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COUNTEREXAMPLE TO A CONJECTURE ON THE STRUCTURE OF BIPARTITE PARTITIONABLE GRAPHS

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Abstract

A graph G is called a prism fixer if $\gamma(G \times K_2) = \gamma(G)$, where $\gamma(G)$ denotes the domination number of G. A symmetric γ-set of G is a minimum dominating set D which admits a partition $D = D_1 \cup D_2$ such that $V(G) - N[D_i] = D_j$, $i, j = 1, 2$, $i \neq j$. It is known that G is a prism fixer if and only if G has a symmetric γ-set.

Hartnell and Rall [On dominating the Cartesian product of a graph and K_2, *Discuss. Math. Graph Theory* **24** (2004), 389–402] conjectured that if G is a connected, bipartite graph such that $V(G)$ can be partitioned into symmetric γ-sets, then $G \cong C_4$ or G can be obtained from $K_{2t,2t}$ by removing the edges of t vertex-disjoint 4-cycles. We construct a counterexample to this conjecture and prove an alternative result on the structure of such bipartite graphs.

Keywords: domination, prism fixer, symmetric dominating set, bipartite graph.

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1. Introduction

We follow [6] for domination terminology and [3] for other graph theoretical notation and terminology. Specifically, for any graph $G = (V, E)$ and $v \in V$, the *open neighbourhood* $N(v)$ of v is defined by $N(v) = \{u \in V : uv \in E\}$, and its *closed neighbourhood* $N[v]$ by $N(v) \cup \{v\}$. For $S \subseteq V$, $N(S) = \bigcup_{s\in S} N(s)$ and $N[S] = \bigcup_{s\in S} N[s]$. For $A, B \subseteq V$, $N_A(B) = N(B) \cap A$; when $B = \{u\}$ we write $N_A(u)$ instead of $N_A(B)$. A set $S \subseteq V$ *dominates* G, written $S \succ G$, if every vertex in $V - S$ is adjacent to a vertex in S, i.e., if $V = N[S]$. The *domination number* $\gamma(G)$ of G is defined by $\gamma(G) = \min\{|S| : S \succ G\}$. A *$\gamma$-set* of G is a dominating set of G of cardinality $\gamma(G)$. Further, a γ-set D of G is a *symmetric γ-set* if D has a partition $D = D_1 \cup D_2$ such that $V(G) - N[D_i] = D_j$, $i, j = 1, 2$, $i \neq j$. (Symmetric γ-sets are called *two-colored γ-sets* in [4, 5].)

A set $S \subseteq V$ is a *packing* (also called a 2-packing) of G if $N[u] \cap N[v] = \phi$ for all distinct $u, v \in S$. A dominating set D of G is an *efficient dominating set* (also known as a perfect code, or a perfect single-error-correcting code) if $|D \cap N[v]| = 1$ for each $v \in V(G)$. Thus D is an efficient dominating set if and only if D is a dominating set and a packing. As shown in [1] and [10], respectively, deciding whether a general graph and a bipartite graph, respectively, has an efficient dominating set, is NP-complete.

The cartesian product $G \times K_2$ is also called the *prism* of G. It is easy to see that $\gamma(G) \leq \gamma(G \times K_2) \leq 2\gamma(G)$ for all graphs G. If the lower bound is satisfied, then G is called a *prism fixer*. It is evident from the characterization of prism fixers as graphs that possess symmetric γ-sets (Theorem 2, [5, 7]) that if G is a prism fixer, then $G \times K_2$ has an efficient dominating set, i.e., a perfect code. (Note that the converse of this statement is not true. For example, the hypercube Q_7 is known to have a perfect code [6, Theorem 4.8] and $\gamma(Q_7) = 16$. Also, $Q_7 = Q_6 \times K_2$, but Q_6 is not a prism fixer because $\gamma(Q_6) = 12$ [8].) Thus the desirability of a graph possessing a perfect code serves as partial motivation for studying prism fixers.

Domination in prisms of graphs has been studied in [2, 4, 5, 7, 9]. In particular, the structure of prism fixers and the relation between prism fixers and Vizing's famous conjecture on the domination number of the cartesian products of graphs were investigated in [4, 5].

Conjecture 1 (Vizing's Conjecture) [11]**.** For any graphs G and H, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$.

Hartnell and Rall [4] constructed infinite classes of graphs to show that Vizing's conjecture, if true, is sharp. Many of these graphs have the property that their vertex sets partition into symmetric γ-sets; such a partition is called a *symmetric partition* and graphs with symmetric partitions are said to be *partitionable*. This connection between prism fixers and Vizing's conjecture serves as further motivation for the study of prism fixers. In [5] Hartnell and Rall further investigated the structure of prism fixers and closed with the following conjecture on the structure of bipartite partitionable graphs.

