

**NONSINGULAR UNICYCLIC MIXED GRAPHS  
WITH AT MOST THREE EIGENVALUES  
GREATER THAN TWO\***

SHI-CAI GONG<sup>1,2</sup> AND YI-ZHENG FAN<sup>1</sup>

<sup>1</sup>*School of Mathematics and Computational Science  
Anhui University  
Hefei, Anhui 230039, P.R. China*

<sup>2</sup>*Department of Mathematics and Physics  
Anhui University of Science and Technology  
Anhui, Huainan 232001*

**e-mail:** fanyz@ahu.edu.cn

**e-mail:** gongsc@ahuu.edu.cn

**Abstract**

This paper determines all nonsingular unicyclic mixed graphs on at least nine vertices with at most three Laplacian eigenvalues greater than two.

**Keywords:** unicyclic graph, mixed graph, Laplacian eigenvalue, matching number, spectrum.

**2000 Mathematics Subject Classification:** 05C50, 15A18.

---

\*Supported by National Natural Science Foundation of China (10601001), Anhui Provincial Natural Science Foundation (050460102), NSF of Department of Education of Anhui Province (2004kj027, 2005kj005zd, 2006kj068a), Foundation of Innovation Team on Basic Mathematics of Anhui University, Foundation of Talents Group Construction of Anhui University, Excellent Youth Science and Technology Foundation of Anhui Province of China (06042088).

## 1. Introduction

Let  $G = (V, E)$  be a *mixed graph* with vertex set  $V = V(G)$  and edge set  $E = E(G)$ , which is obtained from a simple graph by orienting some (possibly none or all) of its edges. For each  $e \in E(G)$ , we define the *sign* of  $e$  and denote by  $\text{sgn } e = 1$  if  $e$  is unoriented and  $\text{sgn } e = -1$  if  $e$  is oriented. Set  $a_{ij} = \text{sgn } e$  if there exists an edge  $e$  joining  $v_i$  and  $v_j$ , and  $a_{ij} = 0$  otherwise. Then the resultant matrix  $A = (a_{ij})$  is called the *adjacency matrix* of  $G$ . The *incidence matrix* of  $G$  is an  $n \times m$  matrix  $M = M(G) = (m_{ij})$  whose entries are given by  $m_{ij} = 1$  if  $e_j$  is an unoriented edge incident to  $v_i$  or  $e_j$  is an oriented edge with head  $v_i$ ,  $m_{ij} = -1$  if  $e_j$  is an oriented edge with tail  $v_i$ , and  $m_{ij} = 0$  otherwise. The *Laplacian matrix* of  $G$  is defined as  $L(G) = MM^T$  (see [1] or [10]), where  $M^T$  denotes the transpose of  $M$ . Obviously  $L(G)$  is symmetric and positive semi-definite, and  $L(G) = D(G) + A(G)$  (or see [10, Lemma 2.1]), where  $D(G) = \text{diag}\{d(v_1), d(v_2), \dots, d(v_n)\}$ . Therefore the eigenvalues of  $L(G)$  can be arranged as follows:

$$\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G) \geq 0.$$

We briefly called the eigenvalues and eigenvectors of  $L(G)$  as those of  $G$ , respectively.  $G$  is called *singular* (or *nonsingular*) if  $L(G)$  is singular (or nonsingular).

A mixed graph  $G$  is called *quasi-bipartite* if it does not contain a non-singular cycle, or equivalently,  $G$  contains no cycles with an odd number of unoriented edges (see [1, Lemma 1]). Denote by  $\vec{G}$  the all-oriented graph obtained from  $G$  by arbitrarily orienting every unoriented edge of  $G$  (if one exists), and  $D$  the *signature matrix* with 1 or  $-1$  along its diagonal of a diagonal matrix. Then  $D^T L(G) D$  is the Laplacian matrix of a graph with the same underlying graph as that of  $G$ . So each signature matrix of order  $n$  gives a re-signing of the edges of  $G$  (that is, some oriented edges of  $G$  may turn to be unoriented and vice versa), and preserves the spectrum and the singularity of each cycle of  $G$ . We now use the notation  ${}^D G$  to denote the graph obtained from  $G$  by a re-signing under the signature  $D$ , and assume that the labelling of the vertices of  ${}^D G$  is the same as that of  $G$ .

**Lemma 1.1** ([10, Lemma 2.2], [4, Lemma 5]). *Let  $G$  be a connected mixed graph. Then  $G$  is singular if and only if  $G$  is quasi-bipartite.*

**Theorem 1.2** ([1, Theorem 4]). *Let  $G$  be a connected mixed graph. Then  $G$  is quasi-bipartite if and only if there exists a signature matrix  $D$  such that  $D^T L(G) D = L(\vec{G})$ .*

If  $G$  is nonsingular, the number of edges of  $G$  is at least  $n$  (the number of vertices of  $G$ ), since such graph  $G$  contains at least one nonsingular cycle, then nonsingular unicyclic mixed graphs may be considered as a class of mixed graphs whose edge number is minimal. By Lemma 1.1 and Theorem 1.2, the spectrum of a singular mixed graph is exactly that of a simple graph with the same underlying graph, one can refer to [10, 11, 3, 5]. So in this paper, we consider only the connected nonsingular unicyclic mixed graphs, and determine all those graphs  $G$  on at least 9 vertices with at most three eigenvalues greater than two, i.e.,  $\lambda_4(G) \leq 2$ . Then we could almost give all mixed graphs with at most three eigenvalues greater than two, since we can obtain the eigenvalues by mathematical softwares if  $G$  contains few vertices. A reason for our research can be explained as follows. Consider the *edge version* of the Laplacian matrix of  $G$ ,  $K(G) = M(G)^T M(G) = 2I + A(G^l)$  (see [2]), where  $G^l$  is the line graph of  $G$  (see [10]). Since  $K(G)$  and  $L(G)$  have the same nonzero eigenvalues, the distribution of eigenvalues of  $L(G)$  greater than 2 is the same as that of  $A(G^l)$  greater than 0.

