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MONOCHROMATIC KERNEL-PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

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Abstract

In this paper, we introduce the concept of monochromatic kernel-perfect digraph, and we prove the following two results:

(1) If D is a digraph without monochromatic directed cycles, then D and each $\alpha_v, v \in V(D)$ are monochromatic kernel-perfect digraphs if and only if the composition over D of $(\alpha_v)_{v \in V(D)}$ is a monochromatic kernel-perfect digraph.

(2) D is a monochromatic kernel-perfect digraph if and only if for any $B \subseteq V(D)$, the duplication of D over B, D^B, is a monochromatic kernel-perfect digraph.

Keywords: kernel, kernel by monochromatic paths, composition, duplication.

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1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. Let $S_1, S_2 \subseteq V(D)$, an arc (u_1, u_2) of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in (V(D) - N)$ there exists a zN-arc in D. A digraph D is called a *kernel-perfect digraph* when every induced subdigraph of D has a kernel.

A digraph D is said to be an m-coloured digraph, if its arcs are coloured with m colours without loss of generality $\{1, 2, \ldots, m\}$. A directed path (or a directed cycle) is called *monochromatic* if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ of vertices of D is said to be a *kernel by monochromatic paths* of the m-coloured digraph D, if it satisfies the two following properties, (1) N is independent by monochromatic paths; i.e., for any two different vertices $x, y \in N$, there is no monochromatic directed path between them, and (2) N is *absorbent* by monochromatic paths; i.e., for each $u \in (V(D) - N)$ there exists a uv-monochromatic directed path, for some $v \in N$.

In this paper, we prove that if D is a digraph without monochromatic directed cycles, then (i) D has a kernel by monochromatic paths if and only if any composition over D of a family of digraphs $(\alpha_v)_{v \in V(D)}$ each one of them having a kernel by monochromatic paths, has a kernel by monochromatic paths, and (ii) D has a kernel by monochromatic paths if and only if for any $B \subseteq V(D)$ the duplication of D over B, D^B, has a kernel by monochromatic paths.

As a consequence we obtain the two results mentioned in the abstract.

Clearly, D has a kernel if and only if the m-coloured digraph D, in which every two different arcs have different colours, has a kernel by monochromatic paths. Sufficient conditions for the existence of a kernel in a digraph have been investigated by several authors, namely Von Neumann and Morgenstern [16], Richardson [13], Duchet and Meyniel [5] and Galeana-Sánchez and Neumann-Lara [6]. The concept of a kernel is very useful in applications, and clearly, the concept of a kernel by monochromatic paths generalizes that of kernel. Sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs have also been investigated by several authors; see for example [7, 9, 10, 14, 15, 18].

Definition 1.1. Let D be an arc coloured digraph and $\alpha = (\alpha_v)_{v\in V(D)}$ a family of pairwise vertex disjoint arc coloured digraphs. We define the *composition of* α *over* D, denoted $\sigma(D,\alpha)$, by the following conditions:

(i) $V(\sigma(D,\alpha)) = \bigcup_{v\in V(D)} V(\alpha_v)$.

(ii) $A(\sigma(D,\alpha)) = \big(\bigcup_{v\in V(D)} A(\alpha_v)\big) \cup \{(x,y)$ coloured $i \mid x \in \alpha_u, y \in \alpha_v,$ $(u,v) \in F(D)$ coloured $i\}$.

The composition of a family of graphs $\beta = (G_v)_{v\in V(G)}$ over a graph G was studied in [3] and its definition was extended to digraphs in [17]. The existence of kernels in the composition $\sigma(D,\alpha)$ of a family of digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over a digraph D was studied in [8], and the result was used to prove the existence of kernel-perfect digraphs with an arbitrarily large dichromatic number whose underlying graphs have no triangles.

In this paper, we study the existence of kernels by monchromatic paths in the composition $\sigma(D,\alpha)$ of a family of arc coloured digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over an arc coloured digraph D.