Conjecture 2 [5]**.** If G is a connected, bipartite, partitionable graph, then $G \cong C_4$ or G can be obtained from $K_{2t,2t}$ by removing the edges of t vertex-disjoint 4-cycles.

We provide a counterexample to Conjecture 2 and prove a suitably amended result instead.

2. Prism Fixers and Symmetric γ-Sets

We begin by stating properties of symmetric γ-sets and a characterization of prism fixers.

Proposition 1 [5, 7]**.** *If A is a symmetric γ-set of G, then*

(a) *A is independent;*

(b) *A_i, $i = 1, 2$, is a maximal packing of G;*

(c) *each vertex in $V - A$ is adjacent to exactly one vertex in A_i, $i = 1, 2$;*

(d) *for each vertex $u \in V - A$ there exists a vertex $v \in V - A$ such that $N_A(u) = N_A(v) = \{x, y\}$ (say) and $\langle u, v, x, y \rangle = C_4$;*

(e) *$\delta(G) \geq 2$.*

Theorem 2 [5, 7]**.** *The graph G is a prism fixer if and only if G has a symmetric γ-set.*

Note that C_4 is a prism fixer and, indeed, a bipartite partitionable graph. The following result on bipartite partitionable graphs was proved by Hartnell and Rall.

Proposition 3 [5]. *Let $G \neq C_4$ be a bipartite graph such that $V(G)$ can be partitioned into t symmetric γ-sets $A^1, \ldots, A^t$. Then G is $2(t-1)$-regular, $\gamma(G) = 4k$ for some integer k and for each $i = 1, \ldots, t$, $|A_1^i| = |A_2^i| = 2k$.*

We now define notation for prism fixers that will be used in the rest of the paper. See Figure 1. For a prism fixer G and a symmetric γ-set A of G, let G^* be the graph with vertex set $V(G^*) = A$ and edge set $E(G^*) = \{uv : N_G(u) \cap N_G(v) \neq \phi\}$. Let $F_1^*, \ldots, F_n^*$ be the components of G^*. We say $F_1^*, \ldots, F_n^*$ are the *graphs used in the construction of G with respect to A*. It follows from Proposition 1 that F_i^* is bipartite for each i (regardless of whether G is bipartite or not). Further, for each F_i^* let F_i be the subgraph of G induced by $N_G[V(F_i^*)]$.

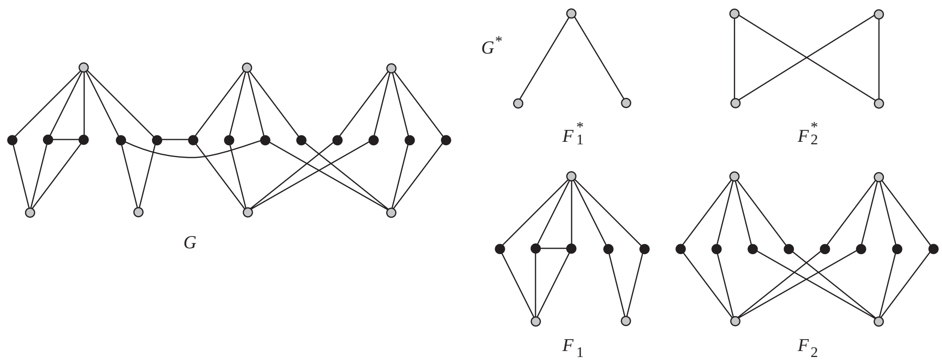

Figure 1. The graphs F_1^*, F_2^* used in the construction of G, and the graphs F_1 and F_2.

3. Counterexample

A counterexample to Conjecture 2 is given by the graph G in Figure 2 with vertex set $V(G) = \{0, 1, \ldots, 15\} \cup \{0', 1', \ldots, 15'\}$ and the following (abbreviated) adjacency list:

v	$N(v)$	v	$N(v)$
$0, 0'$	$4, 4', 5, 5', 6, 6'$	$5, 5'$	$0, 0', 12, 12', 13, 13'$
$1, 1'$	$7, 7', 8, 8', 9, 9'$	$6, 6'$	$0, 0', 10, 10', 14, 14'$
$2, 2'$	$10, 10', 11, 11', 12, 12'$	$7, 7'$	$1, 1', 12, 12', 14, 14'$
$3, 3'$	$13, 13', 14, 14', 15, 15'$	$8, 8'$	$1, 1', 10, 10', 15, 15'$
$4, 4'$	$0, 0', 11, 11', 15, 15'$	$9, 9'$	$1, 1', 11, 11', 13, 13'$

