FRACTIONAL DOMINATION IN PRISMS

MATTHEW WALSH

Department of Mathematical Sciences
Indiana-Purdue University
Fort Wayne, Indiana 46805, USA
e-mail: walshm@ipfw.edu

Abstract

Mynhardt has conjectured that if $G$ is a graph such that $\gamma(G) = \gamma(\pi G)$ for all generalized prisms $\pi G$ then $G$ is edgeless. The fractional analogue of this conjecture is established and proved by showing that, if $G$ is a graph with edges, then $\gamma_f(G \times K_2) > \gamma_f(G)$.

Keywords: fractional domination, graph products, prisms of graphs.

2000 Mathematics Subject Classification: 05C69.

Throughout let us assume that graphs are finite and simple; our notation concurs with [3]. Let $G = (V, E)$ be a graph; the (closed) neighbourhood $N[v]$ of a vertex $v \in V$ consists of $v$ itself and all vertices $u \in V$ such that $u \sim v$. A set $S \subseteq V$ is independent if no two members of $S$ are adjacent; $S$ is dominating if $\cup_{v \in S} N[v] = V$. The size of a smallest dominating set in $G$ is denoted by $\gamma(G)$ and termed the domination number of $G$.

By generalizing “set” to “fuzzy set” in the definition of domination, one can define the concept of fractional domination. A function $f : V \rightarrow [0, 1]$ is a fractional dominating function precisely when $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V$. If one defines the size of a fractional dominating function $f$ by $|f| = \sum_{v \in V} f(v)$ then one can talk about the minimum size of a fractional dominating function of $G$; this is the fractional domination number of $G$ and denoted by $\gamma_f(G)$. Since the characteristic function of a dominating set in $G$ is clearly a fractional dominating function of $G$, $\gamma_f(G) \leq \gamma(G)$.

(Notation will sometimes be abused in the following standard fashions: if $S$ is a set of vertices, then $f(S) = \sum_{v \in S} f(v)$. Thus, $|f| = f(V)$. In the
particular case where the set in question is the closed neighbourhood $N[v]$ of the vertex $v$, the notation is further condensed to $f[v] = f(N[v]).$

An equitable partition $P_1, \ldots, P_k$ of the vertices of a graph $G$ is a partition with the properties that every induced graph $G[P_i]$ is regular, and every induced bipartite graph between two cells $P_i, P_j$ is biregular. The following result can be found in [5].

**Theorem 1.** If $G$ is a graph that admits an equitable partition $\{P_i\}_{i=1}^k$, then there exists a minimum fractional dominating function of $G$ that is constant on each cell $P_i, i = 1, \ldots, k$.

Suppose that $G$ is a graph and $\pi$ a permutation on its vertex set $V$. The generalized prism $\pi G$ is the graph with vertex set $V_{\pi} = V \times \{0, 1\}$, with $(u, i) \sim (v, j)$ when either $i = j$ and $u \sim v$ in $G$, or else $i \neq j$ and $v = \pi(u)$.

When $\pi = 1$, the identity permutation, then the graph $1G = G \times K_2$ is called the prism of $G$.

The following result from [1] is easily shown.

**Lemma 2.** For any graph $G$ and any permutation $\pi$ of its vertex set, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$.

A graph $G$ for which $\gamma(G) = \gamma(\pi G)$ for any permutation $\pi$ is a universal $\gamma$-fixer; if $2\gamma(G) = \gamma(\pi G)$ for all $\pi$, then $G$ is a universal $\gamma$-doubler. In [4] it is conjectured that the only universal $\gamma$-fixers are graphs without edges.

This paper is concerned with the fractional analogue of the conjecture mentioned above. To develop this, some elementary tools are needed.

As discussed in [2], for a function $f : V \to [0, 1]$ define the sets $B_f = \{v \in V : f[v] = 1\}$ and $P_f = \{v \in V : f(v) > 0\}$.

**Lemma 3** [2]. A dominating function $f$ is a minimal dominating function if and only if $B_f$ dominates $P_f$.

If $f$ is a fractional dominating function of the prism $\pi G$, then define the condensation $f_\pi : V(G) \to [0, 1]$ of $f$ by

$$f_\pi(v) = \min\{1, f((v, 0)) + f((\pi(v), 1))\}$$

for all $v \in V(G)$.

**Lemma 4.** If $f$ is a fractional dominating function on $\pi G$, then its condensation $f_\pi$ is a fractional dominating function on $G$ with $|f_\pi| \leq |f|$. 

**Proof.** Let \( v \in V(G) \) and consider \( \sum_{u \in N_G[v]} f_\pi(u) \). If \( f_\pi(v) = 1 \) then clearly this sum exceeds 1; otherwise, for each \( u \in N_G(v) \) we have that \( f_\pi(u) \geq f((u, 0)) \), and \( f_\pi(v) = f((v, 0)) + f((\pi(v), 1)) \). Hence
\[
\sum_{u \in N_G[v]} f_\pi(u) = f_\pi(v) + \sum_{u \in N_G(v)} f_\pi(u) \\
\geq f((v, 0)) + f((\pi(v), 1)) + \sum_{u \in N_G(v)} f((u, 0)) \\
= \sum_{x \in N_{\pi G}((v, 0])} f(x) \\
\geq 1.
\]
A similar calculation shows that \( |f_\pi| \leq |f| \).