The duplication of a vertex of a graph was introduced in [4], and [11] gives the definition of the duplication of a subset of vertices of a graph as a generalization of the duplication of a vertex of a graph. This definition can be applied to arc coloured digraphs as follows:

Definition 1.2. Let D be an arc coloured digraph, B a proper subset of $V(D)$ and let B'_D a digraph isomorphic to $D[B]$ with $V(B'_D)\cap V(D) = \emptyset$. A vertex belonging to B'_D and corresponding to a vertex $x \in B$ will be denoted by x'. *The duplication of* D *over* B is the arc coloured digraph denoted D^B and defined as follows:

$$V(D^B) = V(D) \cup V(B'_D)$$

and

$$A(D^B) = A(D) \cup A(B'_D) \cup A_0 \cup A$$

in which $A_0 = \{(x',y)$ coloured $i \mid x' \in V(B'_D), y \in V(D)$ and $(x,y) \in A(D)$ coloured $i\}$. $A_1 = \{(y,x')$ coloured $i \mid y \in V(D), x' \in V(B'_D)$ and $(y,x) \in A(D)$ coloured $i\}$.

We will denote $B' = V(B'_D)$. A vertex $x' \in B'$ (resp., a subset $S' \subseteq B'$) we will call the copy of the vertex $x \in B$ (resp., the copy of the subset $S \subseteq B$).

The vertex x (resp., the subset S) will be named the original of the vertex x' (resp., of the subset S').

We will denote by Proy the function Proy: $V(\sigma(D,\alpha)) \rightarrow V(D)$ such that $\text{Proy}(x) = v$ if and only if $x \in V(\alpha_v)$.

The existence of kernels in the duplication of a digraph D has been studied in [2]. In this paper, we study the existence of kernels by monochromatic paths in the duplication of an arc coloured digraph D over a proper subset of vertices of $V(D)$.

The composition and the duplication are two operations in digraphs which have been considerated several times, see for example [3, 12, 17], and they constitute a powerful tool in the construction of many examples and counterexamples in digraphs.

Also we consider an extension of the concept of kernel perfectness of a digraph and obtain a large variety of monochromatic kernel-perfect digraphs.

2. Kernels by Monochromatic Paths in the Composition over D, and in the Duplication of D over B

We start this section with a lemma which will be useful in the proof of Theorem 2.1. Its proof is easy and will be omitted.

Lemma 2.1. *Let D be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of pairwise vertex disjoint digraphs. If $T = (x_0, x_1, \ldots, x_n)$ is a directed path in $\sigma(D,\alpha)$ such that $\{x_0, x_n\} \subseteq V(\alpha_v)$ for some $v \in V(D)$, then* Proy(T) *is a join of directed cycles of D or a single vertex of D.*

Theorem 2.1. *Let D be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of arc coloured pairwise vertex disjoint digraphs. A set $N^* \subseteq V(\sigma(D,\alpha))$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$ if and only if there exists a kernel by monochromatic paths of D, say $N \subseteq V(D)$, such that $N^* = \bigcup_{v \in N} N_v$, in which N_v is a kernel by monochromatic paths of α_v.*

Proof. Let $N \subseteq V(D)$ be a kernel by monochromatic paths of D and N_v a kernel by monochromatic paths of α_v, $v \in N$. We will prove that $N^* = \bigcup_{v \in N} N_v$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$.

(a) N^* is absorbent by monochromatic paths.

Let $z \in (V(\sigma(D,\alpha)) - N^*)$. There exists $v_0 \in V(D)$ such that $z \in V(\alpha_{v_0})$. When $v_0 \in N$, we have $N_{v_0} \subseteq N^*$, in which N_{v_0} is a kernel by monochromatic paths of α_{v_0}; and there exists a zN_{v_0}-monochromatic directed path (as $z \in (V(\alpha_0) - N_{v_0}))$.

When $v_0 \notin N$, we have $v_0 \in (V(D) - N)$ and thus, there exists a monochromatic directed path contained in D, say $T = (v_0, v_1, \ldots, v_{n-1}, u)$ with $u \in N$ (because N is a kernel by monochromatic paths of D); since $z \in V(\alpha_0)$; taking $z_i \in V(\alpha_i)$ and $z_u \in N_u$, we have $T' = (z, z_1, z_2, \ldots, z_{n-1}, z_u)$; a zz_u-monochromatic directed path in $\alpha(D,\sigma)$ with $z_u \in N_u \subseteq N^*$.