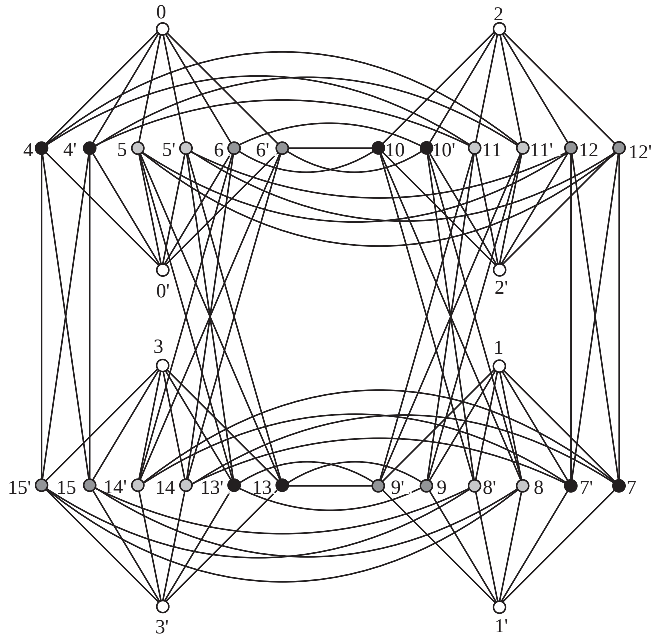

Figure 2. A counterexample to Conjecture 2.

Note that G is a connected, bipartite graph. We have verified by computer that $\gamma(G) = 8$; an analytical proof is not difficult, just tedious. Moreover, $V(G)$ can be partitioned into the γ-sets $A^1 = \{0, 0', 1, 1', 2, 2', 3, 3'\}$, $A^2 = \{4, 4', 7, 7', 10, 10', 13, 13'\}$, $A^3 = \{5, 5', 8, 8', 11, 11', 14, 14'\}$ and $A^4 = \{6, 6', 9, 9', 12, 12', 15, 15'\}$, which are easily seen to be symmetric γ-sets. Also note that if G could be obtained from $K_{16,16}$ by removing 8 vertex-disjoint 4-cycles, then $\deg v = 14$ for all $v \in V(G)$. However, $\deg v = 6$ for all $v \in V(G)$ and thus Conjecture 2 does not hold for G.

4. Structural Results

However, a revised statement of Conjecture 2 does hold. Denote the disjoint union of n copies of the graph H by nH and note that lC_4 is a spanning subgraph of $K_{2l,2l}$. We shall prove:

Theorem 4. *Let G be a connected, bipartite, partitionable graph. Then there exist pairwise edge-disjoint subgraphs $H_1 \cong \cdots \cong H_\lambda \cong lC_4$ of $K_{2l,2l}$ such that G can be obtained from $K_{2l,2l}$ by removing the edges in $\bigcup_{i=1}^{\lambda} E(H_i)$.*

We first prove several other results about the structure of bipartite partitionable graphs. The first result concerns the way in which one γ-set in a symmetric partition $\mathcal{P}$ dominates another γ-set in $\mathcal{P}$.

Proposition 5. *Let G be a bipartite, partitionable graph, $\mathcal{P}$ a symmetric partition of $V(G)$, $A, B \in \mathcal{P}$ and $x \in A$. If $u_1, u_2 \in B \cap N(x)$, then $N_A(u_1) = N_A(u_2)$.*

Proof. Note that $A \cap B = \phi$ since $\mathcal{P}$ is a partition. Without loss of generality, assume $x \in A_1$ and $u_1 \in B_1$.

Suppose to the contrary that $N_A(u_1) \neq N_A(u_2)$; say $N_A(u_1) = \{x, y_1\}$ and $N_A(u_2) = \{x, y_2\}$. Note that $y_1, y_2 \in A_2$, hence $y_1, y_2 \notin B$. Let $S = \{b \in B_1 : ub \in E(G)$ for some $u \in N(x) \cap N(y_1)\}$ and $T = \{a \in A_2 : ab \in E(G)$ for some $b \in S\}$. Since $S \subseteq B$, $A \cap S = \phi$. Since B_1 is a packing (Proposition 1(b)), no two vertices in S share a neighbour. Also, every vertex in S has exactly one neighbour in A_2 and hence in T. Therefore $|S| = |T|$. Finally, note that the only vertices not dominated by $A - T$ are the vertices in T and that $S \succ N(x) \cap N(y_1) - \{u_1\}$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$. Then there exist vertices $b \in S$ and $u \in N(x) \cap N(y_1)$ such that $ab, bu \in E(G)$. If $a = y_1$, then $b = u_1$ and x, b, u, x is an odd cycle in G; a contradiction since G is bipartite. If $a \neq y_1$, then $b \neq u_1$ and there exists a vertex $w \in N(x) \cap N(a)$; thus $w \neq b, u$. But then x, u, b, a, w, x is an odd cycle in G; a contradiction. Therefore $N(x) \cap N(a) = \phi$ for all $a \in T$ and thus $y_1 \notin T$.