**Corollary 5.** For any graph \( G \) and any permutation \( \pi \) of its vertex set, \( \gamma_f(G) \leq \gamma_f(\pi G) \leq 2\gamma_f(G) \), and these bounds are sharp.

**Proof.** The lower bound follows from Lemma 4. To show the upper bound, let \( f \) be a minimum fractional dominating function of \( G \). Then the function \( f' : V(\pi G) \rightarrow [0, 1] \) defined by \( f'((u, i)) = f(u) \) is fractional dominating with \( |f'| = 2|f| \).

An example of the lower bound occurs when \( G \) contains no edges and \( \pi \) is an arbitrary permutation: \( \gamma_f(G) = \gamma_f(\pi G) = |V(G)| \). For the upper bound, let \( G = K_{1,n} \) for \( n \geq 2 \) and let \( \pi \) be any automorphism of \( G \); then \( \gamma_f(G) = 1 \) and \( \gamma_f(\pi G) = 2 \).

The fractional version of Mynhardt’s question is then: *For which graphs \( G \) is it true that, for any permutation \( \pi \) of \( V(G) \), \( \gamma_f(\pi G) = \gamma_f(G) \)? Such a graph would naturally be termed a universal \( \gamma_f \)-fixer. As it turns out, this question can be answered without considering any permutations other than the identity.*

**Lemma 6.** Let \( f \) be fractional dominating on \( 1G \) with condensation \( f_1 \) such that \( |f_1| = |f| \). Then for any vertex \( v \in V(G) \), \( f_1[v] = f((v, 0)) + f((v, 1)) - f_1(v) \).

**Proof.** Since \( |f_1| = |f| \) is follows that \( f_1(v) = f((v, 0)) + f((v, 1)) \) for all vertices \( v \). The result then follows from a simple computation using the fact that \( f((v, i)) = f(\{(u, i) : u \in N_G[v]\} + f((v, 1 - i)) \) for \( i = 0, 1 \).
Lemma 7. Let 1G be the prism of a simple graph G with vertex set V = \{v_1, \ldots, v_n\}. Then the collection of sets \{(v_i, 0), (v_i, 1)\}_{i=1}^n forms an equitable partition of the vertices of 1G.

Proof. Let \(P_i\) denote the set containing the images of \(v_i\) in the prism. Each \(1G[P_i]\) consists of a single edge (and is thus 1-regular); the bipartite graph between \(P_i\) and \(P_j\) will either be edgeless (if \(v_i\) and \(v_j\) are not adjacent) or 1-regular.

Theorem 8. Let \(G\) be a graph such that \(\gamma_f(1G) = \gamma_f(G)\). Then \(G = \overline{K_n}\) for some positive integer \(n\).

Proof. Let \(G\) be a graph such that \(\gamma_f(1G) = \gamma_f(G)\), and suppose that \(f\) is a minimum fractional dominating function of 1G with condensation \(f_1\). Let us assume (by Theorem 1 and Lemma 7) that for any \(v \in V(G)\), \(f((v, 0)) = f((v, 1))\). By Lemma 4 \(f_1\) is a fractional dominating function of \(G\) with \(|f|_1 \leq \gamma_f(1G) = \gamma_f(G)\), and hence \(f_1\) is in fact a minimum fractional dominating function of \(G\). Further, by this equality we know that \(f((v, 0)) + f((v, 1)) \leq 1\), and hence that \(f(x) \leq \frac{1}{2}\) for any vertex \(x \in V(1G)\).

Suppose that \(v\) is a vertex in \(G\) such that \(f_1(v) = 0\). Then by Lemma 6, \(f_1(N[v]) = f(N[(v, 0)]) + f(N[(v, 1)])\), and since \(f\) is fractional dominating in 1G the two right-hand terms are each at least 1; hence, \(f_1(N[v]) \geq 2\) for any vertex \(v\) receiving a weight of 0.

Let \(v^* \in V(G)\) be such that \(f_1[v^*] = 1\); such a vertex exists from Lemma 3. It follows that \(f[(v^*, 0)] = \frac{1}{2}f_1[v^*] + \frac{1}{2}f_1(v^*) = \frac{1}{2} + \frac{1}{2}f_1(v^*) \geq 1\) since \(f\) is dominating; hence \(f_1(v^*) \geq 1\) so \(f_1(v^*) = 1\). Moreover, \(f_1(u) = 0\) for all \(u \sim v^*\).

