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WIENER INDEX OF GENERALIZED STARS AND THEIR QUADRATIC LINE GRAPHS *

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Abstract

The Wiener index, W, is the sum of distances between all pairs of vertices in a graph G. The quadratic line graph is defined as $L(L(G))$, where $L(G)$ is the line graph of G. A generalized star S is a tree consisting of $\Delta \geq 3$ paths with the unique common endvertex. A relation between the Wiener index of S and of its quadratic graph is presented. It is shown that generalized stars having the property $W(S) = W(L(L(S))$ exist only for $4 \leq \Delta \leq 6$. Infinite families of generalized stars with this property are constructed.

Keywords and phrases: distance in a graph, Wiener index, star, iterated line graph.

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1. Introduction

In this paper we are concerned with finite undirected connected graphs without loops and multiple edges. The vertex and edge sets of G are $V(G)$ and $E(G)$, respectively, $n = |V(G)|$ and $q = |E(G)|$. The maximal vertex degree of a graph is denoted by Δ. If u and v are vertices of G, then the number of edges in the shortest path connecting them is said to be their distance and is denoted by $d(u, v)$. Terms not defined here can be found in [26].

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The sum $W(G)$ of distances between all pairs of vertices of the graph G is the Wiener index of G [33]:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v).$$

The same quantity is known also as the distance of a graph or graph transmission [17, 29]. This graph invariant belongs to the molecular structure-descriptors, called topological indices, that are successfully used for the design of molecules with desired properties, including pharmacologic and biological activity. Mathematical properties and chemical applications of the Wiener index have been intensively studied in the last thirty years (see books [6, 22, 32] and selected reviews [1, 5, 7, 9, 10, 17, 18, 23, 28, 29, 31]).

The line graph $L(G)$ of a graph G has vertices corresponding to the edges of G and two vertices are adjacent in $L(G)$ if their corresponding edges of G have a common endvertex. A graph $L(L(G)) = L^2(G)$ is called the *quadratic line graph* of G. The concept of line graph has found various applications in chemical research. Parameters of line graphs have been applied for the evaluation of structural complexity of molecular graphs and for design of novel topological indices [2, 3, 24, 25].

Buckley has shown that Wiener indices of an arbitrary n-vertex tree T and its line graph are always distinct [4]. Namely,

$$W(L(T)) = W(T) - \binom{n}{2}.$$

The following question naturally arises: does there exist a tree T with the property

$$W(L^2(T)) = W(T)\,? \tag{1}$$

A positive answer to this question has been reported in [8]. The number of such trees up to 26 vertices is presented in [13]. Several infinite families of trees with property (1) has been constructed in [13, 14, 15]. By construction, these trees have only one long growing path and two or four branching vertices. The following problem was posed in [13]: construct an infinite family of trees satisfying equality (1) such that they have several paths growing from its centers. A vertex v is said to be a *branching vertex* if $\deg(v) \geq 3$. It is interesting to determine the minimal number of branching vertices in trees having property (1).

A generalized star S is a tree consisting of several paths, called branches, with the unique common endvertex. The number of branches is equal to the maximal vertex degree Δ of a generalized star. An example of such a star with branches of length 2, 3, 3, 3, 7 is shown in Figure 1. In this paper, a simple relation between $W(S)$ and $W(L^2(S))$ is established. Based on this relation, we show that stars with Δ branches and property (1) exist only for $4 \le \Delta \le 6$. Infinite families of such generalized stars for $\Delta = 5, 6$ are constructed.

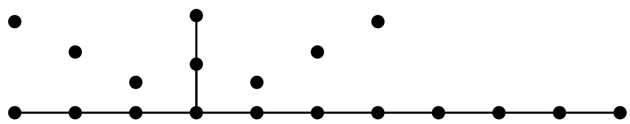

Figure 1. A generalized star with $\Delta = 5$.

2. Relation Between Wiener Indices of a Generalized Star and Its Quadratic Line Graph

The distance of a vertex v, $d_G(v)$, is the sum of distances between v and all other vertices of G, that is, $d_G(v) = \sum_{u \in V(G)} d_G(v, u)$. Then the Wiener index can be rewritten as

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v).$$

It is a well-known fact that the path P and the star S with n vertices have the extremal values of the Wiener index among all n-vertex trees [17]. Their Wiener indices are equal to

$$W(S) = (n-1)^2 = \Delta^2 \quad \text{and} \quad W(P) = n\,(n^2-1)/6 = \binom{n+1}{3}.$$

The distance of endvertex of the n-vertex path P is equal to $d_P(v) = n(n-1)/2 = \binom{n}{2}$.