## 2. Preliminaries

**Lemma 2.1** ([11, Lemma 2.2]). *Let  $G$  be a mixed graph on  $n$  vertices and let  $e$  be an (oriented or unoriented) edge of  $G$ . Then*

$$\lambda_1(G) \geq \lambda_1(G - e) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G) \geq \lambda_n(G - e).$$

We now extend some known results on eigenvalues distribution of simple graphs to mixed graphs.

**Theorem 2.2.** *Let  $G$  be a connected mixed graph on  $n$  vertices, and let  $\mu(G)$  be the matching number of  $G$ . Then*

- (i)  $m_G(2, +\infty) \geq \mu(G)$  if  $n > 2\mu(G)$ ;
- (ii)  $m_G(2, +\infty) \geq \mu(G) - 1$  if  $n = 2\mu(G)$ .

**Proof.** Let  $M \subseteq E(G)$  be a matching of  $G$  with maximum cardinality  $\mu(G)$ . There exists a spanning tree  $T$  of  $G$  which contains the matching  $M$ ,

and a signature matrix  $D$  such that  ${}^D T$  is all-oriented in the graph  ${}^D G$  by Theorem 1.2. Note that  $\mu(T) = \mu(G)$ . Then by Lemma 2.1 and the result of [8, Theorem 3], for the case of  $n > 2\mu(G)$ ,

$$m_G(2, +\infty) = m_{{}^D G}(2, +\infty) \geq m_{{}^D T}(2, +\infty) \geq \mu(G).$$

For the case of  $n = 2\mu(G)$ , by [8, Theorem 2],  $\lambda_{\mu(G)}({}^D T) = 2$ , and  $m_{{}^D T}(2, +\infty) = \mu(G) - 1$  from the fact that any integral eigenvalue greater than 1 of a tree has multiplicity one [7]. Then the result (ii) can be obtained similarly. ■

**Corollary 2.3.** *Let  $G$  be a nonsingular unicyclic mixed graph on at least 9 vertices. If  $m_G(2, +\infty) \leq 3$ , then  $G$  contains no cycles with length greater than 6.*

**Proof.** It follows from Theorem 2.2 that  $\mu(G) \geq 4$  cannot happen. ■

A pendent vertex of  $G$  is a vertex of degree 1, a quasi-pendant vertex is a vertex adjacent to a pendent vertex. Denote by  $\eta(G)$  the number of quasi-pendant vertices of  $G$ .

**Lemma 2.4.** *Let  $G$  be a connected mixed graph. Then  $m_G[0, 1) \geq \eta(G)$  and  $m_G(2, +\infty) \geq \eta(G)$ .*

**Proof.** Let  $v_1, v_2, \dots, v_{\eta(G)}$  be all quasi-pendant vertices of  $G$ . By Lemma 2.1, there exists a signature matrix  $D$  such that the pendant edges of  ${}^D G$  are all oriented. Let  $L_i$  ( $i = 1, 2, \dots, \eta(G)$ ) be the principal submatrix of  $L({}^D G)$  corresponding to the vertex  $v_i$  and all pendent vertices incident to it, which permutes to the following matrix form:

$$L'_i = \begin{bmatrix} d(v_i) & -1 & \cdots & -1 \\ -1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix}.$$

It is seen that the left-up  $2 \times 2$  principal submatrix of  $L'_i$  has one eigenvalue less than 1 and one eigenvalue greater than 2 as  $d(v_i) \geq 2$ . By Courrant-Fischer interlacing theorem [9, Theorem 4.3.15],  $L'_i$  has at least one eigenvalue less than 1 and at least one eigenvalue greater than 2. As  $\bigoplus_{i=1}^{\eta(G)} L'_i$

is a principal submatrix of  $L(DG)$ ,  $L(G)$  has at least  $\eta(G)$  eigenvalues less than 1 and at least  $\eta(G)$  eigenvalues greater than 2 (including multiplicity). ■

### 3. Main Results

Let  $G$  be a connected graph with the property

$$(3.1) \quad \lambda_4(G) \leq 2.$$

The property (3.1) is hereditary, because as a direct consequence of Lemma 2.1, for any (not necessarily induced) subgraph  $U(\subseteq G)$  also satisfies (3.1). The inheritance(hereditary) of property (3.1) implies that there exist minimal connected graphs that do not obey (3.1); such graphs are called *forbidden subgraphs* for  $\lambda_4(G) \leq 2$ . It is easy to verify that the graphs  $H_1(1, 2, 2)$ ,  $H_1(1, 1, 3)$ ,  $H_2(3, 3)$ ,  $H_2(4, 2)$ ,  $H_3(2, 3)$ ,  $H_3(3, 2)$  and  $H_4$  listed in Figure 3.1 are forbidden subgraphs for  $\lambda_4(G) \leq 2$ , where  $K_p^c$  (the complement graph of a complete graph on  $p$  vertices) is a graph consisting of  $p$  isolated vertices.