It follows that the only vertices not dominated by $A' = A - T - \{x, y_1\}$ are the vertices of $T \cup \{x, y_1\} \cup (N(x) \cap N(y_1))$. But then $A'' = A' \cup S \cup \{u_1\} \succ G$ and $|A''| = |A| - |T| - 2 + |S| + 1 = |A| - 1 = \gamma - 1$; a contradiction. ■

Using Proposition 5 we now prove that if G is a bipartite, partitionable graph, then with respect to any γ-set in a symmetric partition of G, $F_i^* = K_2$ for all i.

Theorem 6. *Let G be a bipartite, partitionable graph and $\mathcal{P}$ a symmetric partition of $V(G)$. If $A \in \mathcal{P}$ and $F_1^*, \ldots, F_n^*$ are the graphs used in the construction of G with respect to A, then $F_i^* = K_2$ for all $i \in \{1, \ldots, n\}$.*

Proof. Suppose to the contrary that $F_1^* \neq K_2$. Then without loss of generality there exists a vertex $x \in A_1 \cap V(F_1)$ such that $N_{A_2}(N(x)) \supseteq \{y, z\}$, $y \neq z$.

Let $B \in \mathcal{P} - \{A\}$; thus $x, y, z \notin B$. By Proposition 1(c), x has exactly two neighbours in B; say $N_B(x) = \{v, w\}$. We may assume without loss of generality that $v \in N(y)$. Then by Proposition 5, $N_A(v) = N_A(w)$ and so $w \in N(y)$. Without loss of generality $v \in B_1$ and $w \in B_2$. Since $N_B(x) = \{v, w\}$,

$$N(x) \cap N(z) \cap B = \phi. \tag{1}$$

Therefore each vertex $u \in N(x) \cap N(z)$ has exactly one neighbour in B_1.

Let $S = \{b \in B_1 : bu \in E(G)$ for some $u \in N(x) \cap N(z)\}$ and $T = \{a \in A_2 : ab \in E(G)$ for some $b \in S\}$. Since $A \cap B = \phi$, $S \cap A = \phi$, so every vertex in S has exactly one neighbour in A_2 and since B_1 is a packing, no two vertices of S share the same neighbour. It follows that $|S| = |T|$. Note that $S \succ N(x) \cap N(z)$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$; say $w \in N(x) \cap N(a)$. Then there exist vertices $b \in S$, $u \in N(x) \cap N(z)$ such that $ab, bu \in E(G)$. By (1), $b \notin N(x) \cap N(z)$. If $a \neq z$, then x, w, a, b, u, x is an odd cycle in G; a contradiction. If $a = z$, then z, b, u, z is an odd cycle in G; a contradiction. Therefore $N(x) \cap N(a) = \phi$ for all $a \in T$ and it also follows that $z \notin T$.

Now the only vertices not dominated by $A' = A - T - \{x, z\}$ are the vertices of $T \cup \{x, z\} \cup (N(x) \cap N(z))$. But then letting $u \in N(x) \cap N(z)$, we have $A'' = A' \cup S \cup \{u\} \succ G$ and $|A''| = |A| - |T| - 2 + |S| + 1 = |A| - 1 = \gamma - 1$; a contradiction. Therefore $F_i^* = K_2$ for all $i \in \{1, \dots, n\}$. ■

In our final lemma before the proof of Theorem 4 we compare the cardinalities of the sets $A_i \cap V_j$, $i, j = 1, 2$, where G has bipartition (V_1, V_2) and $A = A_1 \cup A_2$ is a set in a symmetric partition of $V(G)$.

Lemma 7. *Let G be a bipartite, partitionable graph with bipartition (V_1, V_2) and symmetric partition $\mathcal{P}$. If $A \in \mathcal{P}$, then*

(a) $|A_1 \cap V_i| = |A_2 \cap V_i|$, $i = 1, 2$,

(b) $|A_i \cap V_1| = |A_i \cap V_2|$, $i = 1, 2$.

