ON DISTANCE LOCAL CONNECTIVITY AND VERTEX DISTANCE COLOURING

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Abstract

In this paper, we give some sufficient conditions for distance local connectivity of a graph, and a degree condition for local connectivity of a k-connected graph with large diameter. We study some relationships between t-distance chromatic number and distance local connectivity of a graph and give an upper bound on the t-distance chromatic number of a k-connected graph with diameter d.

Keywords: degree condition, distance local connectivity, distance chromatic number.

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1. Introduction

By a graph we mean a simple undirected graph. We use [2] for terminology and notation not defined here. Let dist$_G$(x, y) denote the distance between vertices x and y in G. An x, y-path is a path between vertices x and y in G. Let d = max dist$_G$(xy) : x, y ∈ V(G) denote the diameter of G. An x, y-path P is called diameter-path, if dist$_G$(x, y) = d and |E(P)| = d. Let d$_G$(x) denote the degree of a vertex x in G, δ(G) the minimum degree of G and ∆(G) the maximum degree of G. For a nonempty set U ⊆ V(G), the induced subgraph on U is denoted by $\langle U \rangle$. For a nonempty

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set $A \subset V(G)$, $G - A$ denotes the subgraph of $G$ that we obtain by deleting all vertices of $A$ and all edges adjacent to at least one vertex of $A$. Let $\sigma_k(G) = \min\{\sum_{i=1}^k d_G(x_i)\mid \{x_1, \ldots, x_k\} \subset V(G), \text{ independent}\}$. The square of a graph $G$, denoted by $G^2$, is the graph in which $V(G^2) = V(G)$ and $E(G^2) = E(G) \cup \{\{u, v\}\mid \text{dist}_G(u, v) = 2\}$.

Let $N_G(x) = \{y \in V(G), xy \in E(G)\}$, let $N_G[x] = N_G(x) \cup \{x\}$. The set $N_G(x)$ is called the neighbourhood of the first type of $x$ in $G$. We say that $x$ is a locally connected vertex of $G$, if $\langle N_G(x) \rangle$ is connected. We say that $G$ is a locally connected graph, if every vertex of $G$ is locally connected. Chartrand and Pippert [3] proved the following Ore-type condition for local connectivity of graphs:

**Theorem A** [3]. Let $G$ be a connected graph of order $n$. If

$$d_G(u) + d_G(v) > \frac{4}{3}(n - 1)$$

for every pair of vertices $u, v \in V(G)$, then $G$ is locally connected.

Let $N_2(x)$ be a subgraph induced by the set of edges $uv$, such that

$$\min\{\text{dist}_G(x, u), \text{dist}_G(x, v)\} = 1.$$

The subgraph $N_2(x)$ is called the neighbourhood of the second type of $x$ in $G$. We say that $x$ is an $N_2$-locally connected vertex of $G$, if $N_2(x)$ is connected.

We say that $G$ is $N_2$-locally connected, if every vertex of $G$ is $N_2$-locally connected.

Now define the distance neighbourhood of the first type of a vertex of $G$ as in [5]. Let $m$ be a positive integer and let $x$ be an arbitrary vertex of a graph $G$. The $N_1^m$-neighbourhood of $x$ in $G$, denoted by $N_1^m(x)$, is the set of all vertices $y \in V(G), y \neq x$, such that $\text{dist}_G(x, y) \leq m$. Let $N_1^m[x] = N_1^m(x) \cup \{x\}$. A vertex $x$ is called $N_1^m$-locally connected if $\langle N_1^m(x) \rangle$ is connected. A graph $G$ is said to be $N_1^m$-locally connected if every vertex of $G$ is $N_1^m$-locally connected.

The distance local connectivity of the second type is analogously defined as the neighbourhood of the second type. Let $m$ be a positive integer and let $x$ be an arbitrary vertex of a graph $G$. The $N_2^m$-neighbourhood of $x$, denoted by $N_2^m(x)$, is the subgraph induced by all edges $\{u, v\}$ of $G, u \neq x, v \neq x$, with $\min\{\text{dist}_G(x, u), \text{dist}_G(x, v)\} \leq m$. We say that $x$ is $N_2^m$-locally connected in $G$ if $N_2^m(x)$ is connected. A graph $G$ is said to be $N_2^m$-locally connected if every vertex of $G$ is $N_2^m$-locally connected in $G$. 
Let $t$ be a positive integer. The $t$-distance chromatic number of a graph $G$, denoted $\chi^{(t)}(G)$, is the minimum number of colours required to colour all vertices of $G$ in such a way that any two vertices $x, y$ with $\text{dist}_G(x, y) \leq t$ have distinct colours. Let $\chi(x)$ denote the colour of a vertex $x$ in $G$. Recall that the vertex distance colouring was introduced by Kramer and Kramer in [7] and [8]. In the 90’s, several results on vertex distance colourings were presented, cf. Baldí in [1], Skupień in [11], Chen et al. in [4].

The following result was proved by Jendrol’ and Skupień in [6].

**Theorem B** [6]. Given a planar graph $G$, let $D = \max\{8, \Delta(G)\}$. Then the $t$-distance chromatic number of $G$ is

$$\chi^{(t)}(G) \leq 6 + \frac{3D + 3}{D - 2}((D - 1)^{t-1} - 1).$$

Madaras and Marcinová strengthened this condition in [9].

