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ON UNIQUELY PARTITIONABLE RELATIONAL STRUCTURES AND OBJECT SYSTEMS

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Abstract

We introduce *object systems* as a common generalization of graphs, hypergraphs, digraphs and relational structures. Let \mathbf{C} be a concrete category, a *simple object system* over \mathbf{C} is an ordered pair $S = (V, E)$,

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where $E = \{A_1, A_2, \dots, A_m\}$ is a finite set of the objects of \mathbf{C} , such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements and $V \supseteq \bigcup_{i=1}^m V(A_i)$. To generalize the results on graph colourings to simple object systems we define, analogously as for graphs, that an additive induced-hereditary property of simple object systems over a category \mathbf{C} is any class of systems closed under isomorphism, induced-subsystems and disjoint union of systems, respectively. We present a survey of recent results and conditions for object systems to be uniquely partitionable into subsystems of given properties.

Keywords: graph, digraph, hypergraph, vertex colouring, uniquely partitionable system.

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1. INTRODUCTION

A graph G is said to be uniquely n -colorable if any two proper n -colorings of G induce the same partition of the vertex set $V(G)$. The concept of uniquely n -colorable graphs was introduced in [12]. Different generalizations of this concept may be found e.g. in [1, 2, 5, 16, 20, 22]. Let us recall the basic notions which are needed to summarize the basic results. Let \mathcal{I} denotes the class of all simple graphs. A graph property \mathcal{P} is any nonempty proper isomorphism closed subclass of \mathcal{I} . A graph property is said to be induced-hereditary if it is closed under taking induced subgraphs and additive if it is closed under disjoint union of graphs (see [4]). Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be graph properties, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition of a graph G is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that each partition class V_i induces a subgraph $G[V_i]$ of property $\mathcal{P}_i, i = 1, 2, \dots, n$. A graph G is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable if G has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graphs and by $U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n)$ the set of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graphs. In the case $\mathcal{P}_1 = \mathcal{P}_2 = \dots = \mathcal{P}_n = \mathcal{P}$, we write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n = \mathcal{P}^n$, we say that $G \in \mathcal{P}^n$ is (\mathcal{P}, n) -partitionable and the graphs belonging to $U(\mathcal{P}^n)$ are called (see e.g. [2, 16]) uniquely (\mathcal{P}, n) -partitionable. A property \mathcal{P} is said to be *reducible* if there exist properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise \mathcal{P} is called *irreducible*. For example, the property \mathcal{O} — "to be an edgeless graph" related to regular colouring is irreducible and the smallest additive induced-hereditary reducible property is the class \mathcal{O}^2 of all bipartite graphs. The notion of reducible property have been introduced in [16], where

it was proved that for any reducible property \mathcal{P} there are no uniquely (\mathcal{P}, n) -partitionable graphs. D. Achlioptas, J.I. Brown, D.G. Corneil, M.S.O. Molloy in [1] proved the existence of uniquely $(n, -G)$ -partitionable graphs for $n \geq 2$, where $-G$ denotes the property "do not contain G as an induced subgraph". Using the results of [18], the problem of the existence of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graphs for additive induced-hereditary graph properties was solved in [6] as follows:

Theorem 1. *Let \mathcal{P} be an additive induced-hereditary graph property. Then for $n \geq 2$, $U(\mathcal{P}^n) \neq \emptyset$ if and only if \mathcal{P} is irreducible.*

Let us recall, that the requirement for additivity is substantial. A counterexample for nonadditive induced-hereditary properties is described in [17, 19]. The proof of the Theorem 1 is based on the Unique Factorization Theorem for additive induced-hereditary graph property (see [7, 8, 9, 14, 18, 19]). Some results on graph properties can be generalized and proved analogously for digraphs, hypergraphs, posets and other structures (see e.g. [13]). The extension of the Unique Factorization Theorem and the existence of uniquely partitionable digraphs have been presented in [23].

