

SUBGRAPH DENSITIES IN HYPERGRAPHS

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Abstract

Let $r \geq 2$ be an integer. A real number $\alpha \in [0, 1)$ is a jump for r if for any $\epsilon > 0$ and any integer $m \geq r$, any r -uniform graph with $n > n_0(\epsilon, m)$ vertices and density at least $\alpha + \epsilon$ contains a subgraph with m vertices and density at least $\alpha + c$, where $c = c(\alpha) > 0$ does not depend on ϵ and m . A result of Erdős, Stone and Simonovits implies that every $\alpha \in [0, 1)$ is a jump for $r = 2$. Erdős asked whether the same is true for $r \geq 3$. Frankl and Rödl gave a negative answer by showing an infinite sequence of non-jumps for every $r \geq 3$. However, there are still a lot of open questions on determining whether or not a number is a jump for $r \geq 3$. In this paper, we first find an infinite sequence of non-jumps for $r = 4$, then extend one of them to every $r \geq 4$. Our approach is based on the techniques developed by Frankl and Rödl.

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1. INTRODUCTION

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r -subsets of V . An r -uniform graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq \binom{V}{r}$ of edges. In particular, an r -uniform graph is called a *graph* if $r = 2$ and an r -uniform *hypergraph* if $r \geq 3$. We abbreviate r -uniform graph

to r -graph. The *density* of an r -graph G is defined by $d(G) = \frac{|E(G)|}{\binom{|V(G)|}{r}}$. An r -graph H is a *subgraph* of an r -graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if $E(H) = E(G) \cap \binom{V(H)}{r}$.

By a simple argument (c.f. Katona, Nemetz, Simonovits [8]), the average of densities of all induced subgraphs of an r -graph G with $m \geq r$ vertices is $d(G)$. Therefore, there exists a subgraph of G with m vertices and density $\geq d(G)$. A natural question is whether there exists a subgraph of G with m vertices and density $\geq d(G) + c$, where $c > 0$ is a constant? To be more precise, the concept of ‘jump’ was introduced.

Definition 1.1. Given $r \geq 2$, a real number $\alpha \in [0, 1)$ is a jump for r if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer $m, m \geq r$, there exists an integer $n_0(\epsilon, m)$ such that any r -graph with $n > n_0(\epsilon, m)$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$. A real number $\alpha \in [0, 1)$ is called a non-jump for r if α is not a jump for r .

Erdős and Stone ([4]) proved that every $\alpha \in [0, 1)$ is a jump for $r = 2$. It easily follows from the following classical result.

For an integer $l \geq r$, an r -graph $G = (V, E)$ is called *complete l -partite* if V admits a partition into l classes such that an r -subset of V is an edge if and only if it contains at most one vertex from each class.

Theorem 1.1 (c.f. [4]). *Suppose l is a positive integer. For any $\epsilon > 0$ and any positive integer m , there exists $n_0(m, \epsilon)$ such that any graph G on $n > n_0(m, \epsilon)$ vertices with density $d(G) \geq 1 - \frac{1}{l} + \epsilon$ contains a copy of the complete $(l + 1)$ -partite graph with partition classes of size m .*

Note that the density of a complete $(l + 1)$ -partite graph with partition classes of size m is greater than $1 - \frac{1}{l+1}$ (approaches $1 - \frac{1}{l+1}$ when $m \rightarrow \infty$).

For $r \geq 3$, Erdős proved that every $\alpha \in [0, r!/r^r)$ is a jump. It directly follows from the following:

Theorem 1.2 (c.f. [2]). *For any $\epsilon > 0$ and any positive integer m , there exists $n_0(\epsilon, m)$ such that any r -graph G on $n > n_0(\epsilon, m)$ vertices with density $d(G) \geq \epsilon$ contains a copy of the complete r -partite r -graph with partition classes of size m .*

Note that the density of a complete r -partite r -graph with partition classes of size m is greater than $r!/r^r$ (approaches $r!/r^r$ when $m \rightarrow \infty$).

Furthermore, Erdős proposed the following jumping constant conjecture.

Conjecture 1.3. Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$.

In [6], Frankl and Rödl disproved this conjecture by showing the following result.

Theorem 1.4 (c.f. [6]). *Suppose $r \geq 3$ and $l > 2r$. Then $1 - \frac{1}{r-1}$ is not a jump for r .*

Using the techniques developed by Frankl and Rödl in [6], some other non-jumps were given in [7, 10, 11] and [12]. However, there are still a lot of open questions on determining whether or not a number is a jump for $r \geq 3$. A well-known question of Erdős is to determine whether or not $\frac{r!}{r^r}$ is a jump. At this moment, the smallest known non-jump for $r \geq 3$ is $\frac{5r!}{2r^r}$ given in [7]. Another question raised in [7] is whether there is an interval of non-jumps for $r \geq 3$. By the definition of the ‘jump’, if a number a is a jump, then there exists a constant $c > 0$ such that every number in $[a, a + c)$ is a jump. Consequently, if there is a set of non-jumps whose limits form an interval (number a is a limit of a set A if there is a sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in A$ such that $\lim_{n \rightarrow \infty} a_n = a$), then no number in this interval is a jump. We do not know whether such a ‘dense enough’ set of non-jumps exists or not. In this paper we intend to find more non-jumps in addition to the known non-jumps in [6, 7, 10, 11] and [12]. Our approach is still based on the techniques developed by Frankl and Rödl in [6].