Since $W(P) - W(L^2(P)) = W(S) \neq 0$ for $n \ge 2$, we assume that a generalized star has $\Delta \ge 3$ branches of length $k_1, k_2, \ldots, k_\Delta \ge 1$.

Proposition 1. *Let S be a generalized star with q edges and Δ branches of length $k_1, k_2, \ldots, k_\Delta$. Then*

$$W(S) = q\sum_{i=1}^{\Delta}\binom{k_i+1}{2} - 2\sum_{i=1}^{\Delta}\binom{k_i+1}{3}. \tag{2}$$

Proof. The Wiener index of a tree can be calculated through its maximal path-subtrees (called segments) in which all internal vertices have degree 2 in the tree. If all internal vertices and all edges of a segment B of length k are deleted from a n-vertex tree T, we have two connected components with $n_1(T \setminus B)$ and $n_2(T \setminus B)$ vertices, $n_1(T \setminus B) + n_2(T \setminus B) = n - k + 1$. Then

$$W(T) = \sum_i n_1(T \setminus B_i)\, n_2(T \setminus B_i)\, k_i + \frac{1}{6}\sum_i k_i(k_i-1)(3n-2k_i+1),$$

where the summations go over all segments of T [9].

Every branch B of length k of a generalized star S with q edges forms a segment and $n_1(S \setminus B)\, n_2(S \setminus B) = 1 \cdot (q+1-k)$. Then

$$\begin{aligned} W(S) &= \sum_{i=1}^{\Delta}(q-k_i+1)k_i + \frac{1}{6}\sum_{i=1}^{\Delta}k_i(k_i-1)(3q-2k_i+4) \\ &= \frac{1}{6}\sum_{i=1}^{\Delta}k_i(k_i+1)(3q-2k_i+2) = q\sum_{i=1}^{\Delta}\binom{k_i+1}{2} - 2\sum_{i=1}^{\Delta}\binom{k_i+1}{3}. \end{aligned}$$

■

Other way to prove Proposition 1 consists in application of recurrent formulas for the Wiener index from [5, 30].

Further, by the star we mean the generalized star.

Now we describe the structure of $L^2(S)$ for a star S with $\Delta \geq 3$ branches of arbitrary lengths. The graph $L^2(S)$ consists of the *core* G_0 and Δ paths P^j, $j = 1, 2, \ldots, \Delta$, attached to G_0. The core is the quadratic line graph of the star with Δ branches of length 1, that is, the core contains Δ complete graphs $K_{\Delta-1}$ and every core's vertex belongs exactly to two $K_{\Delta-1}$. The order of the core is equal to $\Delta(\Delta-1)/2$. A terminal vertex of a path P^j connects with all vertices of one complete graph $K_{\Delta-1}$ in the core.

Theorem 1. *Let S be a star with q edges and Δ branches of length $k_1, k_2, \ldots, k_\Delta$. Then*

$$W(L^2(S)) = W(S) + \frac{1}{2}\binom{\Delta-1}{2}\left[\sum_{i=1}^{\Delta}k_i^2 + q\right] - q^2 + 6\binom{\Delta}{4}. \tag{3}$$

Proof. Denote by V_0 the vertex set of the core G_0. Let the path P^j is attached to complete graph $K^j_{\Delta-1}$ in $L^2(S)$, $j = 1, 2, \ldots, \Delta$. Further, the subscript in the notation of complete graphs will be omitted. Let $VP = \cup_{j=1}^{\Delta} V(P^j)$ and $N = \sum_{j=1}^{\Delta} p_j$, where $p_j = |V(P^j)|$. From the definition of the Wiener index, we have

$$2W(L^2(S)) = \sum_{v\in V_0} d(v) + \sum_{v\in VP} d(v).$$

Now we calculate the both sums of this equality.

(1) Let $v \in V_0$ and $v \in V(K^n) \cap V(K^m)$. Then

$$d(v) = \sum_{x\in V_0} d(v,x) + \sum_{x\in VP} d(v,x). \tag{4}$$

Note that the core has diameter 2. Then for the first sum of (4) we have

$$\begin{aligned} \sum_{x\in V_0} d(v,x) &= \sum_{x\in V(K^n)\cup V(K^m)} d(v,x) \quad + \sum_{x\in V_0\setminus\{V(K^n)\cup V(K^m)\}} d(v,x) \\ &= 2(\Delta-2) + 2[\Delta(\Delta-1)/2 - 2(\Delta-2) - 1] = 2\binom{\Delta-1}{2}. \end{aligned} \tag{5}$$

