\}$ be a $\beta(N[a])$-set with $\ell \geq 3$. For each pair $\{a_i, a_j\}$ of nonadjacent vertices of $N(a)$, there exists, by Subcase 2.1, a non-trivial clique-component $H_{ij}$ of $G'$ contained in $N(a_i) \cap N(a_j)$. Since $G$ is $C_4$-free, $V(H_{ij}) \cap N(a_i)$ and $V(H_{ij}) \cap N(a_j)$ partition $V(H_{ij})$ and the $\ell(\ell - 1)/2$ cliques $H_{ij}$ are different. Then any maximal independent set $I$ of $G$ containing $\{a_1, a_2, \ldots, a_t\}$ satisfies $\beta(G) - |I| \geq \ell(\ell - 1)/2 + 1 - \ell > 0$, contradicting $i(G) = \beta(G)$. Therefore $\beta(N[a]) = 2$.

Let $u$ be a vertex of $N(a)$ adjacent to $c$ but not to $b$, if any. As above, let $H'$ be the clique-component of $G'$ contained in $N(b) \cup N(u)$. If $H' \neq H$, then $N(b)$ contains at least one vertex in $V(H)$, one vertex in $V(H')$ and $a$, that is $\beta(N[b]) \geq 3$, in contradiction to the choice of $a$. Therefore $H' = H$ and $N(u) \cap V(H) = N(c) \cap V(H)$. Similarly, $N(v) \cap V(H) = N(b) \cap V(H)$ for every vertex $v$ of $N(a)$ adjacent to $b$ but not to $c$. Hence every vertex $z$ in $N[a] \cup H$ satisfies $\beta(N[z] \cap (V(H) \cup N[a])) = 2$ and by the choice of $a$, $N[z] \subseteq V(H) \cup N[a]$ and $N[a] \cup V(H)$ forms a component $L$ of $G$. For each vertex $z$ of $L$, the vertices of $L$ which are not adjacent to $z$ form a clique. The clique $V(L) \setminus N[a]$ is $H$. Let $\mathcal{B}, \mathcal{C}, \mathcal{X}, \mathcal{Y}$ be respectively the cliques $V(L) \setminus N[b]$, $V(L) \setminus N[c]$, $V(L) \setminus N[x]$, $V(L) \setminus N[y]$. Let $B = C \cap \mathcal{Y}$, $A = \mathcal{Y} \cap \mathcal{X}$, $C = \mathcal{X} \cap \mathcal{B}$, $Y = B \cap H$, $X = H \cap \mathcal{C}$. Then $a \in A$, $b \in B$, $c \in C$, $x \in X$ and $y \in Y$. Since $G$ is $C_4$-free, $A \cup B = \emptyset$ and $(A, B)$ form a partition of $\mathcal{Y}$. Similarly $(A, C)$, $(C, Y)$, $(Y, X)$ and $(X, B)$ respectively form a partition of $\mathcal{X}$, $\mathcal{B}$, $V(H)$ and $C$. Finally if, say, $|A| = 1$, then every maximal 2-independent set of $G$ containing $\{x, y, a\}$ contains no other vertex in $L$ and thus at most $2(\mu_1 - 1) + 4\mu_2 + 3 = 2(\lambda - 1) + 1$ vertices, in contradiction to $i_2(G) = 2\lambda$. Therefore each of the five cliques $A, B, X, Y, C$ has at least two vertices and $L$ is a graph of $\mathcal{F}$. The components of $G'$ are $\mu_1$ non-trivial cliques and $\mu_2$ graphs of $\mathcal{F}$ with $\mu_1 + 2\mu_2 = \lambda - 1$. Hence the components
of $G$ are $\lambda_1 = \mu_1 - 1$ non-trivial cliques and $\lambda_2 = \mu_2 + 1$ graphs of $\mathcal{F}$ with $\lambda_1 + 2\lambda_2 = \mu_1 + 2\mu_2 + 1 = \lambda$. This completes the proof.

\section{Bounds on $i_2$}

In [4], it is proved that every graph $G$ of maximum degree $\Delta \geq 1$ satisfies $i_k(G) \geq (n + k - 1)/(\Delta + 1)$ for $1 \leq k \leq n - 1$ and examples of extremal graphs are given for $k \geq 3$. Here we slightly improve the bound when $k = 2$ and characterize the extremal graphs.