We first work in the case $r = 4$ and find a sequence of non-jumps for $r = 4$. In Sections 3 and 4, we prove the following result.

Theorem 1.5. *Let $l \geq 2$ be an integer. Then $1 - \frac{7}{l^2} + \frac{10}{l^3}$ is not a jump for $r = 4$.*

In Section 5 we extend a special case of Theorem 1.5 ($l = 4$) to all $r \geq 4$. The following result will be proved.

Theorem 1.6. *For $r \geq 4$, $\frac{23r!}{3r^r}$ is not a jump for r .*

Note that when $r = l = 4$, Theorems 1.6 and 1.5 coincide.

In the next section, we introduce the Lagrangian of an r -graph and some other tools to be applied in our proofs.

2. LAGRANGIANS AND OTHER TOOLS

We first give a definition of the Lagrangian of an r -graph. More studies of Lagrangians were given in [5, 6, 9] and [13].

Definition 2.1. For an r -graph G with vertex set $\{1, 2, \dots, m\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \dots, x_m) \in R^m$, define

$$\lambda(G, \vec{x}) = \sum_{\{i_1, \dots, i_r\} \in E(G)} x_{i_1} x_{i_2} \dots x_{i_r}.$$

x_i is called the *weight* of vertex i .

Definition 2.2. Let $S = \{\vec{x} = (x_1, x_2, \dots, x_m) : \sum_{i=1}^m x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \dots, m\}$. The Lagrangian of G , denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{x} \in S$ is called an *optimal vector* for $\lambda(G)$ if $\lambda(G, \vec{x}) = \lambda(G)$.

We note that if H is a subgraph of an r -graph G , then for any vector \vec{x} in S , $\lambda(H, \vec{x}) \leq \lambda(G, \vec{x})$. We formulate this as follows.

Fact 2.1. Let H be a subgraph of an r -graph G . Then

$$\lambda(H) \leq \lambda(G).$$

For an r -graph G and $i \in V(G)$ we define G_i to be the $(r-1)$ -uniform graph on $V - \{i\}$ with edge set $E(G_i)$ given by $e \in E(G_i)$ if and only if $e \cup \{i\} \in E(G)$.

We call two vertices i, j of an r -graph G equivalent if for all $f \in \binom{V(G) - \{i, j\}}{r-1}$, $f \in E(G_i)$ if and only if $f \in E(G_j)$.

The following lemma (proved in [6]) will be useful when calculating Lagrangians of certain graphs.

Lemma 2.2 (c.f. [6]). *Suppose G is an r -graph on vertices $\{1, 2, \dots, m\}$.*

1. *If vertices i_1, i_2, \dots, i_t are pairwise equivalent, then there exists an optimal vector $\vec{y} = (y_1, y_2, \dots, y_m)$ of $\lambda(G)$ such that $y_{i_1} = y_{i_2} = \dots = y_{i_t}$.*
2. *Let $\vec{y} = (y_1, y_2, \dots, y_m)$ be an optimal vector of $\lambda(G)$ and $y_i > 0$. Let \hat{y}_i be the restriction of \vec{y} on $\{1, 2, \dots, m\} \setminus \{i\}$. Then $\lambda(G_i, \hat{y}_i) = r\lambda(G)$.*

We also note that for an r -graph G with m vertices, if we take $\vec{u} = (u_1, \dots, u_m)$, where each $u_i = 1/m$, then

$$\lambda(G) \geq \lambda(G, \vec{u}) = \frac{|E(G)|}{m^r} \geq \frac{d(G)}{r!} - \epsilon$$

for $m \geq m'(\epsilon)$.

On the other hand, we introduce the blow-up of an r -graph G which will allow us to construct r -graphs with large number of vertices and densities close to $r!\lambda(G)$.

Definition 2.3. Let G be an r -graph with $V(G) = \{1, 2, \dots, m\}$ and (n_1, \dots, n_m) be a positive integer vector. Define the (n_1, \dots, n_m) blow-up of G , $(n_1, \dots, n_m) \otimes G$ as an m -partite r -graph with vertex set $V_1 \cup \dots \cup V_m$, $|V_i| = n_i$, $1 \leq i \leq m$, and edge set $E((n_1, \dots, n_m) \otimes G) = \{\{v_{i_1}, v_{i_2}, \dots, v_{i_r}\} : v_{i_k} \in V_{i_k} \text{ for } 1 \leq k \leq r, \{i_1, i_2, \dots, i_r\} \in E(G)\}$. We abbreviate $(n, n, \dots, n) \otimes G$ to $\vec{n} \otimes G$.

We make the following easy Remark used in [10].