Proof. (a) Let $F_1^*, \dots, F_n^*$ be the graphs used in the construction of G with respect to A. Then by Theorem 6, $F_i^* = K_2$ for all i. Thus each vertex $x \in A_1 \cap V_1$ has a unique vertex $y \in A_2 \cap V_1$ such that $N(x) = N(y)$ and therefore $|A_1 \cap V_1| = |A_2 \cap V_1|$. Similarly for V_2, we have $|A_1 \cap V_2| = |A_2 \cap V_2|$.

(b) Note that $\bigcup_{x\in A_1\cap V_1} N(x) = V_2 - A$ and A_1 is a packing. By Proposition 3, G is $2(t-1)$-regular (where $t = |\mathcal{P}|$), hence

$$|A_1 \cap V_1| = \frac{|V_2 - A|}{2(t-1)}$$

and similarly

$$|A_1 \cap V_2| = \frac{|V_1 - A|}{2(t-1)}.$$

Let $H = \langle V - A\rangle$. Then H is bipartite with bipartition $(H_1, H_2) = (V_1 - A, V_2 - A)$. Since every vertex in $V - A$ is adjacent in G to exactly two vertices of A, $\deg_H v = \deg_G v - 2$ for all $v \in V(H)$. Since G is regular, H is also regular. Hence $|H_1| = |H_2|$ and so $|V_1 - A| = |V_2 - A|$. It follows that $|A_1 \cap V_1| = |A_1 \cap V_2|$. A similar argument shows that $|A_2 \cap V_1| = |A_2 \cap V_2|$. ■

We are now ready to prove Theorem 4. For vertices $a, b, c, d \in V(K_{2l,2l})$ with $a, c \in V_1$, $b, d \in V_2$, we write the 4-cycle a, b, c, d, a in $K_{2l,2l}$ simply as $abcd$.

***Proof of Theorem* 4.** Let G have bipartition (V_1, V_2) and symmetric partition $\mathcal{P} = \{A^1, \dots, A^t\}$. By Proposition 3 and Lemma 7, G is a spanning subgraph of $K_{2l,2l}$ for some l. If $G = C_4$, let $\lambda = 0$ and we are done. So assume $G \ncong C_4$ (thus $t \geq 3$). Let $F^*_{i,1}, \dots, F^*_{i,n}$ be the graphs used in the construction of G with respect to A^i. By Theorem 6, $F^*_{i,j} = K_2$ for all i, j. Let $a = |A^i_1 \cap V_1| = |A^i_1 \cap V_2| = |A^i_2 \cap V_1| = |A^i_2 \cap V_2| (= \frac{\gamma}{4})$. For $i \in \{1, \dots, t\}$, $q \in \{1, 2\}$, let

$$A^i_1 \cap V_q = \left\{v^i_{1,q}, v^i_{2,q}, \dots, v^i_{a,q}\right\} \text{ and } A^i_2 \cap V_q = \left\{w^i_{1,q}, w^i_{2,q}, \dots, w^i_{a,q}\right\}$$

so that $N(v^i_{j,q}) = N(w^i_{j,q})$ for all j.

For each $i = 1, \dots, t$, we first define a mutually disjoint sets, each containing a mutually disjoint 4-cycles with vertex sets in A^i and edge sets in $E(\overline{G})$. For each $k \in \{1, \dots, a\}$, define

$$\mathcal{C}^i_k = \left\{v^i_{p,1} v^i_{p+k (\mathrm{mod}\ a),2} w^i_{p,1} w^i_{p+k (\mathrm{mod}\ a),2} : 1 \leq p \leq a\right\}.$$

For the graph in Figure 2 the sets $\mathcal{C}^1_1$ (solid black lines) and $\mathcal{C}^1_2$ (broken black lines) are shown in Figure 3. Since A^i is independent, all of the edges in each of the 4-cycles in $\mathcal{C}^i_k$ are in $E(\overline{G})$. Also,

(2) for each k, every vertex of A^i is in exactly one 4-cycle of $\mathcal{C}^i_k$

and

$$\mathcal{C}^i_k \cap \mathcal{C}^i_{k'} = \phi \text{ when } k \neq k'. \tag{3}$$