\textbf{Theorem 12.} Let $G$ be a connected graph of order $n \geq 2$ and maximum degree $\Delta$. Then $i_2(G) \geq (n + 2)/(\Delta + 1)$, with equality if and only if $G = P_2$ or $G$ is obtained from a double star $S_{\Delta-1,\Delta-1}$ by adding zero or more edges between its leaves without creating a vertex of degree larger than $\Delta$.

\textbf{Proof.} If $n = 2$, then $i_2(P_2) = 2 = (n + 2)/(\Delta + 1)$. If $n = 3$ then $G = P_3$ or $C_3$ and $i_2(G) = 2 > (n + 2)/(\Delta + 1)$. So assume that $n \geq 4$ and $\Delta \geq 2$ since $G$ is connected. Let $S$ be an $i_2(G)$-set, $p$ be the number of edges in $G[S]$ and $t$ the number of edges joining the vertices in $S$ and $V - S$. Assume first that $p \geq 1$. Then since the $p$ edges are independent, $t \leq 2p(\Delta - 1) + (|S| - 2p)\Delta$. Also since $S$ dominates $V - S$, $t \geq |V - S|$. It follows that $|V - S| \leq t \leq 2p(\Delta - 1) + (|S| - 2p)\Delta$. Thus

$$i_2(G) = |S| \geq (n + 2p)/(\Delta + 1) \geq (n + 2)/(\Delta + 1).$$

If further $i_2(G) = (n + 2)/(\Delta + 1)$, then we must have equality throughout the above inequality chain, in particular we have $p = 1$, every vertex of $G[S]$ has degree $\Delta$ and every vertex of $V - S$ is adjacent to exactly one vertex of $S$. If $S$ contains an isolated vertex say $u$, then $S \cup \{v\}$ is a 2-independent set of $G$, where $v \in V - S$ is any neighbor of $u$, contradicting the maximality of $S$. Therefore $S$ contains only two adjacent vertices, each of degree $\Delta$, and $G$ has the structure described in the theorem. The converse is easy to show.

Now assume that $p = 0$. Then $t \leq \Delta|S|$. If $V - S$ contains any vertex, say $w$, that has only one neighbor in $S$ then $S \cup \{w\}$ is a 2-independent set of $G$, a contradiction with the maximality of $S$. Thus each vertex of $V - S$ has at least two neighbors in $S$ and hence $t \geq 2|V - S|$. It follows that $\Delta|S| \geq t \geq 2|V - S|$ and so $i_2(G) \geq 2n/(\Delta + 2)$. Notice that $2n/(\Delta + 2) \geq (n + 2)/(\Delta + 1)$ for $n \geq 4$ with equality if and only if $n = 4$ and $\Delta = 2$. 

If further $i_2(G) = (n + 2)/(\Delta + 1)$ then $n = 4$, $\Delta = 2$ and every vertex of $V - S$ has exactly two neighbors in $S$. Thus $G$ is a cycle $C_4$ which is obtained from a double star $S_{1,1}$ by adding an edge joining the two leaves.

In [1], Blidia et al. have given an upper bound on $i_2(G)$ for every nontrivial connected bipartite graph.

**Theorem 13.** If $G$ is a connected nontrivial bipartite graph with $s(G)$ support vertices, then $i_2(G) \leq (n + s(G))/2$.

When $G$ is a cactus, this upper bound can be extended to non-bipartite graphs. First we give a lemma related to matchings in cactus.

**Lemma 14.** In every cactus $G$ with $k$ odd cycles, there exists a matching of size $k$ containing exactly one edge in each odd cycle of $G$.

**Proof.** We proceed by induction on the number of odd cycles. Clearly the property is true for $k = 0$ and $k = 1$. Let $k \geq 2$. Assume the property true for cactus with less than $k$ odd cycles and let $G$ be a cactus with $k$ odd cycles. Let $C = x_1 x_2 \cdots x_{2p+1}$ with $p \geq 1$ be an odd cycle of $G$. For each $x_i \in V(C)$, let $A_i = N(x_i) \setminus V(C)$. By the definition of cactus, all the sets $A_i$ are disjoint. Let $G'$ be the graph obtained from $G$ by contracting the cycle $C$ into one vertex $c$. More precisely, $V(G') = (V(G) \setminus V(C)) \cup \{c\}$ and for $1 \leq i \leq 2p+1$, the edges between $x_i$ and $A_i$ are replaced by the edges between $c$ and $A_i$.