Remark 2.3 (c.f. [10]). Let G be an r -graph with m vertices and $\vec{y} = (y_1, \dots, y_m)$ be an optimal vector of $\lambda(G)$. Then for any $\epsilon > 0$, there exists an integer $n_1(\epsilon)$, such that for any integer $n \geq n_1(\epsilon)$,

$$(1) \quad d([\lfloor ny_1 \rfloor, \lfloor ny_2 \rfloor, \dots, \lfloor ny_m \rfloor] \otimes G) \geq r!\lambda(G) - \epsilon.$$

Let us also state a fact relating the Lagrangian of an r -graph to the Lagrangian of its blow-up used in [6, 7, 10, 11] and [12] as well).

Fact 2.4 (c.f. [6]). $\lambda(\vec{n} \otimes G) = \lambda(G)$.

The following lemma proved in [6] gives a necessary and sufficient condition for a number α to be a jump. We need a definition to describe it.

Definition 2.4. For $\alpha \in [0, 1)$ and a family \mathcal{F} of r -graphs, we say that α is a threshold for \mathcal{F} if for any $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ such that any r -graph G with $d(G) \geq \alpha + \epsilon$ and $|V(G)| > n_0$ contains some member of \mathcal{F} as a subgraph. We denote this fact by $\alpha \rightarrow \mathcal{F}$.

Lemma 2.5 (c.f. [6]). *The following two properties are equivalent.*

1. α is a jump for r .
2. $\alpha \rightarrow \mathcal{F}$ for some finite family \mathcal{F} of r -graphs satisfying $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathcal{F}$.

We also need the following lemma proved in [6].

Lemma 2.6 (c.f. [6]). *For any $\sigma \geq 0$ and any integer $k \geq r$, there exists $t_0(k, \sigma)$ such that for every $t > t_0(k, \sigma)$, there exists an r -graph A satisfying:*

1. $|V(A)| = t$,
2. $|E(A)| \geq \sigma t^{r-1}$,
3. For all $V_0 \subset V(A)$, $r \leq |V_0| \leq k$ we have $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$.

The general approach in proving Theorems 1.5 and 1.6 is sketched as follows: Let α be a number to be proved to be a non-jump. Assuming that α is a jump, we will derive a contradiction by the following steps.

Step 1. Construct an r -uniform hypergraph (in Theorem 1.5, $r = 4$) with the Lagrangian close to but slightly smaller than $\frac{\alpha}{r!}$, then use Lemma 2.6 to add an r -graph with enough number of edges but sparse enough (see properties 2 and 3 in this Lemma) and obtain an r -graph with the Lagrangian $\geq \frac{\alpha}{r!} + \epsilon$ for some positive ϵ . Then we ‘blow up’ this r -graph to an r -graph, say H with large enough number of vertices and density $> \alpha + \frac{\epsilon}{2}$ (see Remark 2.3). If α is a jump, then by Lemma 2.5, α is a threshold for some finite family \mathcal{F} of r -graphs with Lagrangians $> \frac{\alpha}{r!}$. So H must contain some member of \mathcal{F} as a subgraph.

Step 2. We show that any subgraph of H with the number of vertices not greater than $\max\{|V(F)|, F \in \mathcal{F}\}$ has the Lagrangian $\leq \frac{\alpha}{r!}$ and derive a contradiction.

It is easy to construct an r -graph satisfying the property in Step 1, but it is certainly nontrivial to construct an r -graph satisfying the properties in both Steps 1 and 2. In fact, whenever we find such a construction, we can obtain a corresponding non-jump. This method was first developed by Frankl and Rödl in [6], then it was used in [7, 10, 11] and [12] to find more non-jumps by giving this type of construction. The technical part in the proof is to show that the construction satisfies the property in Step 2 (Lemma 3.1).

3. PROOF OF THEOREM 1.5

In this Section, we focus on $r = 4$ and give a proof of Theorem 1.5. Let $\alpha = 1 - \frac{7}{l^2} + \frac{10}{l^3}$. Let t be a large enough integer determined later. We first define a 4-graph $G(l, t)$ on l pairwise disjoint sets V_1, \dots, V_l , each of cardinality t . The edge set of $G(l, t)$ consists of all 4-subsets taking exactly one vertex from each of V_i, V_j, V_k, V_s ($1 \leq i < j < k < s \leq l$), all 4-subsets taking 2 vertices from V_i , 1 vertex from V_j and 1 vertex from V_k ($1 \leq i \leq l$, $1 \leq j < k \leq l$ and i, j, k are pairwise distinct), and all 4-subsets taking 3 vertices from V_i and 1 vertex from V_{i+1} ($1 \leq i \leq l$ and $V_{l+1} = V_1$). When $l = 2$ or 3 , some of them are vacant.

Note that the density of $G(l, t)$ is close to α if t is large enough. In fact,

$$(2) \quad \begin{aligned} |E(G(l, t))| &= \binom{l}{4} t^4 + l \binom{l-1}{2} \binom{t}{2} t^2 + l \binom{t}{3} t \\ &= \frac{\alpha}{24} l^4 t^4 - c_0(l) t^3 + o(t^3), \end{aligned}$$

where $c_0(l)$ is positive (we omit giving the precise calculation here). Let $\vec{u} = (u_1, \dots, u_{lt})$, where $u_i = 1/(lt)$ for each $i, 1 \leq i \leq lt$, then

$$\lambda(G(l, t)) \geq \lambda(G(l, t), \vec{u}) = \frac{|E(G(l, t))|}{(lt)^4} = \frac{\alpha}{24} - \frac{c_0(l)}{l^4 t} + o\left(\frac{1}{t}\right)$$

which is close to $\frac{\alpha}{24}$ when t is large enough.