**Theorem C** [9]. Let $G$ be a planar graph, let $D = \max\{8, \Delta(G)\}$. Then

$$\chi^{(t)}(G) \leq 6 + \frac{2D + 12}{D - 2}((D - 1)^{t-1} - 1).$$

2. **Distance Local Connectivity of a Graph in $k$-Connected Graphs**

The concept of the local connectivity of a graph was introduced in 1970’s. Ryjáček used the concept of the local connectivity of a vertex in [10] for local completing in his closure concept for claw-free graphs. This closure concept gave a solution for several hamiltonian problems. A degree condition is one of the easily verified conditions. Chartrand and Pippert in [3] proved a degree condition for the local connectivity of connected graphs (see Theorem A). The same degree condition can guarantee the local connectivity of any vertex of a connected locally connected graph. In this chapter, degree conditions for the local connectivity of a $k$-connected graph with a large diameter will be presented as a strengthening of the result of Chartrand and Pippert. Holub and Xiong in [5] proved degree conditions for distance local connectivity of 2-connected graphs. As a strengthening of this condition, degree conditions for distance local connectivity of a $k$-connected graph with a large diameter will be shown, too.
Theorem 1. Let \( k \geq 2 \) be an integer, \( G \) be a \( k \)-connected graph of order \( n \). Let \( d \) be the diameter of \( G \), let \( d \geq 5 \). If

\[
d_G(u) + d_G(v) > \frac{4}{3}(n - kd + 5k - 3)
\]

for every pair of vertices \( u, v \in V(G) \), then \( G \) is locally connected.

Theorem 2. Let \( k \geq 2 \) be an integer, \( G \) be a \( k \)-connected graph of order \( n \). Let \( d \) be the diameter of \( G \), \( m \) be an integer such that \( 2 \leq m \leq \frac{1}{2}(d - 7) \). If

1. \( \sigma_t \geq n - kd - 2mk + 6k - t \), where \( t = \frac{2}{3}m + 1 \) if \( m \equiv 0 \) (mod 3),
2. \( \sigma_t \geq n - kd - 2mk + 6k - 2 - t \), where \( t = \frac{2}{3}(m-1) + 3 \) if \( m \equiv 1 \) (mod 3),
3. \( \sigma_t \geq n - kd + 2mk + 4k - 1 - t \), where \( t = \frac{2}{3}(m-2) + 3 \) if \( m \equiv 2 \) (mod 3),

then \( G \) is \( N^m_1 \)-locally connected.

Before proofs of these two theorems, some auxiliary statements will be shown.

Lemma 1. Let \( k \geq 2 \) be an integer, \( G \) be a \( k \)-connected graph and \( x \) be an arbitrary vertex of \( G \). Let \( d \) be the diameter of \( G \), let \( d \geq 5 \). If \( x \) does not belong to any diameter-path in \( G \), then there are at least \( kd - 5k + 2 \) vertices \( y \) such that \( \text{dist}_G(x, y) > 2 \).

Proof. Let \( P \) denote a diameter-path in \( G \), let \( u, v \) be the end vertices of \( P \). Since \( G \) is \( k \)-connected, there are at least \( k \) vertex-disjoint \( u, v \)-paths in \( G \) by Menger’s theorem. Choose \( P_1, \ldots, P_k \) with minimum sum of their lengths. Note that \( |E(P_i)| \geq d \), \( i = 1, \ldots, k \). Now it will be shown that there are at least \( d - 3 \) vertices at the required distance from \( x \) on each of \( P_i \), \( i = 1, \ldots, k \). Let \( M_j = \{ y \in P_j | \text{dist}_G(x, y) \leq 2 \} \), \( j = 1, \ldots, k \). For each path of \( P_i \), \( i = 1, \ldots, k \), there are two following cases:

Case 1. If \( M_j = \emptyset \), then there are at least \( d + 1 \) vertices at the required distance from \( x \) on \( P_j \).

Case 2. If \( M_j \neq \emptyset \), then let \( a_j \in M_j \) such that \( \text{dist}_G(a_j, u) = \min_{m \in M_j} \text{dist}_G(m, u) \) and let \( b_j \in M_j \) such that \( \text{dist}_G(b_j, v) = \min_{m \in M_j} \text{dist}_G(m, v) \). Since \( x \) does not belong to any diameter path, we have

\[
\text{dist}_G(u, a_j) + \text{dist}_G(a_j, x) + \text{dist}_G(x, b_j) + \text{dist}_G(b_j, v) \geq d + 1.
\]
Since \( \text{dist}_G(a_j, x) \leq 2 \) and \( \text{dist}_G(b_j, x) \leq 2 \), we obtain

\[
\text{dist}_G(u, a_j) + \text{dist}_G(b_j, v) \geq d - 3.
\]

Hence there are at least \( d - 3 \) vertices at the required distance from \( x \) on \( P_j \).

On the paths \( P_i, i = 1, \ldots, k \), there are at least \( k(d - 3) \) vertices at the required distance from \( x \) in \( G \). Since \( u \) and \( v \) can be counted only once, there are at least \( k(d - 5) + 2 \) different vertices at the required distance from \( x \) in \( G \).