By Lemma 3, if \(f_1(w) > 0\) then there exists \(v^* \in N[w]\) such that \(f_1[v^*] = 1\); if \(v^* \in N(w)\) then \(f_1(w) = 0\), contradicting our premise, and hence \(w = v^*\) and so \(f_1(w) = 1\). Therefore, \(f_1\) is the characteristic function of an independent 2-dominating set of \(G\). (A 2-dominating set \(S\) is one where, for every vertex \(u \notin S\), \(|N(u) \cap S| \geq 2\). The 2-dominance comes from the fact that \(f\) only takes the values 0 and \(\frac{1}{2}\); any vertex in 1G which receives a weight of 0 must therefore be adjacent to two vertices in the support of \(f\), and this carries over into the condensation.)

So let \(d = d_G(v^*) \) for some vertex \(v^*\) such that \(f_1(v^*) = 1\), and suppose that \(d > 0\). Pick some \(w \in V(G)\) that is distance 2 from \(v\) such that \(f_1(w) > 0\); this exists by fact that the support of \(f_1\) is 2-dominating.
Define the function $f^*: V(G) \to [0, 1]$ as follows:

$$f^*(v) = \begin{cases} 
0 & \text{if } v = v^*, \\
\frac{1}{d} & \text{if } v \sim v^*, \\
1 - \frac{1}{d} & \text{if } v = w, \\
f_1(v) & \text{otherwise.}
\end{cases}$$

$f^*$ is a fractional dominating function of $G$: If $v$ is a vertex such that $f_1(v) = 1$, then clearly $f^*(v) = 1$. Otherwise $f_1(v) = 0$ and hence $f_1[v] \geq 2$ as $v$ has at least two neighbours with weight 1. If its only two such neighbours are $v^*$ and $w$, then $f^*[v] = f^*(v) + f^*(w) = 1$; otherwise, it is clear that $f^*[v] \geq 1$.

But $|f^*| < |f_1|$, so the latter is not minimum, and hence $\gamma_f(G) < \gamma_f(1G)$. This fails only when there is no $v^*$ with neighbouring vertices, and hence only when $G$ contains no edges.

**Corollary 9.** The only universal $\gamma_f$-fixers are the edgeless graphs.

One consequence of this result to the original conjecture is that if $G$ is a $\gamma$-fixer with respect to the identity permutation and not empty then it must be the case that $\gamma_f(G) < \gamma(G)$, and hence this must be true of any universal $\gamma$-fixer.

Much of the power in the proof of Theorem 8 comes from the fact that the equitable partition in $1G$ guaranteed by Lemma 7 allows us to restrict our choice of fractional dominating functions significantly. This can be exploited for more general permutations $\pi$.

**Theorem 10.** Let $G$ be a graph that admits the equitable partition $P_1, \ldots, P_k$, and let $\pi$ be a permutation of $V(G)$ that fixes each $P_i$ setwise. Then $\gamma_f(G) = \gamma_f(\pi G)$ if and only if $G$ is edgeless.

**Proof.** The images $\{(v,j) : v \in P_i, j \in \{0,1\}\}$ of the partition cells $P_i$ form an equitable partition in $\pi G$, so we find a minimum fractional dominating function $f$ of $\pi G$ that is constant on each of these sets. Using this, we can show (analogously to Lemma 6) that if $f_\pi$ is the condensation of $f$ to $G$, then $f_\pi(N[v]) = f(N[(v,0)]) + f(N[(\pi v,1)]) - f_\pi(v)$. The proof then echoes that of Theorem 8.
Finally, here is a construction for $\gamma_f$-fixers with respect to restricted classes of permutations. Construct the corona $\text{cor}(G)$ of a graph $G$ by adjoining a pendant vertex to every node of $G$.

**Theorem 11.** For any graph $G$, let $V = V(G)$ and $V^* = V(\text{cor}(G)) - V$. Let $\pi$ be any permutation of $V(\text{cor}(G))$ such that $\pi(V) = V^*$. Then $\gamma_f(\text{cor}(G)) = \gamma_f(\pi \text{cor}(G))$.

**Proof.** Since the closed neighbourhoods of pendant vertices in $\text{cor}(G)$ are disjoint, $\gamma_f(\text{cor}(G)) = |V|$. Define $f$ on $V(\pi \text{cor}(G))$ by

$$ f((v, i)) = \begin{cases} \frac{1}{2} & \text{if } v \in V, \\ 0 & \text{if } v \in V^*. \end{cases} $$

Then $f$ is fractional dominating, and $|f| = |V|$.

An example of this construction is shown in Figure 1, with $P_4 = \text{cor}(P_2)$.

![Figure 1. $P_4$ and its prism $\pi P_4$, where $\pi = (12)(34)$, with minimum fractional dominating functions.](image)

The author would like to thank the anonymous referees for their helpful suggestions, and also R. Rubalcaba who commented on an early version of this paper.

**References**

Fractional Domination in Prisms


Received 28 September 2006
Revised 24 April 2007
Accepted 25 April 2007