We will use Lemma 2.6 to add a 4-graph to $G(l, t)$ so that the Lagrangian of the resulting 4-graph is $> \frac{\alpha}{24} + \epsilon(t)$ for some $\epsilon(t) > 0$. The precise argument is given below.

Suppose that α is a jump. In view of Lemma 2.5, there exists a finite collection \mathcal{F} of 4-graphs satisfying the following:

- (i) $\lambda(F) > \frac{\alpha}{24}$ for all $F \in \mathcal{F}$, and
- (ii) α is a threshold for \mathcal{F} .

Set $k_0 = \max_{F \in \mathcal{F}} |V(F)|$ and $\sigma_0 = c_0(l)$. Let $r = 4$ in Lemma 2.6 and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.6. Take an integer $t > \max(t_0, t_1)$, where t_1 is determined in (3) given later. For each $i, 1 \leq i \leq l$, take a 4-graph $A_{k_0, \sigma_0}^i(t)$ satisfying the conditions in Lemma 2.6 with $V(A_{k_0, \sigma_0}^i(t)) = V_i$. The 4-graph $G^*(l, t)$ is obtained by adding all $A_{k_0, \sigma_0}^i(t)$ to the 4-uniform

hypergraph $G(l, t)$. Then

$$\lambda(G^*(l, t)) \geq \lambda(G^*(l, t), \vec{u}) = \frac{|E(G^*(l, t))|}{(lt)^4}.$$

In view of the construction of $G^*(l, t)$ and equation (2), we have

$$(3) \quad \frac{|E(G^*(l, t))|}{(lt)^4} \geq \frac{|E(G(l, t))| + l\sigma_0 t^3}{(lt)^4} \stackrel{(2)}{\geq} \frac{\alpha}{24} + \frac{c_0(l)}{2l^4 t}$$

for $t \geq t_1$. Consequently,

$$(4) \quad \lambda(G^*(l, t)) \geq \frac{\alpha}{24} + \frac{c_0(l)}{2l^4 t}$$

for $t \geq t_1$.

Now suppose $\vec{y} = (y_1, y_2, \dots, y_{lt})$ is an optimal vector of $\lambda(G^*(l, t))$. Let $\epsilon = \frac{6c_0(l)}{l^4 t}$ and $n > n_1(\epsilon)$ as in Remark 2.3. Then 4-graph $S_n = (\lfloor ny_1 \rfloor, \dots, \lfloor ny_{lt} \rfloor) \otimes G^*(l, t)$ has density larger than $\alpha + \epsilon$. Since α is a threshold for \mathcal{F} , some member F of \mathcal{F} is a subgraph of S_n for $n \geq \max\{n_0(\epsilon), n_1(\epsilon)\}$. For such $F \in \mathcal{F}$, there exists a subgraph M of $G^*(l, t)$ with $|V(M)| \leq |V(F)| \leq k_0$ so that $F \subset \vec{n} \otimes M$. By Fact 2.1 and Fact 2.4, we have

$$(5) \quad \lambda(F) \stackrel{\text{Fact 2.1}}{\leq} \lambda(\vec{n} \otimes M) \stackrel{\text{Fact 2.4}}{=} \lambda(M).$$

Theorem 1.5 will follow from the following lemma to be proved in Section 4.

Lemma 3.1. *Let $G^*(l, t)$ be a 4-graph constructed the same way as above with k_0, σ_0, t replaced by any k, σ, t satisfying $t > t_0(k, \sigma)$ as given in Lemma 2.6 respectively. Let M be any subgraph of $G^*(l, t)$ with $|V(M)| \leq k$. Then*

$$(6) \quad \lambda(M) \leq \frac{1}{24}\alpha$$

holds.

Applying Lemma 3.1 to (5), we have

$$\lambda(F) \leq \frac{1}{24}\alpha$$

which contradicts our choice of F , i.e., contradicts the fact that $\lambda(F) > \frac{1}{24}\alpha$ for all $F \in \mathcal{F}$. ■

To complete the proof of Theorem 1.5, what remains is to show Lemma 3.1.

4. PROOF OF LEMMA 3.1

By Fact 2.1, we may assume that M is an induced subgraph of $G^*(l, t)$. For each $s, 1 \leq s \leq l$, let

$$U_s = V(M) \cap V_s = \{v_1^s, v_2^s, \dots, v_{k_s}^s\}.$$

We will apply the following Claim proved in [6].