**Proof of Theorem 1.** Suppose \( G \) is not locally connected. Then there is a vertex \( x \) such that \( x \) is not locally connected in \( G \). There are at least two components of \( \langle N_G(x) \rangle \). Let \( G_1 \) denote a smallest component of \( \langle N_G(x) \rangle \) and let \( G_2 \) be the union of all the other components of \( \langle N_G(x) \rangle \). Let \( g_1 = |V(G_1)| \), let \( g_2 = |V(G_2)| \). Let \( Z = \{ y \in V(G); \text{dist}_G(x, y) = 2 \} \), let \( z = |Z| \). Let \( p = |\{ y \in V(G); \text{dist}_G(x, y) > 2 \}|. \)

**Case 1.** Suppose that \( x \) does not belong to any diameter-path in \( G \). By Lemma 1, the number \( p \geq kd - 5k + 2 \). Clearly \( n = g_1 + g_2 + z + p + 1 \). Choose arbitrary vertices \( u \) and \( v \) such that \( u \in V(G_1) \) and \( v \in V(G_2) \). By the assumptions of Theorem 1

\[
d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).
\]

Since \( d_G(x) = g_1 + g_2 \) and \( d_G(u) \leq g_1 + z = n - 1 - p - g_2 \leq n - 1 - g_2 - kd + 5k - 2 \), we obtain

\[
g_1 + g_2 + n - g_2 - 1 - kd + 5k - 2 > \frac{4}{3}(n - kd + 5k - 3).
\]

Clearly \( g_1 > \frac{1}{3}(n - kd + 5k - 3) \) and \( g_2 > \frac{1}{3}(n - kd + 5k - 3) \) since \( g_2 \geq g_1 \). Therefore

\[
z < \frac{1}{3}(n - kd + 5k - 3).
\]

For vertices \( u \) and \( v \)

\[
d_G(u) + d_G(v) \leq g_1 + z + g_2 + z < \frac{4}{3}(n - kd + 5k - 3),
\]

a contradiction.
Case 2. Suppose that \( x \) belongs to a diameter-path \( P \). Let \( e, f \) be the end vertices of \( P \). Since \( G \) is \( k \)-connected, there are at least \( k \) vertex-disjoint \( e, f \)-paths in \( G \). Choose \( P_1, \ldots, P_k \) with a minimum sum of their lengths. For each of \( P_i, i = 1, \ldots, k \) the following cases can happen.

Subcase 2.1. \( V(P_i) \cap Z = \emptyset \). Then there are at least \( d + 1 \) vertices on \( P_i \) at distance at least 3 from \( x \) in \( G \).

Subcase 2.2. \( V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset \), but \( V(P_i) \cap Z \neq \emptyset \). Let \( d_i = |V(P_i) \cap Z| \). If \( d_i \leq 4 \), then there are at least \( d - 3 \) vertices on \( P_i \) at distance at least 3 from \( x \) in \( G \).

Now suppose that \( d_i \geq 5 \). If there is a vertex \( w \in V(G_1) \cup V(G_2) \) such that \( w \) is adjacent to every vertex of \( V(P_i) \cap Z \), then there are at least \( d - 2 \) vertices at distance at least 3 from \( x \) in \( G \) since \( \text{dist}_G(e, f) \geq d \). If none of the vertices of \( V(G_1) \cup V(G_2) \) is adjacent to every vertex of \( V(P_i) \cap Z \), then

\[
\begin{align*}
&d_G(u) \leq g_1 + z - (d_i - 3) \leq g_1 + z - 1, \quad \forall u \in V(G_1), \\
&d_G(v) \leq g_2 + z - (d_i - 3) \leq g_2 + z - 1, \quad \forall v \in V(G_2).
\end{align*}
\]

Subcase 2.3. \( V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset \). Let \( d_i^1 = |V(P_i) \cap V(G_1)| \), \( d_i^2 = |V(P_i) \cap V(G_2)| \) and \( d_i = |V(P_i) \cap Z| \). Note that \( d_i \geq 2 \). The following two possibilities have to be considered.

(i) \( d_i^1 = 0 \) or \( d_i^2 = 0 \). Up to symmetry, suppose that \( d_i^2 = 0 \). If \( d_i^1 = 1 \) and \( d_i = 2 \), then there are at least \( d - 3 \) vertices on \( P_i \) at distance at least 3 from \( x \) in \( G \).

Now suppose that \( d_i^1 = 1 \) and \( d_i > 2 \). If there is a vertex \( w \in V(G_1) \) such that \( w \) is adjacent to every vertex of \( V(P_i) \cap Z \), then there are at least \( d - 2 \) vertices at distance at least 3 from \( x \) in \( G \) since \( \text{dist}_G(e, f) \geq d \). If there is no vertex \( w \in V(G_1) \) adjacent to every vertex of \( V(P_i) \cap Z \), then

\[
\begin{align*}
&d_G(u) \leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \quad \forall u \in V(G_1).
\end{align*}
\]