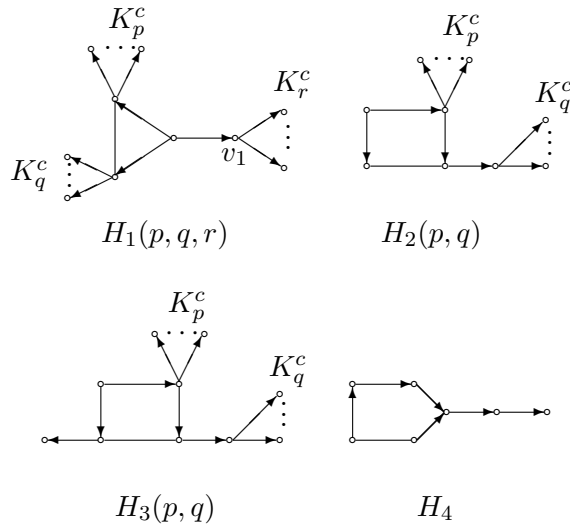


Figure 3.1. Forbidden Graphs

The nonsingular cycle of length  $k$  will be denoted by  $C_k$ , the set of neighbors of  $v$  in  $G$  will be denoted by  $N(v)$ , the cardinality of the set  $S$  will be denoted

by  $|S|$ . If  $G$  is a connected nonsingular unicyclic mixed graph, by Lemma 1.1 and Theorem 1.2, the spectrum of  $G$  is exactly that of a graph  $G'$ , which has the same underlying graph with  $G$  and which contains all oriented except an (arbitrary) unoriented edge on the cycle. So in the following, to convenience our discussion, we always consider the graph  $G(|G| \geq 9)$  with all oriented except an (arbitrary) unoriented edge on the cycle.

Let  $G_1 = G_1(p, q, r, s, t)$ ;  $G_2 = G_2(p)$ ;  $G_3 = G_3(p, q)$ ;  $G_4 = G_4(p, q)$ ;  $G_5 = G_5(p, q, r)$ ;  $G_6 = G_6(p, q, r)$ ;  $G_7 = G_7(p, q, r, s)$  listed in Figure 3.2 be unicyclic mixed graphs on at least nine vertices, where  $p \geq 0, q \geq 0, r \geq 0, s \geq 0$  and  $t \geq 0$ . They will play an important role in our discussion.

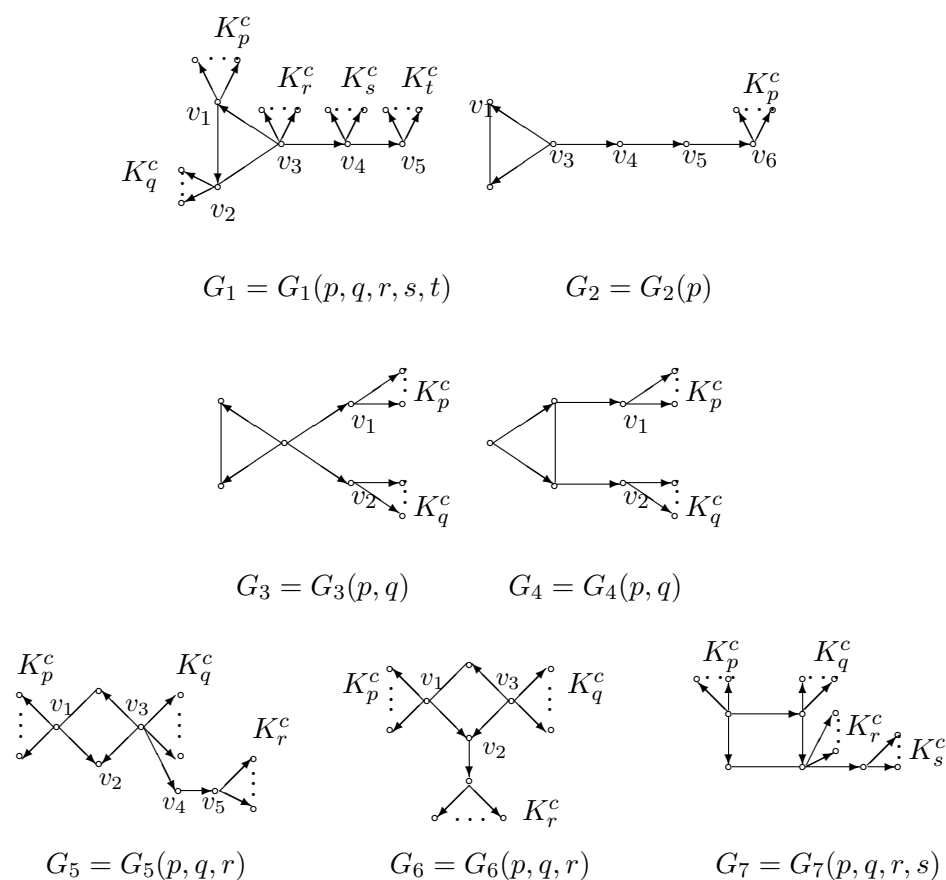


Figure 3.2

**Lemma 3.1.** *Let  $G$  be a connected nonsingular unicyclic mixed graph on at least 9 vertices. If  $m_G(2, +\infty) \leq 3$ , then  $G$  contains cycle of length less than 7, and  $G$  is one of the following types:*

- (1) *types  $U_1$  of Figure 3.3,  $G_1(p, q, 0, 0, 0)$ ,  $G_1(0, 0, 0, s, t)$ ,  $G_1(p, 0, r, s, 0)$ ,  $G_1(p, 0, r, 0, t)$ ,  $G_2$ ,  $G_3$  and  $G_4$  of Figure 3.2, if  $G$  contains cycle  $C_3$ ;*
- (2) *types  $U_2$  of Figure 3.3,  $G_5$ ,  $G_6(p, q, 1)$ ,  $G_6(1, 1, r)$ ,  $G_7(p, 0, r, s)$ ,  $G_7(0, q, 0, s)$  ( $q \leq 2$  and  $s \geq 0$ , or  $q \geq 0$  and  $s \leq 1$ ), or graphs  $G_6(2, 1, 2)$  and  $G_7(0, 3, 0, 2)$  of Figure 3.2, if  $G$  contains cycle  $C_4$ ;*
- (3) *types  $U_3$  (or  $U_4$ ) of Figure 3.3, if  $G$  contains cycle  $C_5$  (or  $C_6$ ).*

**Proof.** By Theorem 2.2 and Corollary 2.3, we have

$$(3.2) \quad \mu(G) \leq 3,$$

and  $G$  contains exactly one cycle  $C_i$  for some  $i$  ( $3 \leq i \leq 6$ ).