For $j \in \{1, \ldots, t\} - \{i\}$, each vertex of A^i has exactly two neighbours in A^j. For i fixed and each $p \in \{1, \ldots, a\}$, let $A^j \cap N(v^i_{p,q}) = \{r^j_{p,q}, s^j_{p,q}\} = A^j \cap N(w^i_{p,q})$. For each $i \in \{1, \ldots, t\}$ and each $j \in \{1, \ldots, t\} - \{i\}$, we now define $a - 1$ mutually disjoint sets, each containing $2a$ mutually disjoint 4-cycles with vertex sets in $A^i \cup A^j$ and edge sets in $E(\overline{G})$. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{C}^{(i,j)}_k = \left\{ v^i_{p,q} r^j_{p+k(\text{mod } a),q} w^i_{p,q} s^j_{p+k(\text{mod } a),q} : 1 \leq p \leq a, 1 \leq q \leq 2 \right\}.$$

For the graph in Figure 2 the set $\mathcal{C}^{(1,2)}_1$ (with solid black lines for $q = 1$ and broken black lines for $q = 2$) is shown in Figure 4. Since $r^j_{p+k(\text{mod } a),q}, s^j_{p+k(\text{mod } a),q} \notin N(\{v^i_{p,q}, w^i_{p,q}\})$ for all $k \in \{1, \ldots, a-1\}$, it follows that all of the edges in each of the 4-cycles of $\mathcal{C}^{(i,j)}_k$ are in $E(\overline{G})$. Also note that

(4) every vertex of $A^i \cup A^j$ is in exactly one 4-cycle of $\mathcal{C}^{(i,j)}_k$,

$$\mathcal{C}^{(i,j)}_k \cap \mathcal{C}^{(i,j)}_{k'} = \phi \text{ when} k \neq k', \tag{5}$$

and for each $i \in \{1, \ldots, t\}$, $j \in \{1, \ldots, a\}$, $q \in \{1, 2\}$,

$$\begin{aligned} & N_{K_{2l,2l}}(v^i_{j,q}) - N_G(v^i_{j,q}) \\ & = \left(\bigcup_{p=1}^{a} \{v^i_{p,q+1(\text{mod } 2)}, w^i_{p,q+1(\text{mod } 2)}\} \right) \cup \left(\bigcup_{\substack{h=1 \\ h \neq i}}^{t} \bigcup_{\substack{p=1 \\ p \neq j}}^{a} \{r^h_{p,q}, s^h_{p,q}\} \right). \end{aligned} \tag{6}$$

Thus the vertices "missing" from the neighbourhood of $v^i_{j,q}$ are precisely the vertices adjacent to $v^i_{j,q}$ in the 4-cycles contained in all of the $\mathcal{C}^i_k$ and the $\mathcal{C}^{(i,j)}_k$. We now consider two cases depending on the parity of t.

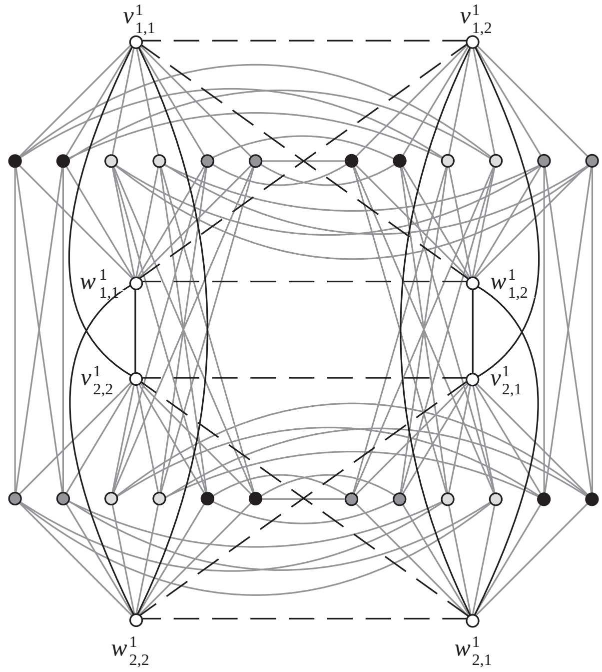

Figure 3. Sets $\mathcal{C}_1^1$ (solid black lines) and $\mathcal{C}_2^1$ (broken lines) for the graph in Figure 2.