Now suppose that \( d_i^1 > 1 \). If there is a vertex \( w \in V(G_1) \) such that \( w \) is adjacent to every vertex of \( V(P_i) \cap Z \), then there are at least \( d - 2 \) vertices at distance at least 3 from \( x \) in \( G \) since \( \text{dist}_G(e, f) \geq d \). If there is no vertex \( w \in V(G_1) \) adjacent to every vertex of \( V(P_i) \cap Z \), then

\[
\begin{align*}
&d_G(u) \leq g_1 + z - (d_i^1 - 2) - (d_i - 1) \leq g_1 + z - 1, \quad \forall u \in V(G_1).
\end{align*}
\]
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(ii) $d_i^1 > 0$ and $d_i^2 > 0$. If $P_i$ is a diameter-path containing $x$, then there are at least $d - 4$ vertices on $P_i$ at distance at least 3 from $x$ in $G$. If $P_i$ does not contain $x$, then $d_i \geq 3$. If there is a vertex $w \in V(G_1) \cup V(G_2)$ such that $w$ is adjacent to every vertex of $V(P_i) \cap Z$, then there are at least $d - 2$ vertices at distance at least 3 from $x$ in $G$ since $\text{dist}_G(e, f) \geq d$. If there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$, then

\[
\begin{align*}
    d_G(u) &\leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \quad \forall u \in V(G_1), \\
    d_G(v) &\leq g_2 + z - (d_i - 2) \leq g_2 + z - 1, \quad \forall v \in V(G_2).
\end{align*}
\]

Let $l_1$ denote the number of such the paths $P_1, \ldots, P_k$, for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset, d_i \geq 5$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 = 1, d_i^2 = 0, d_i > 2$ and there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 > 1, d_i^2 = 0$ and there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 d_i^2 \neq 0, x \notin V(P_i)$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$.

Let $l_2$ denote the number of such the paths $P_1, \ldots, P_k$, for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset, d_i \geq 5$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 = 0, d_i^2 = 1, d_i > 2$ and there is no vertex $w \in V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 > 0, d_i^2 = 0$ and there is no vertex $w \in V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset, d_i^1 d_i^2 \neq 0, x \notin V(P_i)$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$.

Let $l = l_1 + l_2$. Then there are at least $kd - 5k + 2 - l - 1$ vertices at distance at least 3 from $x$ in $G$ and

\[
\begin{align*}
    d_G(u) &\leq g_1 + z - l_1, \quad \forall u \in V(G_1), \\
    d_G(v) &\leq g_2 + z - l_2, \quad \forall v \in V(G_2).
\end{align*}
\]
Suppose that $l_2 \geq l_1$. By the assumptions, for every $u \in V(G_1)$

$$d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).$$

Since $d_G(x) = g_1 + g_2$ and $d_G(u) \leq g_1 + z - l_1 \leq n - 1 - g_2 - l_1 - kd + 5k - 2 + l$,
we have

$$g_1 + g_2 + n - g_2 - l_1 - kd + 5k - 3 + l > \frac{4}{3}(n - kd + 5k - 3).$$

Clearly

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l$$
and

$$g_2 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l,$$

since $g_2 \geq g_1$. Thus

$$z < \frac{1}{3}(n - kd + 5k - 3) + 2l_1 - l.$$

For vertices $u$ and $v$, it holds that

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + l_1 - l_2,$$
a contradiction, since $l_2 \geq l_1$. Hence suppose that $l_1 > l_2$. Then we get

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l \geq \frac{1}{3}(n - kd + 5k - 3) - l_1.$$

Thus

$$g_2 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l,$$

$$z < \frac{1}{3}(n - kd + 5k - 3) + l - l_1 + l_1 - l = \frac{1}{3}(n - kd + 5k - 3).$$

Then

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + 1.$$
Hence
\[ d_G(u) + d_G(v) \leq \frac{4}{3}(n - kd + 5k - 3), \]
a contradiction. \( \blacksquare \)

The following example shows that the conditions of Theorem 1 are sharp.

**Example.** Let \( K_1, \ldots, K_{k_1} \) be \( k_1 \) cliques of order \( k \), let \( L_1, \ldots, L_{k_2} \) be \( k_2 \) cliques of order \( k \). Let \( K_0, L_0 \) be two cliques of order \( l_1 > 2k - 1 \), let \( M \) be a clique of order \( l_1 - k \). All considered cliques \( K_i, L_i \) are vertex-disjoint. Construct a graph \( G \) by joining a new vertex \( x \) with each vertex of \( K_0 \cup L_0 \), a new vertex \( u \) with each vertex of \( K_{k_1} \) and a new vertex \( v \) with each vertex of \( L_{k_2} \). Now join each vertex of \( K_i \) with each vertex of \( K_{i-1} \) for \( i = 1, \ldots, k_1 \), each vertex of \( L_i \) with each vertex of \( L_{i-1} \) for \( i = 1, \ldots, k_2 \) and each vertex of \( K_0 \cup L_0 \) with each vertex of \( M \). Clearly the prescribed graph \( G \) is \( k \)-connected and the vertex \( x \) is not locally connected. The diameter of \( G \) is \( d = k_1 + k_2 + 4 \). It holds that
\[ n = 1 + 2l_1 + l_1 - k + (k_1 + k_2)k + 2 = 3l_1 + (d - 5)k + 3. \]
Thus
\[ 3l_1 = n - kd + 5k - 3. \]
Furthermore
\[ d_G(x) = 2l_1, \]
\[ d_G(y) = 2l_1, \quad \forall y \in K_0, \]
\[ d_G(z) = 2l_1, \quad \forall z \in L_0. \]
Hence for every pair \( a, b \) of vertices of \( N_G[x] \) holds that
\[ d_G(a) + d_G(b) = 4l_1 = \frac{4}{3}(n - kd + 5k - 3). \]
and \( x \) is not locally connected.

The following lemma is a proposition analogous to Lemma 1 for the \( N_1^n \)-local connectivity of a vertex of a graph.