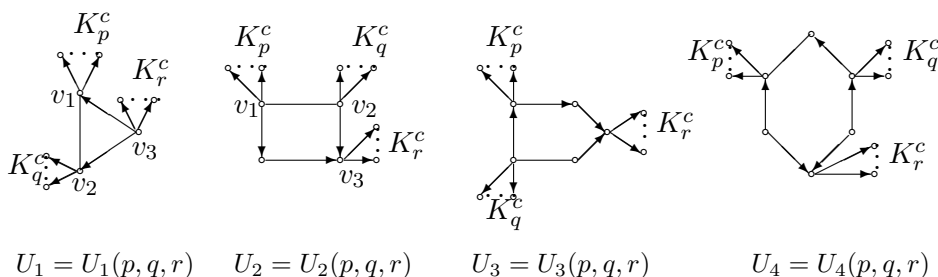


Figure 3.3.  $p \geq 0, q \geq 0, r \geq 0$

We discuss the problem in the following cases.

*Case 1.*  $G$  contains cycle  $C_3$ . Let  $U_1 = U_1(p, q, r)$  (assuming that  $p \geq q \geq r$ ) be the subgraph of  $G$  induced by the vertices of  $C_3$  together with those vertices incident to this cycle (see Figure 3.3).

(1.1)  $r \geq 2$ . Then  $G = U_1$ .

(1.2)  $r = 1$ . If  $G \neq U_1$ , by (3.2)  $G$  contains a subgraph isomorphic to  $H_1(1, 1, 1)$ , and each pendant vertex of  $H_1(1, 1, 1)$  is also the pendant vertex of  $G$ . Without loss of generality, let  $G_1(1, 1, 0, 0, 0)$  be the subgraph of  $G$ . By the fact that  $G$  contains at least 9 vertices and  $H_1(1, 2, 2)$ ,  $H_1(1, 1, 3)$  of Figure 3.1 are forbidden subgraphs,  $G$  has the structure of type  $G_1(p, q, 0, 0, 0)$  of Figure 3.2.

(1.3)  $r = 0, q \geq 1$ . If  $p = 1$  (necessarily  $q = 1$ ), then  $G$  has the structure of type  $G_4(p, q)$  of Figure 3.2. If  $p \geq 2$ , by (3.2) there exists at most one pendant vertex adjacent to  $v_1$  or  $v_2$  in  $U_1$ , denote by  $v_4$ , which joins vertices of  $G$  except those of  $U_1$ . If  $v_4$  joins exactly one vertex of  $V(G) \setminus V(U_1)$ , then  $G$  has the structure of type  $G_1(p, 0, r, 0, t)$ ; otherwise  $G$  has the structure of type  $G_1(p, 0, r, s, 0)$  of Figure 3.2.

(1.4)  $r = 0, q = 0, p \geq 3$ . Then there exists at most one pendant vertex adjacent to  $v_1$  in  $U_1$ , also denoted by  $v_4$ , which joins vertices of  $G$  except those of  $U_1$ . If  $|N(v_4) \setminus \{v_1\}| \geq 2$ , then  $G$  has structure of type  $G_1(0, 0, r, s, 0)$ ; and if  $|N(v_4) \setminus \{v_1\}| = 1$ , then  $G$  must be of the type  $G_1(0, 0, r, 0, t)$ .

(1.5)  $r = 0, q = 0, p = 2$ . If there exists at most one pendant vertex incident to  $v_1$  in  $U_1$ , which joins vertices of  $G$  except those of  $U_1$ , then the discussion is similar to the case (4). Otherwise, the two pendant vertices incident to  $v_1$  in  $U_1$  have their own neighbors in  $G$  except  $v_1$ . Hence the structure of  $G$  must be of type  $G_3(p, q)$ .

(1.6)  $r = 0, q = 0, p = 1$ . Then the longest path  $P$  of the subgraph  $G - U_1$  has length not greater than 2 by (3.2). If the length of  $P$  is 2, then  $G$  has the structure of type  $G_2(p)$ ; and if the length of  $P$  is at most 1, then the structure of  $G$  must be of type  $G_1(0, 0, 0, s, t)$ .

*Case 2.*  $G$  contains cycle  $C_4$ . Let  $U_2$  be the subgraph of  $G$  induced by the vertices of  $C_4$  together with all vertices incident to the cycle, see Figure 3.3 (assuming that  $p \geq r$ ). Note that by (3.2) there exists at most one pendant vertex of  $U_2$  adjacent to vertices of  $V(G) \setminus V(U_2)$ . We discuss the problem in following subcases.

(2.1) Each of  $p, q, r$  is nonzero. If  $q \geq 2$ , then  $G = U_2$  by (3.2). If  $q = 1$  and  $p \geq 2$ , then, in the graph  $U_2$ , only the pendant vertex adjacent to  $v_2$  has neighbors in  $G - U_2$ . As the graphs  $H_2(3, 3)$ ,  $H_2(4, 2)$  and  $H_3(2, 3)$ ,  $H_3(3, 2)$  of Figure 3.1 are forbidden,  $G$  is of type  $G_6(2, 1, r)$  with  $r \leq 2$ , or of type  $G_6(p, q, 1)$ . If  $q = 1$  and  $p = 1$  (necessarily  $r = 1$ ), then, in the graph  $U_2$ , only the pendant vertex adjacent to  $v_2$  or  $v_4$  has neighbors in  $G - U_2$ , and  $G$  is of type  $G_6(1, 1, r)$ .