(b) N^* is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist $x_0, x_n \in N^*$ and a x_0x_n-monochromatic directed path, say $T = (x_0, x_1, \ldots, x_n)$ contained in $\sigma(D,\alpha)$. We consider two possible cases:

Case (b.1). $\{x_0, x_n\} \subseteq V(\alpha_v)$, for some $v \in V(D)$.

When $T \subseteq \alpha_v$, we have that T is an x_0x_n-monochromatic directed path contained in α_v, with $\{x_0, x_n\} \subseteq N_v$, a contradiction.

When $T \nsubseteq \alpha_v$, we have from Lemma 2.1 that Proy(T) is a join of monochromatic directed cycles contained in D, contradicting our hypothesis on D.

Case (b.2). $x_0 \in \alpha_v$ and $x_n \in \alpha_u$ with $u \neq v$.

In this case, it follows from the definition of N^* that $x_0 \in N_v$ and $x_n \in N_u$ with $\{u, v\} \subseteq N$. Since T is monochromatic, We have that Proy(T) contains a vu-monochromatic path, which is contained in D, contradicting that N is a kernel by monochromatic paths of D. We conclude that N^* is a kernel by monochromatic paths.

Now let N^* be a kernel by monochromatic paths of $\sigma(D,\alpha)$. We will prove that $N = \{v \in V(D) \,|\, N^* \cap \alpha_v \neq \emptyset\}$ is a kernel by monochromatic paths of D and $N^* \cap V(\alpha_v) = N_v$ is a kernel by monochromatic paths of α_v, for each $v \in N$.

N is absorbent by monochromatic paths.

Let $v \in (V(D) - N)$ and $z_0 \in V(\alpha_v)$; since $v \notin N$ we have that $z_0 \notin N^*$; thus there exists a monochromatic directed path $T = (z_0, \ldots, z_n)$ with $z_n \in N^*$; now, $z_n \in V(\alpha_u)$ for some $u \in V(D)$; moreover, from the definition of N we have $u \in N$ and then Proy(T) contains a vu-monochromatic directed path with $u \in N$.

N is independent by monochromatic paths.
We proceed by contradiction, suppose that there exist $v_0, v_n \in N$ and a v_0v_n-monochromatic directed path $T = (v_0, v_1, \ldots, v_n)$ contained in D. Since $v_0, v_n \in N$ there exist $z_0 \in V(\alpha_{v_0}) \cap N^*$ and $z_n \in V(\alpha_{v_n}) \cap N^*$; now taking any vertex $z_i \in V(\alpha_{v_i})$ for each $1 \leq i \leq n-1$; we have from the definition of $\sigma(D, \alpha)$ that $(z_0, z_1, \ldots, z_n)$ is a z_0z_n-monochromatic directed path with $\{z_0, z_n\} \subseteq N^*$, a contradiction.

Now; let $v \in V(D)$ be such that $N^* \cap V(\alpha_v) \neq \emptyset$. We will prove that $N_v = N^* \cap V(\alpha_v)$ is a kernel by monochromatic paths of α_v.

N_v is independent by monochromatic paths.

We proceed by contradiction. Suppose that there exist $u, x \in N_v$, $u \neq x$, and a monochromatic directed path T between them, with $T \subseteq \alpha_v$; clearly, $T \subseteq \sigma(D, \alpha)$ and $\{u, x\} \subseteq N^*$, a contradiction (as N^* is independent by monochromatic paths in $\sigma(D, \alpha)$).

N is absorbent by monochromatic paths.

Let $u \in (V(\alpha_v) - N)$ clearly $u \in (V(\sigma(D, \alpha)) - N^*)$; thus there exists $z \in N^*$ and a uz-monochromatic directed path $T \subseteq \sigma(D, \alpha)$. Let $T = (u = u_0, u_1, \ldots, u_n = z)$, we will prove that $T \subseteq \alpha_v$. When $u_n = z \in V(\alpha_v)$; it follows from Lemma 2.1 that Proy(T) is a single vertex i.e., $T \subseteq \alpha_v$; otherwise D contains a monochromatic directed cycle, contradicting our hypothesis on D. When $u_n \notin V(\alpha_v)$; we have $u_n \in \alpha_w$ for some $w \in V(D)$ and $w \in N^*$. Now take $x \in N_v^*$ (recall $N^* \cap \alpha_v \neq \emptyset$); it follows from the definition of $\alpha(D, \alpha)$ that $(x, u_1, u_2, \ldots, u_n = z)$ is a xz-monochromatic directed path in $\sigma(D, \alpha)$ with $x \neq z$, $x, z \in N^*$, a contradiction. ■