**Lemma 2.** Let \( k \geq 2 \) be an integer, \( G \) be a \( k \)-connected graph. Let \( d \) be the diameter of \( G \) and \( m \leq \frac{1}{2}(d - 1) \) be an integer. Then, for each vertex \( x \) of \( G \), there are at least \( kd - 2km + 2 \) vertices at distance at least \( m \) from \( x \) in \( G \).
Proof. Let $P$ denote a diameter-path in $G$, let $u, v$ be the end vertices of $P$. Since $G$ is $k$-connected, there are at least $k$ vertex-disjoint $u,v$-paths in $G$ by Menger’s theorem. Choose $P_1, \ldots, P_k$ with minimum sum of their lengths. Note that $|E(P_i)| \geq d$, $i = 1, \ldots, k$. Now it will be shown that there are at least $d - 2m + 2$ vertices at the required distance from $x$ on each of $P_i$, $i = 1, \ldots, k$. Let $M_j = \{ y \in P_j | \text{dist}_G(x, y) \leq m - 1 \}$, $j = 1, \ldots, k$.

For each path of $P_i$, $i = 1, \ldots, k$, there are two following cases:

Case 1. If $M_i = \emptyset$, then there are at least $d + 1$ vertices at the required distance from $x$ on $P_i$.

Case 2. If $M_i \neq \emptyset$, then let $a_i \in M_i$ such that $\text{dist}_G(a_i, u) = \min_{m \in M_i} \text{dist}_G(m, u)$ and let $b_i \in M_i$ such that $\text{dist}_G(b_i, v) = \min_{m \in M_i} \text{dist}_G(m, v)$. Clearly

$$\text{dist}_G(u, a_i) + \text{dist}_G(a_i, x) + \text{dist}_G(x, b_i) + \text{dist}_G(b_i, v) \geq d.$$

Since $\text{dist}_G(a_i, x) \leq m - 1$ and $\text{dist}_G(b_i, x) \leq m - 1$, we have

$$\text{dist}_G(u, a_i) + \text{dist}_G(b_i, v) \geq d - 2m + 2.$$

Hence there are at least $d - 2m + 2$ vertices at the required distance from $x$ on $P_i$.

On the paths $P_i$, $i = 1, \ldots, k$ there are at least $k(d - 2m + 2)$ vertices at the required distance from $x$ in $G$. Since $u$ and $v$ can be counted only once, there are at least $kd - 2km + 2$ different vertices at the required distance from $x$ in $G$.

Let $C$ be a cycle, $x \in V(C)$ and $\bar{C}$ be an orientation of $C$. Let $x^{-(i)}$ denote the $i$-th predecessor of $x$ on $C$ and $x^{+(i)}$ denote the $i$-th successor of $x$ on $C$ in the orientation $\bar{C}$.

Lemma 3 [5]. Let $G$ be a 2-connected graph, $x \in V(G)$, and $m$ be a positive integer. If $x$ is not $N^m_1$-locally connected, then there is an induced cycle $C$ of length at least $2m + 2$ such that, in an orientation of $C$,

- $\text{dist}_G(x^{-(i)}, x) = i$ and $\text{dist}_G(x^{+(i)}, x) = i$, $i = 1, \ldots, m$,
- $\text{dist}_G(y, x) > m$, for every $y \in V(C) \setminus \{x, x^{-(1)}, \ldots, x^{-(m)}, x^{+(1)}, \ldots, x^{+(m)}\}$. 
On Distance Local Connectivity and ...

The following consequence proved by Holub and Xiong we use in the proof of Theorem 2.

**Corollary 1** [5]. Let \( m \geq 2 \) be an integer, \( G \) a 2-connected graph. If \( x \in V(G) \) is not \( N_1^m \)-locally connected, then there is a set \( M \subset V(G) \) such that

1. \( M \) is independent in \((G - x)^2\), \( M \subset N_1^{m+1}(x) \) and \(|M| \geq \frac{2}{3} m + 1\), if \( m \equiv 0 \pmod{3} \),
2. \( M \) is independent in \((G - N_G[x])^2\), \( M \subset (N_1^{m+1}(x) \setminus N_1^1(x)) \) and \(|M| \geq \frac{2}{3} (m - 1) + 1\), if \( m \equiv 1 \pmod{3} \),
3. \( M \) is independent in \( G^2 \), \( M \subset N_1^{m}[x] \) and \(|M| \geq \frac{2}{3} (m - 2) + 2\), if \( m \equiv 2 \pmod{3} \).

**Proof of Theorem 2.** Suppose that \( G \) is not \( N_1^m \)-locally connected. Then there is a vertex \( x \in V(G) \) such that \( x \) is not \( N_1^m \)-locally connected in \( G \). Hence \( \langle N_1^m(x) \rangle \) consists of at least two components. Let \( G_1 \) denote arbitrary component of \( \langle N_1^m(x) \rangle \), let \( G_2 \) denote the union of all the other components of \( \langle N_1^m(x) \rangle \).