(2.2) Exactly two of  $p, q, r$  are nonzero. Then  $U_2$  is of type  $U_2(0, q, r)$  or  $U_2(p, 0, r)$ . For  $U_2$  being the former type, as  $H_2(3, 3)$  and  $H_2(4, 2)$  are forbidden,  $G$  is of type  $G_7(0, q, 0, s)$  ( $q \leq 2$  and  $s \geq 0$ , or  $q \geq 0$  and  $s \leq 1$ ), or graph  $G_7(0, 3, 0, 2)$ ; and for  $U_2$  being the latter type,  $G$  is of type  $G_7(p, 0, r, s)$  or  $G_5(p, q, r)$ .

(2.3) Exactly one of  $p, q, r$  is nonzero. Without loss of generality, let  $U_2$  be the type  $U_2(0, 0, r)$ . Then  $G$  is of type  $G_7(0, 0, r, s)$  or  $G_5(0, q, r)$ .



Case 3.  $G$  contains cycle  $C_5$  or  $C_6$ . By (3.2) and the forbidden graph  $H_4$  of Figure 3.1, the structure of  $G$  must be of type  $U_3$  or  $U_4$  of Figure 3.3. ■

Let  $G = (V, E)$  be a mixed graph with  $V = \{v_1, v_2, \dots, v_n\}$ , and let  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  be a real vector. It will be convenient to adopt the following terminology from [6]:  $x$  is said to give a valuation of the vertices of  $V$ , that is, for each vertex  $v_i$  of  $V$ , we associate the value  $x_i$ , i.e.,  $x(v_i) = x_i$ . Then  $\lambda$  is an eigenvalue of  $G$  with the corresponding eigenvector  $x = (x_1, x_2, \dots, x_n)$  if and only if  $x \neq 0$  and

$$(3.3) \quad [\lambda - d(v_i)]x(v_i) = \sum_{e=\{v_i, v_j\} \in E} (\text{sgn } e)x(v_j), \text{ for } i = 1, 2, \dots, n.$$

**Proposition 3.2.** *Suppose  $G$  is a connected nonsingular unicyclic mixed graph on at least 9 vertices. If  $G$  is one of types  $U_1, G_1(p, q, 0, 0, 0), G_1(0, 0, 0, s, t), G_1(p, 0, r, s, 0), G_1(p, 0, r, 0, t), G_2, G_3$  and  $G_4$  as in Figure 3.2 or 3.3, where  $p \geq 0, q \geq 0, r \geq 0, s \geq 0$  and  $t \geq 0$ , then  $m_G(2, +\infty) \leq 3$ .*

**Proof.** For the graph  $U_1$ , by Lemma 2.1, it suffices to prove the graph  $U_1(m, m, m)$  (denoted still by  $U_1$ ) holds  $m_{U_1}(2, +\infty) \leq 3$ , where  $m = \max(p, q, r) \geq 1$ , since  $U_1(p, q, r) \subseteq U_1(m, m, m)$ . By (3.3) and a direct calculation, we have that all eigenvalues of the graph  $U_1$  distinct from 1 are determined by the equation

$$\Psi_{U_1}(\lambda) = (\lambda^2 - m\lambda - 5\lambda + 4)(\lambda^2 - m\lambda - 2\lambda - 1)^2.$$

Then, by Lemma 2.4,  $m_{U_1}(0, 1) \geq 3$  as  $\eta(U_1) = 3$ . Hence,  $m_{U_1}(2, +\infty) \leq 3$ .

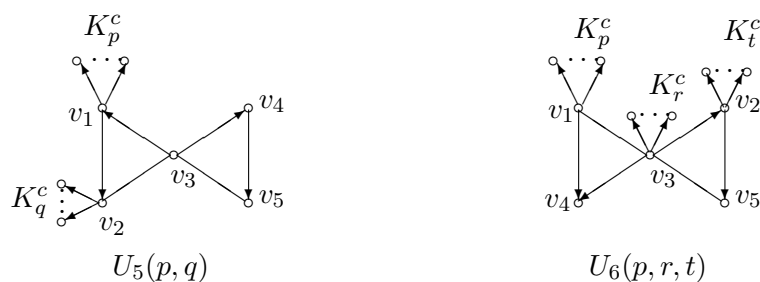


Figure 3.4

For the graphs of types  $G_1(p, q, 0, 0, 0), G_1(0, 0, 0, s, t)$  and  $G_3$ , by Lemma 2.1, it suffices to discuss the graph of type  $U_5(p, q)$  ( $p \geq 0, q \geq 0$  and

$p + q \geq 4$ ) of Figure 3.4, since  $G_1(p, q, 0, 0, 0) = U_5(p, q) - (v_3, v_5)$ ,  $G_1(0, 0, 0, s, t) = U_5(p, q) - (v_2, v_3)$  and the spectrum of  $G_3$  is same to that of the graph  $U_5(p, q) - (v_1, v_2)$ . For  $U_5(p, q)$ , let  $m = \max(p, q) \geq 2$ . Similarly, the eigenvalues of  $U_5(m, m)$  (denoted by  $U_5$ ) distinct from 1 are determined by the equation  $\Psi_{U_5}(\lambda) = f_1(\lambda)g_1(\lambda) = 0$ , where

$$f_1(\lambda) = \lambda^2 - m\lambda - 2\lambda + 1,$$

$$g_1(\lambda) = \lambda^4 - (m + 11)\lambda^3 + (7m + 39)\lambda^2 - (10m + 53)\lambda + 24.$$

Then  $m_{U_5}(1, 2) \geq 1$ , since  $g_1(1) = -4m < 0$ ,  $g_1(2) = 2 > 0$ , and  $m_{U_5}(0, 1) \geq 1$  as  $\eta(U_5) = 2$ , by Lemma 2.4. Hence  $m_{U_5}(2, +\infty) \leq 3$ .