Lemma 2.2. *Let D be an arc coloured digraph, $B \subset V(D)$; D^B the duplication of D over B; and $\psi\colon D[B] \to B'_D$ the isomorphism defined by the duplication (i.e., $\psi(x) = x'$ for any $x \in V(D[B])$); and denote by ϕ the function defined as follows: $\phi\colon D \to D^B - D[B]$*

$$\phi(x) = \begin{cases} x & \text{if } x \notin B, \\ \phi(x) = x' & \text{if } x \in B\,. \end{cases}$$

Then ϕ is an isomorphism such that (x, y) is coloured in i if and only if $(\phi(x), \phi(y))$ is coloured in i; in particular $T \subseteq D$ is a monochromatic directed path if and only if $\phi(T) \subseteq D^B - D[B]$ is a monochromatic directed path.

This Lemma is a direct consequence of the definition of ϕ and the definition of the duplication of D over B.

Theorem 2.2. *Let D be an arc coloured digraph which has no monochromatic directed cycles; $B \subset V(D)$ and D^B the duplication of D over B.*

D has a kernel by monochromatic paths if and only if D^B has a kernel by monochromatic paths.

Proof. Let D, B and D^B be as in the hypothesis and suppose that D has a kernel by monochromatic paths, say N.

We consider two possible cases:

Case 1. $N \cap B = \emptyset$.
In this case, we will prove that N is a kernel by monochromatic paths of D^B.

N is independent by monochromatic paths in D^B.
Let $x, y \in N$; $x \neq y$ and assume for a contradiction that there exists an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D^B.

When $V(T) \cap B = \emptyset$, we have $T \subseteq D^B - D[B]$, and from Lemma 2.1 $\phi^{-1}(T)$ is an xy-monochromatic directed path contained in D, contradicting that N is independent by monochromatic paths. When $V(T) \cap B \neq \emptyset$, we denote $I = V(T) \cap B$; say $I = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, $i_1 < i_2 < \cdots < i_k$, we also denote by $T(I) = (x_0, \ldots, x_{i_1-1}, x'_{i_1}, x_{i_1+1}, \ldots, x_{i_2-1}, x'_{i_2}, \ldots, x_n = y)$ (the succesion obtained from T by substituting x_{i_j}, for x'_{i_j} in T, for each $j \in \{1, \ldots, k\}$). It follows from the definition of D^B that $T(I)$ is a monochromatic directed path contained in $D^B - D[B]$; and from Lemma 2.2 $\phi^{-1}(T(I))$ is an xy-monochromatic directed path contained in D, a contradiction.

N is absorbent by monochromatic paths in D^B.

Let $z \in (V(D^B) - N)$. If $z \notin B'_D$, then $z \in (V(D) - N)$ and there exists a zN-monochromatic directed path, say T, with $T \subseteq D \subseteq D^B$.

If $z \in V(B'_D)$, then there exists $y \in B$ such that $z = y' \in V(B'_D)$; we have $y \notin N$ because $N \cap B = \emptyset$; thus there exists a yN-monochromatic directed path, say $T = (y, x_1, \ldots, x_n)$ and then from definition of D^B we have that $T' = (y', x_1, \ldots, x_n)$ is a zN-monochromatic directed path in D^B.

Case 2. $N \cap B \neq \emptyset$.
Let $Z = N \cap B$, and denote by $Z' = \{z' \in B'_D \mid z \in Z\}$.

We will prove that $N^* = N \cup Z'$ is a kernel by monochromatic paths of D^B.

N^* is independent by monochromatic paths.

Let $x, y \in N^*$, $x \neq y$, and assume for a contradiction that there exists an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D^B. Here we consider several possible cases:

Case 2.a. $x, y \in N$.
Let $I' = V(T) \cap V(B'_D) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ and denote by y_{i_j} the original of x_{i_j} (i.e., $y_{i_j} = \psi^{-1}(x_{i_j})$. Now let T' be the succession obtained from T by substituting each x_{i_j} for y_{i_j}. It follows from the definition of ψ and from the definition of D^B that T' contains an xy-monochromatic directed path contained in D, with $x, y \in N$, a contradiction.