**Case 1.** \( m \equiv 0 \pmod{3} \). By Corollary 1 case (1), there is a set \( M \subset N_1^{m+1}(x) \) such that \(|M| = \frac{2}{3} m + 1\) and \( M \) is independent in \((G - x)^2\). Let \( t = |M| \). Using Lemma 3, the set \( M \) can be chosen in the following way: \( M = \{x_1, x_2, \ldots, x_t\} \), where \( x_{2j-1} = x^{-(3j-2)} \), \( x_{2j} = x^{+(3j-2)} \), \( j = 1, \ldots, \frac{m}{3} \), \( x_t = x^{+(m+1)} \). Let \( A = \{y \in V(G) \mid \text{dist}_G(x, y) > m + 2\} \), let \( a = |A| \). By Lemma 2, the number \( a \geq kd - 2(m + 3)k + 2 \). Since \( M \) is independent in \((G - x)^2\), we have, for every pair \( u, v \in M \setminus \{x\} \),

\[ N_{G - x}(u) \cap N_{G - x}(v) = \emptyset. \]

Since \( x \) is adjacent to at most two vertices of \( M \), we obtain

\[ \sum_{x_i \in M} d_G(x_i) \leq (n - 1) - t - a + 2 = n - t - a + 1. \]

Since \( a \geq kd - 2(m + 3)k + 2 \), we have

\[ \sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 1, \]

a contradiction.
Case 2. $m \equiv 1 \pmod{3}$. By Corollary 1 case (2), there is a set $M \subset N_1^{m+1}(x)$ such that $|M| = \frac{2}{3}(m-1)+1$ and $M$ is independent in $(G - N_G[x])^2$. Let $t = |M|$. Using Lemma 3, the set $M$ can be chosen in the following way: $M = \{x_1, x_2, \ldots, x_t\}$, where $x_{2j-1} = x^{-(3j-1)}, \ x_{2j} = x^{+(3j-1)}, \ j = 1, \ldots, \frac{m-1}{3}, \ x_t = x^{+(m+1)}$. Let $A = \{y \in V(G) \mid \text{dist}_G(x, y) > m + 1\}$, let $a = |A|$. By Lemma 2, the number $a \geq kd - 2(m + 3)k + 2$. Since $M$ is independent in $(G - N_G[x])^2$, we have, for every pair $u, v \in M$,

$$N_G(u) \cap N_G(v) = \emptyset.$$ 

Since each vertex of $N_G(x)$ is adjacent to at most one vertex of $M$, we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq (n - 1) - t - a.$$ 

Since $a \geq kd - 2(m + 3)k + 2$, we have

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 3,$$

a contradiction.

Case 3. $m \equiv 2 \pmod{3}$. By Corollary 1 case (3), there is a set $M \subset N_1^m(x)$ such that $|M| = \frac{2}{3}(m - 2) + 2$ and $M$ is independent in $G^2$. Let $A = \{y \in V(G) \mid \text{dist}_G(x, y) > m + 1\}$, let $a = |A|$. By Lemma 2, the number $a \geq kd - 2(m + 2)k + 2$. Since $M$ is independent in $G^2$, we have, for every pair $u, v \in M$,

$$N_G(u) \cap N_G(v) = \emptyset.$$ 

Let $t = |M|$. Hence

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - a.$$ 

Since $a \geq kd - 2(m + 2)k + 2$, we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 4k - 2,$$

a contradiction. \qed
3. Vertex Distance Colouring

There are several results on $t$-distance chromatic number for planar graphs. In this paragraph, results on $t$-distance chromatic number in $k$-connected, not necessary planar, graphs are presented. Moreover, the relations between distance local connectivity and $t$-distance chromatic number in 2-connected graphs are given. Main results of this section are the following theorems.

Theorem 3. Let $G$ be a $k$-connected graph of order $n$, $d$ be the diameter of $G$. Let $t < d$ be a positive integer. Then the distance-chromatic number

$$\chi^{(t)}(G) \leq \begin{cases} 
  n - 1 & \text{if } t = d - 1, \\
  n - (d - t - 2)k - 2 & \text{if } t < d - 1.
\end{cases}$$

Theorem 4. Let $G$ be a 2-connected graph of order $n$, let $t, k$ be positive integers. If

$$\chi^{(t)}(G) > n - (2k - 1)(t + 1),$$

then $G$ is $N^m_1$-locally connected, where $m = k(t + 1) - 1$.

Theorem 5. Let $G$ be a 2-connected graph of order $n$, $k$ be a positive integer and $t$ be an even positive integer. If

$$\chi^{(t)}(G) > n - 2k(t + 1),$$

then $G$ is $N^m_2$-locally connected, where $m = k(t + 1) + \frac{t}{2} - 1$.

The distance local connectivity number of a 2-connected graph $G$, denoted $dlc(G)$, is the smallest positive integer $m$ for which $G$ is $N^m_1$-locally connected. Since $G$ is 2-connected, the number $dlc(G)$ is well-defined. Note that local connectivity of a graph is the $N^1_1$-local connectivity. The following statement is a straightforward consequence of Theorem 4.

Corollary 2. Let $G$ be a 2-connected graph, let $t$ be a positive integer. If $dlc(G) = m$, then

$$\chi^{(t)}(G) \leq n - (k - 1)(t + 1),$$

where $k = \lceil \frac{2m}{t+1} \rceil$.

Proof of Theorem 3. Let $u, v$ denote the end vertices of a diameter path in $G$. Since $G$ is $k$-connected, there are at least $k$ vertex-disjoint $u, v$-paths $P_1, \ldots, P_k$ in $G$ by Menger’s theorem. Since $\text{dist}_G(u, v) = d$, each of $P_i$, $i = 1, \ldots, k$, has length at least $d$. Let $u_{i,j}$ denote a vertex on $P_i$ such
that \( \text{dist}_G(u, u_{i,j}) = j, i = 1, \ldots, k, j = 1, \ldots, d \). Since \( d > t \), there is at least one vertex \( u_{i,j} \) on \( P_i, i = 1, \ldots, k \), including the end-vertex \( v \), such that \( j > t, j = t + 1, \ldots, d \). If \( d - t = 1 \), then \( u_{i,t+1} = v \) for every \( i \in \{1, \ldots, k\} \).