The main aim of this paper is to introduce a general notion of *object systems* and to show, how the presented results on uniquely partitionable graphs can be generalized for object systems over a given concrete category. We present a survey of results on uniquely partitionable systems of objects and describe the main ideas, which help to prove such general statements. In the first paper on uniquely colorable graphs [12] it has been proved that every k -colorable graph is an induced subgraph of a uniquely k -colorable graph. For additive induced-hereditary graph properties the generalization of this fact was presented in [6]. We will present, that the same is true for object systems.

2. SYSTEMS OF OBJECTS OF A CONCRETE CATEGORY

We use the basic elementary notions of category theory (see [21]). A concrete category \mathbf{C} is a collection of *objects* and *arrows* called *morphisms*. An object in a concrete category \mathbf{C} is a *set with structure*. We will denote the *ground-set* of the object A by $V(A)$. The morphism between two objects is a *structure preserving mapping*. Obviously, the morphisms of \mathbf{C} have to satisfy the axioms of the category theory. The natural examples of concrete categories are: **Set** of sets, **FinSet** of finite sets, **Graph** of graphs,

Poset of partially ordered sets with structure preserving mappings, called homomorphisms of corresponding structures.

In our investigations we will consider concrete categories \mathbf{C} with *isomorphisms* i.e., structure preserving bijections between the ground-sets of objects only. Moreover, we will assume that by renaming (relabeling) the elements of the object A , we obtain an object A^* in \mathbf{C} isomorphic to A in each considered concrete category \mathbf{C} . To avoid formal difficulties, we will assume in what follows, that the elements $V(A)$ of each objects A of \mathbf{C} are labelled by labels taken from a given infinite set \mathcal{V} of cardinality α . Hence, any considered concrete category \mathbf{C} will be small and the objects of \mathbf{C} are "labelled" objects. We also assume, that for each object A of \mathbf{C} , after relabelling of the elements of $V(A)$ by labels from a set $V^* \subset \mathcal{V}$ the obtained "new" object A^* with $V(A^*) = V^*$ belongs to \mathbf{C} too and it is always isomorphic to A . This requirements are natural, they are satisfied e.g. if the concrete category \mathbf{C} is any small full subcategory of the above mentioned categories **Set**, **FinSet**, **Graph**, **Poset**, etc. Let us call such categories *wide*.

For example, a simple finite graph is a finite system of two element sets, a simple finite hypergraph $H = (V, E)$ can be considered as a system of its hyperedges $E = \{e_1, e_2, \dots, e_m\}$, where edges are finite sets and the set of its vertices $V(H)$ is a superset of the union of hyperedges, i.e., $V \supseteq \bigcup_{i=1}^m e_i$. The following definition gives a natural generalization of graphs and hypergraphs.

Let \mathbf{C} be a wide concrete category. A *simple system of objects* of \mathbf{C} is an ordered pair $S = (V, E)$, where $E = \{A_1, A_2, \dots, A_m\}$ is a finite set of the objects of \mathbf{C} , such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements (i.e., there are no loops) and $V \supseteq \bigcup_{i=1}^m V(A_i)$. The class of all simple systems of objects of \mathbf{C} will be denoted by $\mathcal{I}(\mathbf{C})$. The symbols K_0 and K_1 denotes the null system $K_0 = (\emptyset, \emptyset)$ and system consisting of one isolated element, respectively.

For example, graphs can be viewed as systems of objects of a concrete category of two-element sets with bijections as arrows, digraphs are systems of objects of the category of ordered pairs, hypergraphs are finite set systems, etc. Let us remark, that the relational L -structures generalizing graphs, digraphs and k -uniform hypergraphs introduced by Fraïssé in [10] see e.g. [3, 11], are systems of objects on category of relations given by the signature L .