Case 2.b. $x \in N$, $y \in Z'$ and $x \notin B$. In this case, we proceed as in Case 2.a to get a contradiction.

Case 2.c. $x \in N \cap B$, $y \in Z'$.
When x is the original vertex of y, taking the succession T' defined in Case 2.a we have that T' contains a monochromatic directed cycle, contradicting our hypothesis on D; as $T' \subseteq D$.

When x is not the original vertex of y; taking again the succession T' defined in Case 2.a, we have that T' contains an xz-monochromatic directed path, in which z is the original vertex of y and $x, z \in N$ with $x \neq z$, contradicting that N is independent by monochromatic paths.

Case 2.d. $x, y \in Z'$.
Let $\overline{x}$ (resp., $\overline{y}$) be the original vertex of x (resp., y); clearly, in this case T' (defined in Case 2.a) contains an $\overline{x}\overline{y}$-monochromatic directed path which is contained in D; with $\overline{x} \neq \overline{y}$, $\overline{x}, \overline{y} \in N$, a contradiction. So, we conclude that N^* is independent by monochromatic paths.

Now we prove that N^* is absorbent by monochromatic paths.

Let $z \in (V(D^B) - N^*)$. When $z \in B'$, we have $z = y'$ in which $y \in B$ is the original vertex of z. Since N is a kernel by monochromatic paths of D; there exists a yN-monochromatic directed path in D, say, $T = (y = x_0, x_1, \ldots, x_n)$; thus $T' = (y' = z, x_1, \ldots, x_n)$ is a zN^*-monochromatic directed path contained in D^B. When $z \notin B'$, we have $z \in (V(D) - N)$ and there exists a zN-monochromatic directed path contained in D; say, T. Clearly, T is a zN^*-monochromatic directed path contained in D^B.

We conclude that N^* is a kernel by monochromatic paths of D^B. Now suppose that D^B has a kernel by monochromatic paths and let N^* be a kernel by monochromatic paths of D^B. We will prove that D has a kernel by monochromatic paths.

Let Z be such that $Z' = N^* \cap V(B'_D)$ in which Z' is defined by the process introduced in the construction of B'_D, when $Z' = \emptyset$ we define $Z = \emptyset$. Denote by $N = (N^* - Z') \cup Z$. We will show that N is a kernel by monochromatic paths of D.

N is independent by monochromatic paths in D.
Assume by contradiction that there exist $x, y \in N$; $x \neq y$; and an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D.

Let $\overline{x}$ and $\overline{y}$ be defined as follows: $\overline{x} = x$ if $x \in (N^* - Z')$ and $\overline{x}$ is the copy of x if $x \in Z$, $\overline{y} = y$ if $y \in (N^* - Z^*)$ and $\overline{y}$ is the copy of y if $y \in Z$. Clearly, $T' = (\overline{x}, x_1, \ldots, x_{n-1}, \overline{y})$ is a monochromatic directed path in D^B with $\overline{x} \neq \overline{y}$ and $\overline{x}, \overline{y} \in N^*$, a contradiction.

N is absorbent by monochromatic paths in D.

Let $z \in (V(D) - N)$, then from the definition of N, we have $z \in (V(D^B) - N)$, thus there exists a zN^*-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$ contained in D^B. Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} = V(T) \cap V(B'_D)$; y_{i_j} the original vertex of x_{i_j} and T' the succession obtained from T by substituting x_{i_j} for y_{i_j} for each $1 \leq j \leq k$ in T. Clearly, T' contains a zN-monochromatic directed path, and $T' \subseteq D$.

We conclude that N is absorbent by monochromatic paths. ∎

3. Monochromatic Kernel Perfectness of Composition and Duplication

The following definition is a generalization of the concept of kernel perfectness of a digraph.

Definition 3.1. Let D be an arc coloured digraph, D is said to be a monochromatic kernel perfect digraph whenever for every nonempty subset B of vertices of D, the digraph $D[B]$ has a kernel by monochromatic paths.