Denote $V(P^n)$ by V^n, $n = 1, 2, \ldots, \Delta$. Then for the second sum of (4), we have

$$\begin{aligned} \sum_{x\in VP} d(v,x) &= \sum_{x\in V^n\cup V^m} d(v,x) + \sum_{x\in VP\setminus\{V^n\cup V^m\}} d(v,x) \\ &= \left[p_n + p_m + \binom{p_n}{2} + \binom{p_m}{2}\right] + \left[2\sum_{j\neq n,m} p_j + \sum_{j\neq n,m}\binom{p_j}{2}\right] \\ &= 2N + \sum_{j=1}^{\Delta}\binom{p_j}{2} - p_n - p_m. \end{aligned} \tag{6}$$

Therefore, the distance of the vertex $v \in V_0$ is equal to

$$d(v) = 2\binom{\Delta-1}{2} + \sum_{j=1}^{\Delta}\binom{p_j}{2} + 2N - p_n - p_m.$$

Summing the distances of all vertices of the core, we have

$$\sum_{v\in V_0} d(v) = \binom{\Delta}{2}\left[2\binom{\Delta-1}{2} + \sum_{j=1}^{\Delta}\binom{p_j}{2} + 2N\right] - N(\Delta-1).$$

Substituting $p_i = k_i - 1$ into the later equation, we obtain

$$\sum_{v\in V_0} d(v) = \binom{\Delta}{2}(\Delta-2)^2 + \frac{1}{2}\binom{\Delta}{2}\sum_{j=1}^{\Delta} k_j^2 + \frac{1}{4}(\Delta-1)(\Delta-4)\sum_{j=1}^{\Delta} k_j. \tag{7}$$

(2) Consider the path P^n attached to K^n. Assume that the vertex numbering in P^n is consecutive and u_1 is the attachment vertex of P^n. Let $u_i \in V(P^n)$. Then

$$d(u_i) = d_{P^n}(u_i) + \sum_{x\in V_0} d(u_i,x) + \sum_{x\in VP\setminus V^n} d(u_i,x). \tag{8}$$

For the first sum of (8), we have

$$\begin{aligned}\sum_{x\in V_0} d(u_i,x) &= \sum_{x\in V(K^n)} d(u_i,x) + \sum_{x\in V_0\setminus V(K^n)} d(u_i,x)\\ &= i(\Delta-1) + (i+1)[\Delta(\Delta-1)/2 - (\Delta-1)] = \binom{\Delta}{2}i + \binom{\Delta-1}{2}.\end{aligned}$$

Denote by w_1^j the terminal vertex of P^j attached to $K^j_{\Delta-1}$, $j = 1, 2, \ldots, \Delta$. Then for the second sum of (8), we can write

$$\begin{aligned}\sum_{x\in VP\setminus V^n} d(u_i,x) &= \sum_{j\neq n}\sum_{x\in V^j}[\, d(u_i,w_1^j) + d(w_1^j,x)\,]\\ &= (i+1)N - (i+1)p_n + \sum_{j=1}^{\Delta}\binom{p_j}{2} - \binom{p_n}{2}.\end{aligned}$$

Therefore, the sum of distances for all vertices of P^n is equal to

$$\begin{aligned}\sum_{u_i\in V^n} d(u_i,x) &= 2W(P^n) + \binom{\Delta}{2}\sum_{i=1}^{p_n} i + p_n\binom{\Delta-1}{2} + N\sum_{i=1}^{p_n}(i+1)\\ &\quad - p_n\sum_{i=1}^{p_n}(i+1) + p_n\sum_{j=1}^{\Delta}\binom{p_j}{2} - p_n\binom{p_n}{2}.\end{aligned}$$

Summing the later expression over all paths P^n, $n = 1, 2, \ldots, \Delta$, we get

$$\begin{aligned}\sum_{j=1}^{\Delta}\sum_{u_i\in V^j} d(u_i) &= 2\sum_{j=1}^{\Delta} W(P^j) + \binom{\Delta}{2}\sum_{j=1}^{\Delta}\binom{p_j+1}{2} + \binom{\Delta-1}{2}N \\ &+ \left[N\sum_{j=1}^{\Delta}\binom{p_j+1}{2} + N^2\right] - \left[\sum_{j=1}^{\Delta} p_j\binom{p_j+1}{2} + \sum_{j=1}^{\Delta} p_j^2\right] \\ &+ N\sum_{j=1}^{\Delta}\binom{p_j}{2} - \sum_{j=1}^{\Delta} p_j\binom{p_j}{2}. \qquad (9)\end{aligned}$$