For the graphs of types  $G_1(p, 0, r, s, 0)$  and  $G_1(p, 0, r, 0, t)$ , similar to above, it suffices to discuss the graph  $U_6(p, r, t)$  ( $p \geq 0, r \geq 0, t \geq 0$  and  $p+r+t \geq 4$ ) of Figure 3.4, since  $G_1(p, 0, r, s, 0) = U_6(p, r, t) - (v_3, v_5)$  and the spectrum of  $G_1(p, 0, r, 0, t)$  is same to that of the graph  $U_6(p, r, t) - (v_2, v_3)$ . For  $U_6(p, r, t)$ , let  $m = \max(p, r, t) \geq 2$ . Then the eigenvalues of  $U_6(m, m, m)$  (denoted by  $U_6$ ) distinct from 1 are determined by the equation  $\Psi_{U_6}(\lambda) = f_2(\lambda)g_2(\lambda) = 0$ , where

$$f_2(\lambda) = \lambda^3 - (m + 5)\lambda^2 + (2m + 7)\lambda - 3,$$

and

$$g_2(\lambda) = \lambda^5 - (2m + 10)\lambda^4 + (m^2 + 12m + 32)\lambda^3 \\ - (2m^2 - 19m + 46)\lambda^2 + (9m + 31)\lambda - 8.$$

Then, by a discussion similar to that of  $U_5$ ,  $m_{U_6}(2, +\infty) \leq 3$ .

For the graph of type  $G_2$ , by a direct calculation, the eigenvalues of the graph  $G_2$  distinct from 1 are determined by the equation  $\Psi_{G_2}(\lambda) = (\lambda - 1)g_3(\lambda) = 0$ , where

$$g_3(\lambda) = \lambda^6 - (p + 12)\lambda^5 + (10p + 53)\lambda^4 - (33p + 108)\lambda^3 \\ + (41p + 104)\lambda^2 - (15p + 42)\lambda + 4.$$

Then  $2 \leq m_{G_2}(2, +\infty) \leq 4$ , since  $g_3(1) = 2p > 0$ ,  $g_3(2) = -2p < 0$  and  $\eta(G_2) = 1$ . If  $m_{G_2}(2, +\infty) = 4$ , then yield a contradiction to  $\Psi_{G_2}(2) = (2 - 1) \times g_3(2) = -2p < 0$ . Hence  $m_{G_2}(2, +\infty) \leq 3$ .

For the graph of type  $G_4(p, q)$ , by a similar method to that of  $U_1$ , it suffices to consider the graph  $G_4(m, m)$ , where  $m = \max(p, q) \geq 2$ . By a direct

calculation, the eigenvalues of the graph  $G_4(m, m)$  distinct from 1 are determined by the equation

$$\begin{aligned} \Psi_{G_4}(\lambda) &= \lambda^7 - (2m + 12)\lambda^6 + (m^2 + 22m + 54)\lambda^5 - (10m^2 + 88p + 120)\lambda^4 \\ &\quad - (34m^2 - 160m - 144)\lambda^3 - (48m^2 + 164m + 94)\lambda^2 \\ &\quad - (24m^2 - 78m - 31)\lambda - 14m - 4. \end{aligned}$$

Then  $m_{G_4}(1, 2) \geq 1$  as  $\Psi_{G_4}(1) = m^2 > 0$  and  $\Psi_{G_4}(2) = 2 - 2m < 0$ . On the other hand, by Theorem 2.2(i) and Theorem 2.4, we have  $m_{G_4}(2, \infty) \geq 3$  and  $m_{G_4}[0, 1) \geq 2$ . Consequently,  $m_{G_4}(2, +\infty) \leq 3$ , otherwise, it will yield a contradiction to  $\Psi_{G_4}(2) < 0$ . ■

**Proposition 3.3.** *Suppose  $G$  is a connected nonsingular unicyclic mixed graph on at least 9 vertices. If  $G$  is one of types  $U_2$ ,  $G_5$ ,  $G_6(p, q, 1)$ ,  $G_6(p, q, 1)$ ,  $G_6(1, 1, r)$ ,  $G_7(p, 0, r, s)$ ,  $G_7(0, q, 0, s)$  ( $q \leq 2$  and  $s \geq 0$ , or  $q \geq 0$  and  $s \leq 1$ ),  $G_6(2, 1, 2)$  and  $G_7(0, 3, 0, 2)$  listed in Figure 3.2 or 3.3, where  $p \geq 0, q \geq 0, r \geq 0, s \geq 0$ , then  $m_G(2, +\infty) \leq 3$ .*

**Proof.** The result can be verified directly if  $G$  is  $G_6(2, 1, 2)$  or  $G_7(0, 3, 0, 2)$ . For the graph of type  $U_2$ , by (3.3), the eigenvalues of the graph  $U_2(m, m, m)$  (still denoted by  $U_2$ ) distinct from 1 are determined by the equation  $\Psi_{U_2}(\lambda) = f_4(\lambda)g_4(\lambda) = 0$ , where

$$\begin{aligned} f_4(\lambda) &= \lambda^3 - (m + 5)\lambda^2 + (2m + 6)\lambda - 2, \\ g_4(\lambda) &= \lambda^4 - (2m + 6)\lambda^3 + (m^2 + 6m + 11)\lambda^2 - (4m + 8)\lambda + 2, \end{aligned}$$

and  $m = \max(p, q, r) \geq 1$ . Then  $m_{U_2}(2, +\infty) \leq 3$ , since  $f_4(1) = p > 0$ ,  $f_4(2) = -2 < 0$ ,  $g_4(1) = m^2 > 0$ ,  $g_4(2) = 4m^2 - 2 > 0$  and  $\eta(U_2) = 3$ .