Claim 4.1 (c.f. [6]). *If N is the 4-graph formed from M by removing the edges contained in each U_s and inserting the edges $\{\{v_1^s, v_2^s, v_3^s, v_j^s\} : 1 \leq s \leq l, 4 \leq j \leq k_s\}$ then $\lambda(M) \leq \lambda(N)$.*

By Claim 4.1 the proof of Lemma 3.1 will be complete if we show that $\lambda(N) \leq \frac{\alpha}{24}$. Since v_1^s, v_2^s, v_3^s are pairwise equivalent and $v_4^s, \dots, v_{k_s}^s$ are pairwise equivalent we can use Lemma 2.2(part 1) to obtain an optimal vector \vec{z} of $\lambda(N)$ such that

$$z_1^s = z_2^s = z_3^s \stackrel{\text{def}}{=} \rho_s, \quad z_4^s = z_5^s = \dots = z_{k_s}^s \stackrel{\text{def}}{=} \zeta_s.$$

Let w_s be the sum of the total weights in U_s . Let $P = \{s : w_s > 0\}$ and $p = |P|$. Without loss of generality, we may assume that $P = \{1, 2, \dots, p\}$. We may also assume that $p \geq 2$. Otherwise,

$$\lambda(N) = \rho_1^3(1 - 3\rho_1) \leq \frac{1}{256} < \frac{1}{24} \left(1 - \frac{7}{2^2} + \frac{10}{2^3}\right) \leq \frac{1}{24} \left(1 - \frac{7}{l^2} + \frac{10}{l^3}\right) = \frac{\alpha}{24}$$

since $1 - \frac{7}{x^2} + \frac{10}{x^3}$ increases when $x \geq 3$ increases and $1 - \frac{7}{2^2} + \frac{10}{2^3} < 1 - \frac{7}{3^2} + \frac{10}{3^3}$.

So we may assume that $2 \leq p \leq l$. For each $s \in P$ take a vertex $u_s \in U_s$ with positive weight as follows: if $\zeta_s > 0$ then $u_s = v_4^s$ otherwise $u_s = v_1^s$. The vertex u_s receives non-zero weight. Let \hat{z}^s be the restriction of \vec{z} on $V(N_{u_s})$. Then by Lemma 2.2(part 2) we have

$$4\lambda(N) = \lambda(N_{u_s}, \hat{z}^s).$$

Moreover by considering the edges containing vertex u_s we have

$$\begin{aligned}
\lambda(N_{u_s}, \hat{z}^s) &\leq \sum_{1 \leq i < j < k \leq p; i, j, k \neq s} w_i w_j w_k + w_s \sum_{1 \leq i < j \leq p; i, j \neq s} w_i w_j \\
&+ \sum_{1 \leq i < j \leq p; i, j \neq s} \left(\frac{w_j^2}{2} w_i + \frac{w_i^2}{2} w_j \right) + \frac{w_s^2}{2} w_{s+1} \\
(7) \quad &+ \left[\frac{1}{6} (w_{s-1} - 3\rho_{s-1})^3 + \frac{3\rho_{s-1}}{2} (w_{s-1} - 3\rho_{s-1})^2 \right. \\
&\quad \left. + 3\rho_{s-1}^2 (w_{s-1} - 3\rho_{s-1}) + \rho_{s-1}^3 \right] + \rho_s^3,
\end{aligned}$$

where all subscripts are modulo p . Note that

$$\begin{aligned}
&\frac{1}{6} (w_{s-1} - 3\rho_{s-1})^3 + \frac{3\rho_{s-1}}{2} (w_{s-1} - 3\rho_{s-1})^2 + 3\rho_{s-1}^2 (w_{s-1} - 3\rho_{s-1}) + \rho_{s-1}^3 \\
&\leq \frac{(w_{s-1} - 3\rho_{s-1})^3 + 9\rho_{s-1} (w_{s-1} - 3\rho_{s-1})^2 + 27\rho_{s-1}^2 (w_{s-1} - 3\rho_{s-1}) + 27\rho_{s-1}^3}{6} \\
&\quad - \rho_{s-1}^3 = \frac{w_{s-1}^3}{6} - \rho_{s-1}^3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
4p\lambda(N) &= \sum_{s=1}^p \lambda(N_{u_s}, \hat{z}^s) \\
(8) \quad &\leq p \sum_{1 \leq i < j < k \leq p} w_i w_j w_k + \frac{p-2}{2} \sum_{1 \leq i < j \leq p} (w_i^2 w_j + w_j^2 w_i) \\
&\quad + \frac{1}{2} \sum_{s=1}^p w_s^2 w_{s+1} + \frac{1}{6} \sum_{s=1}^p w_s^3.
\end{aligned}$$

If $p = 2$, then

$$8\lambda(N) \leq \frac{w_1^3}{6} + \frac{w_2^3}{6} + \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2} = \frac{(w_1 + w_2)^3}{6} = \frac{1}{6}.$$

This implies that

$$\lambda(N) \leq \frac{1}{48} = \frac{1}{24} \left(1 - \frac{7}{2^2} + \frac{10}{2^3} \right) \leq \frac{1}{24} \left(1 - \frac{7}{l^2} + \frac{10}{l^3} \right) = \frac{\alpha}{24}.$$

So we may assume that $p \geq 3$ from now on. We separate the right hand side of (8) into two parts as follows:

$$(9) \quad f(w_1, w_2, \dots, w_p) = \sum_{1 \leq i < j < k \leq p} w_i w_j w_k + \frac{1}{2} \sum_{s=1}^p w_s^2 w_{s+1}.$$

$$(10) \quad \begin{aligned} g(w_1, w_2, \dots, w_p) &= (p-1) \sum_{1 \leq i < j < k \leq p} w_i w_j w_k \\ &+ \frac{1}{6} \sum_{s=1}^p w_s^3 + \frac{p-2}{2} \sum_{1 \leq i < j \leq p} (w_i^2 w_j + w_j^2 w_i). \end{aligned}$$