Substituting $p_i = k_i - 1$ into equation (9), we obtain

$$\begin{aligned}\sum_{j=1}^{\Delta}\sum_{u_i\in V^j} d(u_i) &= -4\sum_{j=1}^{\Delta}\binom{k_j}{3} + \frac{1}{4}(\Delta^2 - 5\Delta - 4)\sum_{j=1}^{\Delta} k_j^2 + \sum_{j=1}^{\Delta} k_j\left(\sum_{j=1}^{\Delta} k_j^2\right) \\ &- \left(\sum_{j=1}^{\Delta} k_j\right)^2 + \frac{1}{4}(\Delta^2 - \Delta + 8)\sum_{j=1}^{\Delta} k_j - 3\binom{\Delta}{3}. \qquad (10)\end{aligned}$$

Note that

$$-4\sum_{j=1}^{\Delta}\binom{k_j}{3} = -4\sum_{j=1}^{\Delta}\binom{k_j+1}{3} + 2\sum_{j=1}^{\Delta} k_j^2 - 2\sum_{j=1}^{\Delta} k_j\,, \qquad (11)$$

$$\sum_{j=1}^{\Delta} k_j\left(\sum_{j=1}^{\Delta} k_j^2\right) - \left(\sum_{j=1}^{\Delta} k_j\right)^2 = -2q^2 + 2q\sum_{j=1}^{\Delta}\binom{k_j+1}{2}. \qquad (12)$$

Finally, applying expressions (2), (7) and (10) – (12), we can write

$$\begin{aligned}2W(L^2(S)) &= 2q\sum_{j=1}^{\Delta}\binom{k_j+1}{2} - 4\sum_{j=1}^{\Delta}\binom{k_j+1}{3} \\ &- 2q^2 + \binom{\Delta-1}{2}\sum_{j=1}^{\Delta} k_j^2 + \binom{\Delta-1}{2}\sum_{j=1}^{\Delta} k_j + 12\binom{\Delta}{4}.\end{aligned}$$

■

Formula (3) can be applied to calculate the Wiener index of several graphs. For example, since $L(S) \cong K_n$ for the $(n+1)$-vertex star S with branches of length 1 ($n \geq 3$), one immediately obtains that $W(L(K_n)) = n(n-1)^2 (n-2)/4$. Let S be a star with Δ branches of equal length k. Using formula (2) for the Wiener index of S, we have

$$W(L^2(S)) = \Delta\left[\binom{\Delta-1}{2}\binom{k+1}{2} + 3\Delta\binom{k}{3} - 2\binom{k+1}{3} + \frac{3}{2}\binom{\Delta-1}{3}\right].$$

In particular, if $k = \Delta$ then $W(L^2(S)) = \Delta(\Delta-1)(9\Delta^3 - 4\Delta^2 - 25\Delta + 18)/12$.

Corollary 1. *Let S and S' be stars with the same number of edges and branches of lengths $k_1, k_2, \ldots, k_\Delta$ and $k'_1, k'_2, \ldots, k'_\Delta$, respectively. Suppose that for these stars $\sum_{i=1}^{\Delta} k_i^2 = \sum_{i=1}^{\Delta} {k'_i}^2$. Then $W(L^2(S)) = W(L^2(S'))$ if and only if $W(S) = W(S')$.*

As an illustration, consider 4 stars with $q = 90$ edges and $\Delta = 6$ branches of length (6, 15, 15, 15, 15, 24), (7, 11, 14, 16, 19, 23), (8, 11, 11, 19, 19, 22) and (9, 9, 12, 18, 21, 21). These stars and their quadratic line graphs have the same Wiener index $W = 62940$.

3. Existence of Stars with Property (1)

In this section, we obtain necessary conditions for a star and its quadratic line graph to have the same Wiener index.

Theorem 2. *Let S be a star with Δ branches. If $\Delta = 3$, then $W(L^2(S)) < W(S)$. If $\Delta \geq 7$, then $W(L^2(S)) > W(S)$.*

Proof. Let S be a star with Δ branches of lengths $k_1, k_2, \ldots, k_\Delta$.
(1) Let $\Delta = 3$. Then formula (3) reduces to the obvious inequality

$$W(S) - W(L^2(S)) = 2(k_1 + k_2 + k_3)^2 - (k_1 + k_2 + k_3) - (k_1^2 + k_2^2 + k_3^2) > 0.$$