We define colouring \( \chi \) of vertices of \( G \) in such a way that \( \chi(v) = \chi(u) \) and \( \chi(x) \neq \chi(y) \) for all pairs \( x, y \in V(G) \setminus \{u, v\} \). Clearly \( \chi \) is a \( t \)-distance colouring of \( G \) and

\[
\chi^{(t)}(G) \leq n - 1.
\]

Suppose that \( d - t > 1 \). We define a colouring \( \chi \) of vertices of \( G \) in such a way that the vertices of \( N_1^{t+1}(u) \) have distinct colours in \( G \), \( \chi(u) = \chi(u_{i,t+1}) \) and \( \chi(v) = \chi(u_{i,d-t-1}) \) for some \( i \in \{1, \ldots, k\} \). Moreover, if \( d - t > 2 \), then, for every \( i \in \{1, \ldots, k\} \), \( \chi(u_{i,j+t+1}) = \chi(u_{i,j}) \), since \( \text{dist}_G(u_{i,j}, u_{i,j+t+1}) = t + 1, j = 1, \ldots, d - t - 2 \). Clearly \( \chi \) is a \( t \)-distance colouring of \( G \). Hence there are at least \( k(d - t - 2) + 2 \) vertices with previously used colours, implying that

\[
\chi^{(t)}(G) \leq n - k(d - t - 2) - 2.
\]

For the proofs of Theorem 4 and Theorem 5 we need some auxiliary statements. The following lemma is the analogue of Lemma 3.

**Lemma 4.** Let \( G \) be a 2-connected graph, \( x \in V(G) \) and \( m \) be a positive integer. If \( x \) is not \( N_2^m \)-locally connected, then there is an induced cycle \( C \) containing \( x \) of length at least \( 2m + 3 \) such that, in an orientation of \( C \),

\[
\text{dist}_G(x^{-i}, x) = i \quad \text{and} \quad \text{dist}_G(x^{+i}, x) = i, \quad i = 1, \ldots, m + 1,
\]

**Proof.** The vertex \( x \) is not \( N_2^m \)-locally connected. The \( N_2^m \)-neighbourhood of a vertex \( x \) consists of at least two components \( G_1, G_2 \). Since \( G \) is 2-connected, there is a cycle \( C \) containing \( x \), such that \( x^{-1} \in G_1 \) and \( x^{+1} \in G_2 \) in an orientation of \( C \). Choose \( C \) shortest possible with this property. Since \( x \) is not \( N_2^m \)-locally connected, \( |V(C)| \geq 2m + 3 \). It is easy to see that \( C \) has the required property since otherwise there is a shorter cycle.

From the definition of a \( t \)-distance colouring we obtain the following clear observation.

**Proposition 1.** Let \( G \) be a 2-connected graph of order \( n \), let \( t \) be a positive integer, let \( d \) denote the diameter of \( G \). Then \( \chi^{(t)}(G) = n \) if and only if \( d \leq t \).
Corollary 3. Let $G$ be a 2-connected graph of order $n$, let $t$ be a positive integer. If $\chi^{(t)}(G) = n$, then $G$ is $N_1^t$-locally connected.

**Proof.** Suppose that $G$ is not $N_1^t$-locally connected, i.e., there is a vertex $x \in V(G)$ such that $x$ is not $N_1^t$-locally connected in $G$. By Proposition 1, $d \leq t$. By Lemma 3, there is an induced cycle $C$ in $G$ of length at least $2t + 2$, which contradicts the fact that $d \leq t$.

**Proof of Theorem 4.** Suppose that $G$ is not $N_1^m$-locally connected, i.e., there is a vertex $x$ which is not $N_1^m$-locally connected. By Lemma 3 there is an induced cycle $C$ containing $x$, such that $|V(C)| \geq 2m + 2$. Moreover $\text{dist}_G(x, x^{-i}) = \text{dist}_G(x, x^{+i}) = i$ for $i = 1, \ldots, m$. Since $x$ is not $N_1^m$-locally connected, the cycle $C$ can be chosen such that $x^{-1}$ and $x^{+1}$ belong to different components of $(N_1^m(x))$. Clearly $\text{dist}_G(x^{-i}, x^{-j}) = |i - j|$, for $i, j = 0, \ldots, m$ where $x^{-0} = x$.

We define a colouring $\chi$ of vertices of $G$ in such a way that all the vertices $x^{-0}, \ldots, x^{-t}$ have distinct colours, $\chi(x^{-i}) = \chi(x^{-i+t+1})$, $i = 0, \ldots, t$, since $|V(C)| \geq 2(t + 1)$. If $k > 1$, then $\chi(x^{-i+(j+1)(t+1)}) = \chi(x^{-i+(j-1)(t+1)})$ for $i = 0, \ldots, t$ and $j = 1, \ldots, 2k - 1$. All the remaining vertices of $G$ will be coloured with distinct unused colours. Clearly $\chi$ is a $t$-vertex distance colouring in $G$.