For the graph of type  $G_5$ , by Lemma 2.1, it suffices to discuss the graph  $G_5 + e$ , where  $e = (v_3, v)$ ,  $v \in K_r^c$ , is unoriented. Let  $m = \max(p, q, r) \geq 1$ , by (3.3), the eigenvalues of the graph  $G_5(m, m, m + 1) + e$  (still denoted by  $G_5$ ) distinct from 1 are determined by the equation  $\Psi_{G_5}(\lambda) = (\lambda - 2)f_5(\lambda)g_5^2(\lambda) = 0$ , where

$$f_5(\lambda) = \lambda^3 - (m + 7)\lambda^2 + (2m + 10)\lambda - 4, \quad g_5(\lambda) = \lambda^3 - (m + 5)\lambda^2 + (2m + 6)\lambda - 2.$$

Then  $m_{G_5}(1, 2) \geq 3$ , since  $f_5(1) = m > 0$ ,  $f_5(2) = -4 < 0$ ,  $g_5(1) = m > 0$ ,  $g_5(2) = -2 < 0$ . And  $m_{G_5}(0, 1) \geq 3$  as  $\eta(G_5) = 3$  by Lemma 2.4. Hence,  $m_{G_5}(2, +\infty) \leq 3$ .

For the graph of type  $G_6(p, q, 1)$ , it suffices to discuss the graph  $G_6(m, m, 1)$  with  $m = \max(p, q) \geq 1$ , denoted by  $G_{61}$ . By (3.3), the eigenvalues of the graph  $G_{61}$  distinct from 1 are determined by the equation  $\Psi_{G_{61}}(\lambda) = f_{61}(\lambda)g_{61}(\lambda) = 0$ , where

$$\begin{aligned} f_{61}(\lambda) &= \lambda^3 - (m+5)\lambda^2 + (2m+6)\lambda - 2, g_{61}(\lambda) \\ &= \lambda^5 - (m+9)\lambda^4 + (6m+27)\lambda^3 - (9m+33)\lambda^2 + (2m+16)\lambda - 2. \end{aligned}$$

Then  $m_{G_{61}}(1, 2) \geq 2$ , since  $f_{61}(1) = m > 0$ ,  $f_{61}(2) = -2 < 0$ ,  $g_{61}(1) = -2m < 0$ ,  $g_{61}(2) = 2 > 0$ . And, by Lemma 2.3,  $m_{G_{61}}(0, 1) \geq 3$  as  $\eta(G_{61}) = 3$ . Hence,  $m_{G_{61}}(2, +\infty) \leq 3$ .

For the graph of type  $G_6(1, 1, r)$ , denote by  $G_{62}$ . By (3.3), the eigenvalues of the graph  $G_{62}$  distinct from 1 are determined by the equation  $\Psi_{G_{62}}(\lambda) = f(\lambda)g(\lambda) = 0$ , where

$$\begin{aligned} f_{62}(\lambda) &= \lambda^3 - 6\lambda^2 + 8\lambda - 2, \\ g_{62}(\lambda) &= \lambda^5 - (r+9)\lambda^4 + (7r+26)\lambda^3 - (12r+30)\lambda^2 + (4r+14)\lambda - 2. \end{aligned}$$

By a discussion similar to the graph  $G_{61}$ , we have  $m_{G_{62}}(2, +\infty) \leq 3$ .

For the graph of type  $G_7(p, 0, r, s)$ , similarly, the eigenvalues of the graph  $G_7(m, 0, m, m)$  with  $m = \max(p, r, s) \geq 1$  distinct from 1 are determined by the equation  $\Psi_{G_7}(\lambda) = f_7(\lambda)g_7(\lambda) = 0$ , where

$$\begin{aligned} f_7(\lambda) &= \lambda^3 - (m+5)\lambda^2 + (2m+6)\lambda - 2, \\ g_7(\lambda) &= \lambda^5 - (2m+8)\lambda^4 + (m^2+10m+21)\lambda^3 \\ &\quad - (2m^2+14m+24)\lambda^2 + (6m+12)\lambda + 4p - 2. \end{aligned}$$

By a discussion similar to that of the graph  $G_{61}$ , we also have  $m_{G_7}(2, +\infty) \leq 3$ .

For the graph of type  $G_7(0, q, 0, s)$  ( $q \leq 2$  and  $s \geq 0$ , or  $q \geq 0$  and  $s \leq 1$ ), it suffices to discuss the graphs  $G_7(0, 2, 0, s)$  ( $s \geq 2$ ) and  $G_7(0, q, 0, 1)$  ( $q \geq 3$ ), denoted respectively by  $G_{71}$  and  $G_{72}$ . The eigenvalues of  $G_{71}$  distinct from 1 are determined by the equation

$$\begin{aligned} \Psi_{71}(\lambda) &= \lambda^7 - (s+14)\lambda^6 + (73+12s)\lambda^5 - (49s+180)\lambda^4 \\ &\quad + (80s+224)\lambda^3 - (48s+140)\lambda^2 + (8s+40)\lambda - 4 = 0. \end{aligned}$$