Note that

$$f\left(\frac{1}{p}, \frac{1}{p}, \dots, \frac{1}{p}\right) + g\left(\frac{1}{p}, \frac{1}{p}, \dots, \frac{1}{p}\right) = \frac{p}{6} \left(1 - \frac{7}{p^2} + \frac{10}{p^3} \right) \leq \frac{p}{6} \left(1 - \frac{7}{l^2} + \frac{10}{l^3} \right) = \frac{p\alpha}{6}.$$

Therefore, Lemma 3.1 follows from the following two Claims.

Claim 4.2. If function $f(a_1, a_2, \dots, a_p)$ reaches the maximum at (a_1, a_2, \dots, a_p) under the constraints $\sum_{i=1}^p a_i = 1; a_i \geq 0$, then $a_1 = a_2 = \dots = a_p = \frac{1}{p}$.

The proof of Claim 4.2 will be given later.

Claim 4.3. If function $g(a_1, a_2, \dots, a_p)$ reaches the maximum at (a_1, a_2, \dots, a_p) under the constraints $\sum_{i=1}^p a_i = 1; a_i \geq 0$, then $a_1 = a_2 = \dots = a_p = \frac{1}{p}$.

Proof of Claim 4.3. Suppose that $g(a_1, a_2, \dots, a_p)$ reaches the maximum at (a_1, a_2, \dots, a_p) . We first note that $q = |\{i : a_i > 0\}| \geq 3$. If $q = 1$, then by a direct calculation, $g(1, 0, 0, \dots, 0) \leq g(\frac{1}{p}, \frac{1}{p}, \dots, \frac{1}{p})$ when $p \geq 3$. If $q = 2$, without loss of generality, assume that $a_1 > 0$ and $a_2 > 0$, then it is not difficult to show that

$$g(a_1, a_2, 0, \dots, 0) \leq g\left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right) \leq g\left(\frac{1}{p}, \frac{1}{p}, \dots, \frac{1}{p}\right).$$

Now we are going to show that $a_1 = a_2 = \dots = a_p = \frac{1}{p}$. If not, without loss of generality, assume that $a_2 > a_1$, we will show that $g(a_1 + \epsilon, a_2 - \epsilon, a_3, \dots, a_p) - g(a_1, a_2, a_3, \dots, a_p) > 0$ for small enough $\epsilon > 0$ and get a contradiction. In fact

$$\begin{aligned} & g(a_1 + \epsilon, a_2 - \epsilon, a_3, \dots, a_p) - g(a_1, a_2, a_3, \dots, a_p) \\ &= (p-1)[(a_1 + \epsilon)(a_2 - \epsilon) - a_1 a_2](1 - a_1 - a_2) \\ &+ \frac{1}{6} [(a_1 + \epsilon)^3 + (a_2 - \epsilon)^3 - a_1^3 - a_2^3] \\ &+ \frac{p-2}{2} [(a_1 + \epsilon)^2(a_2 - \epsilon) + (a_1 + \epsilon)(a_2 - \epsilon)^2 - a_1^2 a_2 - a_1 a_2^2] \\ &= (a_2 - a_1) \left[p-1 - \left(\frac{p}{2} + \frac{1}{2} \right) (a_1 + a_2) \right] \epsilon + o(\epsilon) > 0 \end{aligned}$$

for small enough $\epsilon > 0$ since the coefficient of ϵ , $(a_2 - a_1)[p-1 - (\frac{p}{2} + \frac{1}{2})(a_1 + a_2)]$ is positive under the assumption that $a_2 > a_1$, $p \geq 3$ and $a_1 + a_2 < 1$ (since $q \geq 3$). This contradicts to the assumption that g reaches the maximum at (a_1, a_2, \dots, a_p) and Claim 4.3 follows. ■

Now we will prove Claim 4.2.

Proof of Claim 4.2. We will use induction on p . If $p = 3$, it is enough to show the following Claim.

Claim 4.4.

$$\begin{aligned} (11) \quad f(a_1, a_2, a_3) &= a_1 a_2 a_3 + \frac{1}{2} a_1^2 a_2 + \frac{1}{2} a_2^2 a_3 + \frac{1}{2} a_3^2 a_1 \\ &\leq f(1/3, 1/3, 1/3) = \frac{5}{54} \end{aligned}$$

holds under the constraints $\sum_{i=1}^3 a_i = 1$; $a_i \geq 0$.

Proof of Claim 4.4. By the theory of Lagrange multipliers (see [1]), if $f(a_1, a_2, a_3)$ attains the maximum at (a_1, a_2, a_3) , then either $\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \frac{\partial f}{\partial a_3}$, i.e.,

$$(12) \quad a_2 a_3 + \frac{1}{2} a_3^2 + a_1 a_2 = a_1 a_3 + \frac{1}{2} a_1^2 + a_2 a_3 = a_1 a_2 + \frac{1}{2} a_2^2 + a_3 a_1,$$

or some $a_i = 0$.