Case 1. t is even. Then K_t is 1-factorable (see [3, Theorem 9.19]). Let $V(K_t) = \{1, \ldots, t\}$ and let $M_1, \ldots, M_{t-1}$ be the edge sets of a 1-factorization of K_t. For each $h \in \{1, \ldots, t-1\}$, we obtain the sets $\mathcal{S}_1^h, \ldots, \mathcal{S}_{a-1}^h$ as follows. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k^h = \bigcup_{ij \in M_h, i<j} \mathcal{C}_k^{(i,j)}.$$

Since M_h is a perfect matching in K_t, it follows from (4) that each vertex of $V(G) = \bigcup_{i=1}^t A^i$ is in exactly one 4-cycle of $\mathcal{S}_k^h$ and thus $\langle \mathcal{S}_k^h \rangle \cong lC_4$. Also, by (5), $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^h = \phi$ when $k \neq k'$. Moreover, each $ij \in E(K_t)$ is in exactly one M_h and so $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^{h'} = \phi$ when $h \neq h'$.

Further, for each $k \in \{1, \ldots, a\}$, define

$$\mathcal{S}_k = \bigcup_{i=1}^t \mathcal{C}_k^i.$$

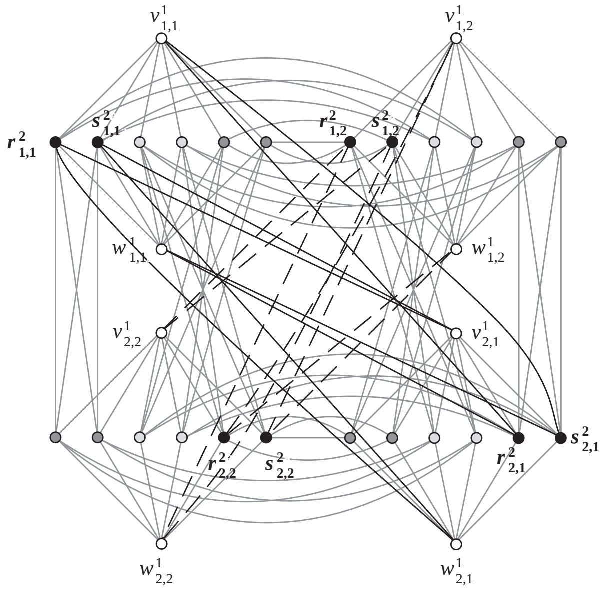

Figure 4. Set $\mathcal{C}_1^{(1,2)}$ for the graph in Figure 2.

By (2), every vertex of $V(G)$ is in exactly one 4-cycle in $\mathcal{S}_k$ and thus $\langle \mathcal{S}_k \rangle \cong lC_4$. Also, by (3), $\mathcal{S}_k \cap \mathcal{S}_{k'} = \phi$ when $k \neq k'$. Let

$$\mathfrak{C} = \left(\bigcup_{k=1}^{a} \langle \mathcal{S}_k \rangle \right) \cup \left(\bigcup_{h=1}^{t-1} \bigcup_{k=1}^{a-1} \left\langle \mathcal{S}_k^h \right\rangle \right).$$

Then $\mathfrak{C}$ consists of $a + (a-1)(t-1) = t(a-1) + 1$ disjoint copies of lC_4. Also, $\bigcup \mathfrak{C}$ is precisely all of the 4-cycles in all of the $\mathcal{C}_k^i$ and $\mathcal{C}_k^{(i,j)}$. Thus by (6), G can be obtained from $K_{2l,2l}$ by removing the edges of the copies of lC_4 in $\mathfrak{C}$.

Case 2. t is odd. Let $M_1, \ldots, M_t$ be the edge sets of a 1-factorization of K_{t+1}, where $V(K_{t+1}) = \{1, \ldots, t+1\}$. For each $h \in \{1, \ldots, t\}$, we obtain the sets $\mathcal{S}_1^h, \ldots, \mathcal{S}_{a-1}^h$ as follows. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k^h = \bigcup_{ij \in M_h, i<j<t+1} \mathcal{C}_k^{(i,j)} \cup \mathcal{C}_a^m \text{ where } m(t+1) \in M_h.$$

Since M_h is a perfect matching in K_{t+1}, (2) and (4) imply that each vertex of $V(G)$ is in exactly one 4-cycle of $\mathcal{S}_k^h$ and thus $\left\langle \mathcal{S}_k^h \right\rangle \cong lC_4$. Since each vertex in $\{1, \ldots, t\}$ is adjacent to vertex $t+1$ in exactly one M_h, (3) and (5) imply that $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^h = \phi$ when $k \neq k'$. Also, $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^{h'} = \phi$ when $h \neq h'$.