Every cycle $C \neq C'$ of $G$ is unchanged in $G'$ if $V(C) \cap V(C') = \emptyset$ or is replaced by a cycle $C'$ of same length and containing $c$ if $|V(C) \cap V(C')| = 1$. Hence $G'$ is a cactus with $k-1$ odd cycles and by the inductive hypothesis, contains a matching $M'$ of size $k-1$ with exactly one edge in each of its odd cycles. All the edges of $M'$ are edges of $G$ except possibly one, say $cy_1$ with $y_1 \in A_1$. In this case, the edge $cy_1$ belongs to an odd cycle $C'_1$ of $G'$ corresponding to an odd cycle $C_1$ of $G$ containing $x_1$. The set $M = M' \cup \{x_2 x_3\}$ if $M' \subseteq E(G)$, $M = (M' \setminus \{cy_1\}) \cup \{x_1 y_1, x_2 x_3\}$ if $cy_1 \in M'$, is a matching of $G$ containing exactly one edge in each of its odd cycles.

**Theorem 15.** If $G$ is a connected nontrivial cactus graph with $k$ odd cycles and $s(G)$ support vertices, then $i_2(G) \leq (n + s(G) + k)/2$ and this bound is sharp.
Proof. Let $G$ be a connected nontrivial cactus graph with $k$ odd cycles and $s(G)$ support vertices. If $k = 0$, then $G$ is a bipartite graph and hence by Theorem 13 the result is valid. So assume that $G$ contains at least one odd cycle. By Lemma 14, there exists in $G$ a matching $M$ of size $k$ containing one edge in each odd cycle of $G$. We subdivide each edge of $M$ by exactly one vertex. Let $D$ be the set of such vertices and $G' = (V', E')$ the resulting graph. Then every vertex of $D$ has degree two and $G'$ is a connected bipartite graph of order $n + k$ with $s(G') = s(G)$ and different from a tree. Let $C$ be a set of leaves of $G'$ so that every support vertex has exactly one leaf in $C$. Clearly $|C| = s(G)$. Let $A$ and $B$ be the two classes of the bipartition of $G'[V' \setminus C]$ with $|A| \leq |B|$. Then $|B| \geq (n + k - s(G'))/2 \geq |A| > 0$. Let $S_A$, $C_A$ denote the set of support vertices and leaves of $G'$ belonging to $A$, respectively, and let $A' = A \setminus (S_A \cup C_A)$. Likewise, we define $S_B$, $C_B$ and $B'$. The 2-independent set $S' = A \cup C$ is maximal in $G'$ since every leaf of $B$ is adjacent to a support vertex of $A$, which has degree one in $G'[S']$, and the other vertices of $B$ have at least two neighbors in $A$. Its order satisfies

$$|S'| = |A \cup C| \leq (n + k - s(G'))/2 + |C| = (n + s(G) + k)/2.$$  

We shall construct a maximal 2-independent set $S$ of $G$ with $|S| \leq (n + s(G) + k)/2$. Let $D_A = D \cap A$, $D_B = D \cap B$, $F_B = N(D_A) \cap B$ and $F_A = N(D_B) \cap A$. Note that each of $G[F_A]$ and $G[F_B]$ consists of disjoint copies of $P_2$. Then $F_B \subset B \setminus (C_B \cup D_B)$, $F_A \subset A \setminus (C_A \cup D_A)$. Thus each component in $G[A \setminus D_A]$ is either an isolated vertex or a path $P_2$. So $A \setminus D_A$ is a 2-independent set but $(A \setminus D_A) \cup C$ may be not 2-independent. This occurs if $F_A \cap S_A \neq \emptyset$. In that case delete from $C$ all leaves adjacent to $F_A \cap S_A$ and let $C' \subseteq C$ be the resulting set. Thus $(A \setminus D_A) \cup C'$ is a 2-independent set. To extend it to a maximal 2-independent set of $G$, we can only add vertices of $B \setminus D_B$ having only one neighbor in $A \setminus D_A$ and this neighbor must be isolated in $G[(A \setminus D_A) \cup C']$. Hence we add at most one endvertex of each edge of $G[F_B]$, that is at most $|D_A|$ vertices. Thus $i_2(G) \leq |(A \setminus D_A) \cup C'| + |D_A| \leq |A \cup C| = |S'|$. This completes the proof.

Odd cycles are examples of graphs attaining the bounds. 

References

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