To generalize the results on generalized colourings of graphs to arbitrary simple systems of objects we need to define *isomorphism of systems*, *disjoint*

union of systems and induced-subsystems, respectively. We can do this in a natural way.

Let $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ be two simple systems of objects of a given concrete category \mathbf{C} . The systems S_1 and S_2 are said to be isomorphic if there is a pair of bijection:

$$\phi : V_1 \longleftrightarrow V_2, \quad \psi : E_1 \longleftrightarrow E_2,$$

such that if $\psi(A_{1i}) = A_{2j}$ then $\phi/V(A_{1i}) : V(A_{1i}) \longleftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ in the category \mathbf{C} . The homomorphism of the systems can be defined in a similar way.

The disjoint union of the systems S_1 and S_2 is the system $S_1 \cup S_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where we assume that $V_1 \cap V_2 = \emptyset$. A system is said to be connected if it cannot be expressed as a disjoint union of two systems.

The subsystem of S_1 induced by the set $U \subseteq V(S_1)$ is $S_1[U]$, with objects $E(S_1[U]) := \{A_{1i} \in E(S_1) | V(A_{1i}) \subseteq U\}$. S_2 is an induced-subsystem of S_1 if it is isomorphic to $S_1[U]$ for some $U \subseteq V(S_1)$.

Using these definitions we can define, analogously as for graphs, that an additive induced-hereditary property of simple systems of objects over a category \mathbf{C} is any class of systems closed under isomorphism, induced-subsystems and disjoint union of systems. Let us denote by $\mathbb{M}^a(\mathbf{C})$ the set of all additive induced-hereditary properties of simple systems of objects of a category \mathbf{C} . In the following we will call simple systems of objects of a category \mathbf{C} shortly *systems*.

In [15] it is proved, that the set $\mathbb{M}^a(\mathbf{C})$ of all additive induced-hereditary properties of simple object-systems over \mathbf{C} partially ordered by set inclusion forms a complete and distributive lattice and the structure of the lattice $\mathbb{M}^a(\mathbf{C})$ is investigated.

3. REDUCIBLE PROPERTIES AND UNIQUE FACTORIZATION THEOREM FOR SYSTEMS

Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be properties of systems, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition of a system S is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(S)$ such that each partition class V_i induces a subsystem $S[V_i]$ of property $\mathcal{P}_i, i = 1, 2, \dots, n$. A system S is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable if S has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable systems

and by $U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n)$ the set of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable systems. A property \mathcal{P} is said to be *reducible* in $\mathbb{M}^a(\mathbf{C})$ if there exist properties $\mathcal{P}_1, \mathcal{P}_2$ in $\mathbb{M}^a(\mathbf{C})$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise \mathcal{P} is called *irreducible*.

Following the same way as for graph properties the Unique Factorization Theorem for reducible properties of systems can be proved:

Theorem 2. *Any reducible additive induced-hereditary property of object systems is uniquely factorizable into irreducible factors (up to the order of factors).*

Let us recall the main steps of the proof, the details can be proved analogously as in [7, 8, 19, 23]. We follow the idea of the proof from [18].

We define the set $\mathcal{S} \subseteq \mathcal{I}(\mathbf{C})$ to be a *generating set* of $\mathcal{P} \in \mathbb{M}^a(\mathbf{C})$ if $S \in \mathcal{P}$ if and only if S is an induced subsystem of some system from \mathcal{S} . The fact that \mathcal{S} is a generating set of \mathcal{P} will be written as $[\mathcal{S}] = \mathcal{P}$. The members of \mathcal{S} are called *generators* of \mathcal{P} .