We have coloured $2k(t + 1)$ vertices of $C$ with only $t + 1$ colours. Since $m = k(t + 1) - 1$, we have coloured $2m + 2$ vertices of $C$ with only $t + 1$ colours, implying that

$$\chi^{(t)}(G) \leq n - (2m + 2) + (t + 1) = n - (2k - 1)(t + 1),$$

a contradiction.

**Proof of Theorem 5.** We will use similar arguments as is the proof of Theorem 4. Suppose that $G$ is not $N_2^m$-locally connected, i.e., there is a vertex $x$ which is not $N_2^m$-locally connected. By Lemma 4 there is an induced cycle $C$ containing $x$, such that $|V(C)| \geq 2m + 3$. Moreover $\text{dist}_G(x, x^{-i}) = \text{dist}_G(x, x^{+i}) = i$ for $i = 1, \ldots, m+1$. Since $x$ is not $N_2^m$-locally connected, the cycle $C$ can be chosen such that $x^{-1}$ and $x^{+1}$ belong to different components of $(N_2^m(x))$. Clearly $\text{dist}_G(x^{-i}, x^{-j}) = |i - j|$, for $i, j = 0, \ldots, m + 1$ where $x^{-0} = x$.

We define a colouring $\chi$ of vertices of $G$ in such a way that all the vertices $x^{-0}, \ldots, x^{-t}$ have distinct colours, $\chi(x^{-i}) = \chi(x^{-i+t+1})$, $i = 0, \ldots, t$,}
since \(|V(C)| \geq 2(t+1)|. If \(k > 1\), then \(\chi(x^{-(i+j(t+1)}) = \chi(x^{-(i+j-1)(t+1)})\) for \(i = 0, \ldots, t\) and \(j = 1, \ldots, 2k\), since \(|V(C)| \geq 2m + 3 = (2k + 1)(t + 1)|. All the remaining vertices of \(G\) will be coloured with distinct unused colours. Clearly \(\chi\) is a \(t\)-vertex distance colouring in \(G\).

Thus we can colour \((2k + 1)(t + 1)|\) vertices of \(C\) with only \(t + 1|\) colours. Hence we have

\[
\chi^{(t)}(G) \leq n - 2k(t + 1),
\]

a contradiction.

Now we give an example which show that conditions of Theorem 3 are sharp. Let \(d\) and \(k \geq 2\) be two positive integers. Consider two vertices \(u\) and \(v\) and \(d - 1\) cliques \(K_1, \ldots, K_{d-1}\) of order \(k\). We construct a graph \(G\) by joining each vertex of \(K_1\) with \(u\), each vertex of \(K_{d-1}\) with \(v\) and each vertex of \(K_i\) with each vertex of \(K_{i+1}\) for each \(i \in \{1, \ldots, d - 2\}\). The diameter of \(G\) is \(d\), the graph \(G\) is \(k\)-connected and the \(t\)-distance chromatic number is equal to

\[
\begin{cases} 
  n - 1 & \text{if } t = d - 1, \\
  n - (d - t - 2)k - 2 & \text{if } t < d - 1.
\end{cases}
\]

For the following two examples the conditions of Theorem 3 give better upper bound on the \(t\)-distance chromatic number than the conditions of Theorem B and C. Let \(d\) be a positive integer. Consider two vertices \(u\), \(v\) and \(d - 1\) cliques \(K_1, \ldots, K_{d-1}\) of order 3. Construct a graph \(G\) by joining each vertex of \(K_1\) with \(u\), each vertex of \(K_{d-1}\) with \(v\). Now pair vertices of \(K_i\) with vertices of \(K_{i+1}\), for each \(i \in \{1, \ldots, d - 2\}\). The structure of \(G\) is shown in Figure 1.

![Figure 1](image_url)

The graph \(G\) is 3-connected, the diameter of \(G\) is \(d\) and \(G\) is planar, because the graph on the following picture (Figure 2) is isomorphic with \(G\).
From Theorem 3 we obtain $\chi(t)(G) \leq 3(t + 1)$ and from Theorem B we get $\chi(t)(G) \leq \frac{9}{2}(7^{t-1} - 1) + 6$. For $t \geq 2$ the upper bound of Theorem 3 is better.

For any positive integer $d$, consider two vertices $u, v$, and $d - 1$ cliques $K_1, \ldots, K_{d-1}$, such that $K_1$ and $K_{d-1}$ are triangles and $K_1, \ldots, K_{d-2}$ are alternatively cliques of orders 3 and 4. Construct a graph $G$ in such a way that we join each vertex of $K_1$ with $u$, each vertex of $K_{d-1}$ with $v$ and we pair vertices of $K_i$ with vertices of $K_{i+1}$, for all $i \in \{1, \ldots, d - 2\}$, in such a way that is shown in Figure 3.

This graph $G$ is 3-connected, the diameter of $G$ is $d$ and $G$ is planar, because the graph on the following picture (Figure 4) is isomorphic with $G$. 
From Theorem 3 we get $\chi^{(t)}(G) \leq 3(t + 1) + 2 + \frac{d-1}{2}$, and, from Theorem B we obtain $\chi^{(t)}(G) \leq \frac{9}{2}((7)^{t-1} - 1) + 6$. Comparing these two values, the upper bound of Theorem 3 is asymptotically better for $t \geq 2$ and $d \ll 7^t$.

References


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