Theorem 3.1. *Let D be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v \in V(D)}$ a family in which the α_v are mutually disjoint arc coloured digraphs.*

D and each α_v, $v \in V(D)$ are monochromatic kernel perfect digraphs if and only if $\sigma(D, \alpha)$ is a monochromatic kernel perfect digraph.

Proof. Theorem 3.1 follows directly from Theorem 2.1 and the two following assertions: (1) The disjoint union of monochromatic kernel perfect digraphs is also a monochromatic kernel perfect digraph. (2) Every connected induced subdigraph of $\sigma(D, \alpha)$ has the form $\sigma(D', \alpha')$ for a suitable D' and $\alpha' = (\alpha'_v)_{v \in V(D)}$ (actually D' is an induced subdigraph of D and α'_v is an induced subdigraph of α_v for each $v \in V(D')$). ■

Theorem 3.2. *Let D be an arc coloured digraph which has no monochromatic directed cycle, $B \subset V(D)$ and D^B the duplication of D over B. Then D is a monochromatic kernel perfect digraph if and only if D^B is a monochromatic kernel perfect digraph.*

Proof. Clearly, an arc coloured digraph D is a monochromatic kernel perfect digraph if and only if each induced subdigraph of D is a monochromatic kernel perfect digraph. Thus if D^B is a monochromatic kernel perfect digraph, then D is a monochromatic kernel perfect digraph.

Now suppose that D is a monochromatic kernel perfect digraph and let $A \subseteq V(D^B)$. We will prove that $D^B[A]$ has a kernel by monochromatic paths. Here we consider two possible cases:

Case 1. $A \cap V(B'_D) = \emptyset$.
In this case, $A \subseteq V(D^B - V(B'_D))$ and $D^B[A] \cong D[A]$ and since $D[A]$ has a kernel by monochromatic paths; it follows that $D^B[A]$ has a kernel by monochromatic paths.

Case 2. $A \cap V(B'_D) \neq \emptyset$.
Let $C' = \{x' \in V(D^B) \,|\, x' \in A \cap V(B'_D)\}$ and $E = A - C'$ be, thus $A = C' \cup E$.

Case 2.1. $E \cap C = \emptyset$. (In Which $C = \psi^{-1}(C')$).
In this case, we have $D^B[E \cup C'] \cong D^B[E \cup C] \cong D[E \cup C]$ and then $D^B[A] \cong D[E \cup C]$ has a kernel by monochromatic paths.

Case 2.2. $E \cap C \neq \emptyset$.
It follows from the hypothesis that $D[E \cup C]$ has a kernel by monochromatic paths, say N.

When $N \cap B = \emptyset$ it follows as in Case 1 of the proof of Theorem 2.2 that N is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over C); therefore N is independent by monochromatic paths in $D^B[E \cup C']$. Since N is absorbent by monochromatic paths in $D[E \cup C]$ it follows that N is absorbent by monochromatic paths in $D^B[E \cup C']$.

(Clearly, to each monochromatic directed path in $D[E \cup C]$, say T there corresponds an unique monochromatic directed path in $D^B[E \cup C']$, T' obtained from T by substituting each vertex x in $V(T) \cap (C - E)$ for its copy x' in C').

When $N \cap B \neq \emptyset$, we denote by $Z = N \cap B$; we have proved in Case 2 of the proof of Theorem 2.2 that $N \cup Z'$ is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over C). So $N \cup Z'$ is independent by monochromatic paths in $D^B[E \cup C']$. Now, let $z \in (V(D^B[E \cup C']) - (N \cup Z'))$; clearly, $z \in (V(D^C[E \cup C] - (N \cup Z')))$ and then there exists a $z \in (N \cup Z')$-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$; if $T \cap (C - E) = \{x_{i_1}, \ldots, x_{i_k}\}$ then let T' be the succession obtained from T by substituting each x_{i_j}, $1 \leq j \leq k$ for its copy x'_{i_j} in C'. Since D has no monochromatic directed cycles we have $x'_{i_j} \notin V(T)$ for each $1 \leq j \leq k$. Therefore from the definition of D^B we have that T' is a monochromatic directed path contained in $D^B[E \cup C']$ from z to $(N \cup Z')$. We conclude that $N \cup Z'$ is a kernel by monochromatic paths of $D^B[E \cup C']$. ■

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