If some $a_i = 0$, then it is easy to verify that $f(a_1, a_2, a_3) \leq \frac{2}{27}$.

Now assume that none of a_1, a_2, a_3 is 0, then (12) holds. In this case,

$$\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \frac{\partial f}{\partial a_3} = a_1 \frac{\partial f}{\partial a_1} + a_2 \frac{\partial f}{\partial a_2} + a_3 \frac{\partial f}{\partial a_3} = 3f(a_1, a_2, a_3).$$

Therefore,

$$\begin{aligned} 9f(a_1, a_2, a_3) &= \frac{\partial f}{\partial a_1} + \frac{\partial f}{\partial a_2} + \frac{\partial f}{\partial a_3} \\ &= 2(a_1a_2 + a_1a_3 + a_2a_3) + \frac{a_1^2 + a_2^2 + a_3^2}{2} \\ &= \frac{1}{2} + (a_1a_2 + a_2a_3 + a_1a_3) \\ &\leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

This implies that $f(a_1, a_2, a_3) \leq \frac{5}{54} = f(1/3, 1/3, 1/3)$ and completes the proof of Claim 4.4. \blacksquare

Now let us apply the induction on p and continue the proof of Claim 4.2. Suppose that $f(a_1, \dots, a_p)$ has the maximum at (a_1, \dots, a_p) . If some $a_i = 0$, say $a_p = 0$, then by induction assumption, $f(a_1, \dots, a_{p-1}, 0) \leq \frac{1}{6}(1 - \frac{3}{p-1} + \frac{5}{(p-1)^2}) < \frac{1}{6}(1 - \frac{3}{p} + \frac{5}{p^2}) = f(1/p, 1/p, \dots, 1/p)$. Therefore, each $a_i > 0$ and $\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \dots = \frac{\partial f}{\partial a_p}$. By a direct calculation, for each $i, 1 \leq i \leq p$,

$$\frac{\partial f}{\partial a_i} = \sum_{1 \leq j < k \leq p; j, k \neq i} a_j a_k + a_i a_{i+1} + \frac{a_{i-1}^2}{2},$$

where all subscripts here are modulo p . Then for each $i, 1 \leq i \leq p$,

$$\frac{\partial f}{\partial a_i} = \sum_{i=1}^p a_i \frac{\partial f}{\partial a_i} = 3f(a_1, \dots, a_p).$$

Therefore,

$$\begin{aligned} 3pf(a_1, \dots, a_p) &= \sum_{i=1}^p \frac{\partial f}{\partial a_i} \\ (13) \quad &= (p-2) \sum_{1 \leq i < j \leq p} a_i a_j + \sum_{i=1}^p \frac{a_i^2}{2} + \sum_{i=1}^p a_i a_{i+1}. \end{aligned}$$

If $p \geq 5$, then we apply $a_i a_{i+1} \leq \frac{a_i^2 + a_{i+1}^2}{2}$ to the above inequality and obtain that

$$\begin{aligned} 3pf(a_1, \dots, a_p) &\leq (p-2) \sum_{1 \leq i < j \leq p} a_i a_j + \sum_{i=1}^p \frac{3a_i^2}{2} \\ &= \frac{3}{2} + (p-5) \sum_{1 \leq i < j \leq p} a_i a_j \\ &\leq \frac{3}{2} + (p-5) \frac{\binom{p}{2}}{p^2} = \frac{p^2 - 3p + 5}{2p}. \end{aligned}$$

Therefore,

$$f(a_1, \dots, a_p) \leq \frac{1}{6} \left(1 - \frac{3}{p} + \frac{5}{p^2} \right) = f(1/p, 1/p, \dots, 1/p).$$

If $p = 4$, then (13) is equivalent to

$$\begin{aligned} 12f(a_1, a_2, a_3, a_4) &= 2 \sum_{1 \leq i < j \leq 4} a_i a_j + \sum_{i=1}^4 \frac{a_i^2}{2} + (a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_1) \\ &= \frac{1}{2} + \sum_{1 \leq i < j \leq 4} a_i a_j + (a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_1) \\ &\stackrel{\text{def}}{=} h(a_1, a_2, a_3, a_4). \end{aligned}$$

It is enough to show that

$$(14) \quad h(a_1, a_2, a_3, a_4) \leq h(1/4, 1/4, 1/4, 1/4) = \frac{9}{8}.$$

In fact, $h(a_1, a_2, a_3, a_4)$ has the maximum either at some $a_i = 0$ or satisfy

$$\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_2} = \frac{\partial h}{\partial a_3} = \frac{\partial h}{\partial a_4}.$$

By a direct calculation, the above equation implies that $a_1 = a_2 = a_3 = a_4$.