Observe that  $\Psi_{71}(1) = 2s > 0$  and  $\Psi_{71}(2) = -4 < 0$  so that  $m_{G_{71}}(1, 2) \geq 1$ . By Lemma 2.4,  $m_{G_{71}}(0, 1) \geq 2$  as  $\eta(G_{71}) = 2$ . Hence,  $\Psi_{71}(\lambda) = 0$  has at most 4 roots greater than two. If it has exactly 4 roots greater than two, then  $\Psi_{71}(2) > 0$  which yields a contradiction to  $\Psi_{71}(2) = -4$ . So we have  $m_{G_{71}}(2, +\infty) \leq 3$ . The eigenvalues of  $G_{72}$  distinct from 1 are determined by the equation

$$\begin{aligned} \Psi_{72}(\lambda) = & \lambda^7 - (q + 13)\lambda^6 + (65 + 10q)\lambda^5 - (35q + 159)\lambda^4 \\ & + (51q + 202)\lambda^3 - (28q + 132)\lambda^2 + (4q + 40)\lambda - 4 = 0. \end{aligned}$$

By discussion in a similar way we also have  $m_{G_{72}}(2, +\infty) \leq 3$ . ■

**Proposition 3.4.** *If  $G$  is of type  $U_3(p, q, r)$  or  $U_4(p, q, r)$  listed in Figure 3.3, where  $p \geq 0, q \geq 0, r \geq 0$ , then  $m_G(2, +\infty) \leq 3$ .*

**Proof.** Obviously, it suffices to discuss the graphs  $U_3(m, m, m)$  and  $U_4(m, m, m)$ , where  $m = \max(p, q, r)$ , still denoted respectively by  $U_3, U_4$ . By (3.3), the eigenvalues of the graphs  $U_3$  and  $U_4$  distinct from 1 are respectively determined by the equations  $\Psi_{U_3}(\lambda) = 0, \Psi_{U_4}(\lambda) = 0$ , where

$$\begin{aligned} \Psi_{U_3}(\lambda) = & (\lambda^3 - m\lambda^2 - 4\lambda^2 + 2m\lambda + 4\lambda - 1) \\ & \times (\lambda^5 - 2m\lambda^4 - 9\lambda^4 + m^2\lambda^3 + 11m\lambda^3 + 28\lambda^3 \\ & - 2m^2\lambda^2 - 16m\lambda^2 - 37\lambda^2 + 7m\lambda + 21\lambda - 4), \\ \Psi_{U_4}(\lambda) = & (\lambda - 2)(\lambda^2 - m\lambda - 3\lambda + 2)(\lambda^3 - m\lambda^2 - 5\lambda^2 + 2m\lambda + 5\lambda - 1)^2. \end{aligned}$$

By a discussion similar to the Proposition 3.2 and 3.3, the result follows. ■

By Proposition 3.1, 3.2, 3.3 and 3.4, we get the main result of this paper directly.

**Theorem 3.5.** *Let  $G = (V, E)$  be a connected nonsingular unicyclic mixed graph on at least 9 vertices. Then  $m_G(2, +\infty) = 3$  if and only if there exists a signature matrix  $D$  such that  ${}^D G$  is one of the following types:*

- (1) types  $U_1, G_1(p, q, 0, 0, 0), G_1(0, 0, 0, s, t), G_1(p, 0, r, s, 0), G_1(p, 0, r, 0, t), G_2, G_3$  and  $G_4$  of Figure 3.2 and 3.3 if  $G$  contains the cycle  $C_3$ ;
- (2) types  $U_2, G_5, G_6(p, q, 1), G_6(1, 1, r), G_7(p, 0, r, s), G_7(0, q, 0, s)$  ( $q \leq 2$  and  $s \geq 0$ , or  $q \geq 0$  and  $s \leq 1$ ), and  $G_6(2, 1, 2), G_7(0, 3, 0, 2)$  of Figure 3.2 or 3.3 if  $G$  contains the cycle  $C_4$ ;

- (3) type  $U_3$  (and type  $U_4$ , respectively) of Figure 3.3 if  $G$  contains the cycle  $C_5$  (and the cycle  $C_6$ , respectively), where  $p \geq 0$ ,  $q \geq 0$ ,  $r \geq 0$ ,  $s \geq 0$ ,  $t \geq 0$ .

## References

- [1] R.B. Bapat, J.W. Grossman and D.M. Kulkarni, *Generalized matrix tree theorem for mixed graphs*, Linear and Multilinear Algebra **46** (1999) 299–312.
- [2] R.B. Bapat, J.W. Grossman and D.M. Kulkarni, *Edge version of the matrix tree theorem for trees*, Linear and Multilinear Algebra **47** (2000) 217–229.
- [3] Y.-Z. Fan, *Largest eigenvalue of a unicyclic mixed graph*, Applied Mathematics A Journal of Chinese Universities (English Series) **19** (2004) 140–148.
- [4] Y.-Z. Fan, *On the least eigenvalue of a unicyclic mixed graph*, Linear and Multilinear Algebra, accepted for publication.
- [5] Y.-Z. Fan, *On spectral integral variations of mixed graphs*, Linear Algebra Appl. **347** (2003) 307–316.
- [6] M. Fiedler, *A property of eigenvectors of nonnegative symmetric matrices and its applications to graph theory*, Czechoslovak Math. J. **25** (1975) 619–633.
- [7] R. Grone, R. Merris and V.S. Sunder, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl. **11** (1990) 218–238.
- [8] J.-M. Guo and S.-W. Tan, *A relation between the matching number and the Laplacian spectrum of a graph*, Linear Algebra Appl. **325** (2001) 71–74.
- [9] R.A. Horn and C.R. Johnson, *Matrix analysis* (Cambridge University Press, 1985).
- [10] X.-D. Zhang and J.-S. Li, *The Laplacian spectrum of a mixed graph*, Linear Algebra Appl. **353** (2002) 11–20.
- [11] X.-D. Zhang and R. Luo, *The Laplacian eigenvalues of a mixed graph*, Linear Algebra Appl. **353** (2003) 109–119.

Received 23 September 2005

Revised 29 November 2006