Further, for each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k = \bigcup_{i=1}^{t} \mathcal{C}_k^i.$$

Then by (2), every vertex of $V(G)$ is in exactly one 4-cycle in $\mathcal{S}_k$ and thus $\langle \mathcal{S}_k \rangle \cong lC_4$. Note that we do not have an $\mathcal{S}_a$ because the sets $\mathcal{C}_a^i$ were included in the $\mathcal{S}_k^h$ above. By (3), $\mathcal{S}_k \cap \mathcal{S}_{k'} = \phi$ when $k \neq k'$. Let

$$\mathfrak{C} = \left(\bigcup_{k=1}^{a-1} \mathcal{S}_k \right) \cup \left(\bigcup_{h=1}^{t} \bigcup_{k=1}^{a-1} \mathcal{S}_k^h \right).$$

Then $\mathfrak{C}$ consists of $a - 1 + t(a-1) = (t+1)(a-1)$ disjoint copies of lC_4. Also, $\bigcup \mathfrak{C}$ is precisely all of the 4-cycles in all of the $\mathcal{C}_k^i$ and $\mathcal{C}_k^{(i,j)}$. Thus by (6), G can be obtained from $K_{2l,2l}$ by removing the edges of the copies of lC_4 in $\mathfrak{C}$. ■

In the proof of Theorem 4, a given bipartite graph whose vertex set partitions into t symmetric γ-sets was obtained by deleting the edges of $t(a-1)+1$ or $(t+1)(a-1)$, depending on whether t is even or odd, pairwise disjoint copies of lC_4 from $K_{2l,2l}$, where $a = \gamma(G)/4$ and $t = \frac{l}{a}$. We close with the following problem.

Problem 1. *Consider $K_{2l,2l}$ and let $a \geq 1$ be a divisor of l such that $t = \frac{l}{a} \geq 3$. For which values of l and a is it possible to remove the edges of $t(a-1)+1$ if t is even, or $(t+1)(a-1)$ if t is odd, pairwise disjoint copies of lC_4 from $K_{2l,2l}$ and obtain a connected, bipartite, partitionable graph?*

Note that it is possible to remove edges as described and obtain a bipartite graph whose vertex set partitions into dominating sets with the same properties as symmetric γ-sets (Proposition 1), except that they are not necessarily γ-sets.

For example, if $l = 6$ and $a = 2$, there are two ways of removing edges of four disjoint copies of $6C_4$ from $K_{12,12}$ to obtain a bipartite graph G whose vertex set partitions into three dominating sets, each of which satisfies Proposition 1 and $F_i^* = K_2$ for each i. In one case $\gamma(G_1) = 4a = 8$ and G_1 is partitionable but not connected. In the other case $\gamma(G_2) = 6$, and the dominating sets in the partition are thus not γ-sets. See Figure 5.

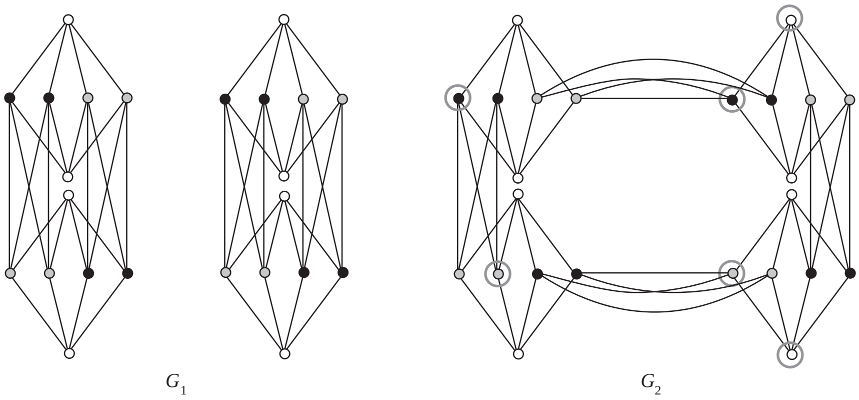

Figure 5. G_1 is partitionable but disconnected; G_2 is not partitionable.

As a final remark we note that the graph G in Figure 2 with $\gamma(G) = 8$ can be obtained as a "duplication" of its induced subgraph $H = \langle\{0, 1, \ldots, 15\}\rangle$; that is, for each vertex $v \in V(H)$ we add a duplicate vertex v', joining v' to all vertices u, u', where $u \in N(v)$ and u' is the duplication of u. The set $\{0, 1, 2, 3\}$ is an efficient dominating set of H, hence $\gamma(H) = 4$ [6, Theorem 4.2]. However, it is not true in general that if G is a duplication of a graph G' with efficient dominating set of size k, then $\gamma(G) = 2k$. It is an obvious upper bound, but the graph G_2 in Figure 5 presents a counterexample to equality in this bound. It is a duplication of C_{12}, which has efficient dominating sets of size 4, but $\gamma(G_2) = 6$ as shown.

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