If $|\{i : a_i = 0, 1 \leq i \leq 4\}| = 3$ or 2 , then (14) is clearly true. If one of a_i is 0, without loss of generality, assuming that $a_4 = 0$, then

$$\begin{aligned} h(a_1, a_2, a_3, 0) &= \frac{1}{2} + 2(a_1a_2 + a_2a_3) + a_1a_3 \leq \frac{1}{2} + 2a_2(1 - a_2) + \frac{(1 - a_2)^2}{4} \\ &= -\frac{7}{4} \left(a_2 - \frac{3}{7} \right)^2 + \frac{15}{14} < \frac{9}{8}. \end{aligned}$$

The proof of Claim 4.2 is completed. ■

5. PROOF OF THEOREM 1.6

Theorem 1.6 extends Theorem 1.5 for the case $l = 4$ to every integer $r \geq 4$. The proof is based on an extension of the 4-graph $G^*(l, t)$ in Section 3 for the case $l = 4$.

Suppose that $\frac{23r!}{3^{r^r}}$ is a jump for $r \geq 4$. In view of Lemma 2.5, there exists a finite collection \mathcal{F} of r -graphs satisfying the following:

- (i) $\lambda(F) > \frac{23}{3^{r^r}}$ for all $F \in \mathcal{F}$, and
- (ii) $\frac{23r!}{3^{r^r}}$ is a threshold for \mathcal{F} .

Set $k_0 = \max_{F \in \mathcal{F}} |V(F)|$. Let $\sigma_0 = c_0(4)$ be the number defined as in Section 3. Let $r = 4$ in Lemma 2.6 and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.6. Take an integer $t > \max(t_0, t_1)$, where t_1 is the number from (3). Now define $G^*(4, t)$ (i.e., $l = 4$) the same way as in Section 3. with the new k_0 . For simplicity, we simply write $G^*(4, t)$ as $G(t)$.

Since Theorem 1.5 holds, we may assume that $r \geq 5$. Based on the 4-graph $G(t)$, we construct an r -graph $G^{(r)}(t)$ on r pairwise disjoint sets $V_1, V_2, V_3, V_4, V_5, \dots, V_r$, each of cardinality t . The edge set of $G^{(r)}(t)$ consists of all r -subsets in the form of $\{u_1, u_2, u_3, u_4, u_5, \dots, u_r\}$, where $\{u_1, u_2, u_3, u_4\}$ is an edge in $G(t)$ and for each $j, 5 \leq j \leq r, u_j \in V_j$. Notice that

$$(15) \quad |E(G^{(r)}(t))| = t^{r-4}|E(G(t))|.$$

Take $l = 4$ in (3), we get

$$(16) \quad |E(G(t))| \geq \frac{23}{3}t^4 + \frac{c_0(l)t^3}{2}.$$

Therefore,

$$\begin{aligned} \lambda(G^{(r)}(t)) &\geq \frac{|E(G^{(r)}(t))|}{(rt)^r} \\ &\stackrel{(15),(16)}{\geq} \frac{23}{3r^r} + \frac{c_0(l)}{2r^r t}. \end{aligned}$$

Similar to the case that Theorem 1.5 follows from Lemma 3.1, Theorem 1.6 follows from the following Lemma.

Lemma 5.1. *Let $M^{(r)}$ be a subgraph of $G^{(r)}(t)$ with $|V(M^{(r)})| \leq k_0$. Then*

$$(17) \quad \lambda(M^{(r)}) \leq \frac{23}{3r^r}$$

holds.

Proof of Lemma 5.1. By Fact 2.1, we may assume that $M^{(r)}$ is an induced subgraph of $G^{(r)}(t)$. Let $M^{(4)}$ be the 4-graph defined on $\cup_{i=1}^4 V_i$ by taking the edge set to be $\{e \cap (\cup_{i=1}^4 V_i), \text{ where } e \text{ is an edge of the } r\text{-graph } M^{(r)}\}$. Note that $|V(M^{(4)})| \leq |V(M^{(r)})| \leq k_0$. Let $\vec{\xi}$ be an optimal vector for $\lambda(M^{(r)})$. Define $U_i = V(M) \cap V_i$ for $1 \leq i \leq r$. Let a_i be the sum of the weights in $U_i, 1 \leq i \leq r$ respectively. Let $\vec{\xi}^{(4)}$ be the restriction of $\vec{\xi}$ on $V(M^{(4)})$. In view of the relationship between $M^{(r)}$ and $M^{(4)}$, we have

$$(18) \quad \lambda(M^{(r)}) = \lambda(M^{(4)}, \vec{\xi}^{(4)}) \times \prod_{i=5}^r a_i.$$

Applying Lemma 3.1 (take $l = 4$ there) with the constraints replaced by $\sum_{i=1}^4 a_i = 1 - \sum_{i=5}^r a_i$, we obtain that

$$\lambda(M^{(4)}, \vec{\xi}^{(4)}) \leq \frac{1}{24} \frac{23}{32} \left(1 - \sum_{i=5}^r a_i\right)^4.$$

Therefore,

$$\lambda(M^{(r)}) \leq \frac{1}{24} \frac{23}{32} \left(1 - \sum_{i=5}^r a_i\right)^4 \prod_{i=5}^r a_i.$$

Since geometric mean is no more than arithmetic mean, we obtain that

$$\lambda(M^{(r)}) \leq \frac{1}{24} \frac{23}{32} 4^4 \frac{1}{r^r} = \frac{23}{3r^r}.$$

This completes the proof of Lemma 5.1. ■

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