Let us define *the operation* $*$:

For given vertex disjoint systems S_1, S_2, \dots, S_n , $n \geq 2$, denote by

$$S_1 * S_2 * \dots * S_n = \{S \in \mathcal{I}(\mathbf{C}) : V(S) = \bigcup_{i=1}^n V(S_i) \text{ and } S[V(S_i)] = S_i\}.$$

The operation " $*$ " is motivated by the following observation. Let us suppose that $S \in \mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and let $V(S) = (V_1, V_2)$ be a $(\mathcal{P}, \mathcal{Q})$ -partition of S . Then by additivity of \mathcal{P} and \mathcal{Q} the class $k.S[V_1] * k.S[V_2] \subseteq \mathcal{R}$ for every positive integer k . Therefore we define, for a given system $S \in \mathcal{S}(\mathcal{R})$, the invariant $dec_{\mathcal{R}}(S)$ as follows:

$dec_{\mathcal{R}}(S) = \max\{n : \text{there exist a partition } (V_1, V_2, \dots, V_n), V_i \neq \emptyset, \text{ such that for each } k \geq 1, k.S[V_1] * k.S[V_2] * \dots * k.S[V_n] \subseteq \mathcal{R}\}$. If $S \notin \mathcal{R}$ we set $dec_{\mathcal{R}}(S)$ to be zero.

A system $S \in \mathcal{P}$ is called \mathcal{P} -*strict* if $S * K_1 \not\subseteq \mathcal{P}$. A system S is said to be \mathcal{R} -*decomposable* if $dec_{\mathcal{R}} \geq 2$ and \mathcal{R} -*indecomposable* otherwise. Thus if the property \mathcal{R} is reducible, every system $S \in \mathcal{R}$ with at least two vertices is \mathcal{R} -decomposable. For any additive reducible induced-hereditary property the converse assertion is also valid.

An induced-hereditary additive property \mathcal{R} is reducible if and only if all the \mathcal{R} -strict systems with at least two vertices are \mathcal{R} -decomposable.

In proofs of this fact the number $dec(\mathcal{R})$ by $dec(\mathcal{R}) = \min\{dec_{\mathcal{R}}(S) : S \in \mathcal{S}(\mathcal{R})\}$ is defined. The main part of the proof is to show that there

exists a generating set $\mathcal{S}^* \subseteq \mathcal{S}(\mathcal{R})$ of \mathcal{R} which consists of systems with decomposability number equal to $\text{dec}(\mathcal{R})$ and these generators are uniquely \mathcal{R} -decomposable. The final step of the proof of Theorem 2 presents the determination of the corresponding irreducible factors.

4. UNIQUELY PARTITIONABLE SYSTEMS

Following the proof of Theorem 2 one can see that in fact any reducible additive induced-hereditary property of systems \mathcal{R} is generated by uniquely partitionable systems. More precisely:

Let \mathcal{S}^* be a generating set of a reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$, $n \geq 2$ consisting of \mathcal{R} -strict systems of decomposability number equal to $\text{dec}(\mathcal{R}) = n$. Since the generators of \mathcal{S}^* are uniquely \mathcal{R} -decomposable into n indecomposable systems, they have exactly one $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition. This implies also that each object system $S \in \mathcal{R}$ is an induced subsystem of a uniquely partitionable system.

Theorem 3. *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$, $n \geq 2$ be a factorization of a reducible property $\mathcal{R} \in \mathbb{M}^a(\mathbf{C})$ into irreducible factors. Then $U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n) \neq \emptyset$ and moreover $[U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n)] = \mathcal{R}$.*

We can present the previous results in the form of criteria of irreducibility.

Theorem 4. *Let \mathcal{P} be an additive induced-hereditary property of systems. Then the following statements are equivalent:*

1. *the property \mathcal{P} is irreducible,*
2. *for every $n \geq 2$ the property \mathcal{P}^n can be generated by a set of uniquely (\mathcal{P}, n) -partitionable systems,*
3. *there exist uniquely (\mathcal{P}, n) -partitionable systems for every $n \geq 2$,*
4. *the property \mathcal{P} can be generated by a set of \mathcal{P} -indecomposable systems,*
5. *there exists a \mathcal{P} -indecomposable \mathcal{P} -strict system.*

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