(2) Let $\Delta \geq 7$. Suppose that $W(L^2(S)) \leq W(S)$. By Theorem 1, one can write

$$\sum_{i=1}^{\Delta} k_i^2 \leq \frac{4q^2}{(\Delta-1)(\Delta-2)} - q - \Delta(\Delta-3). \tag{13}$$

It is a well-known fact that for any numbers $a_1, a_2, \ldots, a_n$ (see, for example, [27])

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right)^{\frac{1}{2}}.$$

Applying the latter relation to $k_1, k_2, \ldots, k_\Delta$, we have

$$\frac{q^2}{\Delta^2} = \frac{(k_1 + k_2 + \ldots + k_\Delta)^2}{\Delta^2} \leq \frac{k_1^2 + k_2^2 + \ldots + k_\Delta^2}{\Delta} < \frac{4q^2}{\Delta(\Delta-1)(\Delta-2)}.$$

Then the following inequality must hold

$$\frac{1}{\Delta} < \frac{4}{(\Delta-1)(\Delta-2)}.$$

However, it is easy to verify that for $\Delta \geq 7$ the last inequality is not valid. The obtained contradiction implies $W(L^2(S)) > W(S)$. ■

Table 1. The smallest stars having property (1).

Δ	q	k_1	k_2	k_3	k_4	k_5	k_6	W	q	k_1	k_2	k_3	k_4	k_5	k_6	W
	27	1	2	3	21	–	–	3417	90	3	7	8	72	–	–	118140
	42	1	2	6	33	–	–	12572	102	2	3	16	81	–	–	175042
4	69	2	6	6	55	–	–	53783	105	4	5	12	84	–	–	187493
	72	1	3	11	57	–	–	62112	105	2	9	10	84	–	–	187553
	90	4	5	9	72	–	–	118128	111	4	9	9	89	–	–	220195
	18	2	3	3	3	7	–	744	30	4	4	4	4	14	–	3360
	24	2	3	3	6	10	–	1766	30	3	3	3	8	13	–	3430
5	24	2	2	5	5	10	–	1770	30	1	4	5	7	13	–	3450
	24	1	4	4	5	10	–	1774	36	4	4	4	7	17	–	5792
	24	1	2	6	6	9	–	1802	36	3	4	6	6	17	–	5796
	50	7	7	7	8	10	11	10776	60	7	8	8	10	13	14	18644
	50	6	7	8	9	9	11	10780	60	6	8	9	11	12	14	18660
6	50	5	8	9	9	9	10	10792	60	6	7	10	12	12	13	18676
	60	8	8	8	9	12	15	18624	60	5	10	10	10	11	14	18680
	60	6	9	10	10	10	15	18640	60	5	8	11	12	12	12	18696

Corollary 2. *Let S be a star with Δ branches. If the Wiener index of S and its quadratic line graph is the same, $W(S) = W(L^2(S))$, then $\Delta \in \{4, 5, 6\}$.*

To prove the existence of stars with property (1), it is naturally to use computer generation of trees and calculation their Wiener index. Parameters of the smallest stars with q edges are presented in Table 1.

Corollary 3. *Let S be a star with q edges, Δ branches and $W(S) = W(L^2(S))$. Then q is divisible by $\binom{\Delta-1}{2}$.*

Proof. Let a star S and its quadratic line graph have the same Wiener index.

If $\Delta = 4$, then equation (3) can be rewritten as $3\sum_{i=1}^{4} k_i^2 = 2q^2 - 3q - 12$. Therefore, q is divisible by 3.

If $\Delta = 5$, then we have $3\sum_{i=1}^{5} k_i^2 = q^2 - 3q - 30$. Hence, q is divisible by 3, that is, $q = 3t$ for some integer t. Since the left-hand side of the equality $\sum_{i=1}^{5} k_i^2 = 3t(t-1) - 10$ is even, $q = \sum_{i=1}^{5} k_i$ must be even and, therefore, $q \equiv 0 \pmod 6$.

If $\Delta = 6$, then $5\sum_{i=1}^{6} k_i^2 = q^2 - 5q - 90$. Hence $q = 5t$ for some integer t. Since the left-hand side of the equality $\sum_{i=1}^{6} k_i^2 = 5t(t-1) - 18$ is even, q is also even. This implies that $q \equiv 0 \pmod{10}$. ■

4. Infinite Families of Stars with Property (1)

By Theorem 1, the existence of stars with property (1) is equivalent to the solvability of some Diophantine equations.

1. $\Delta = 4$. The corresponding Diophantine equation has the form

$$(14)\quad 3(k_1^2 + k_2^2 + k_3^2 + k_4^2) = 2(k_1 + k_2 + k_3 + k_4)^2 - 3(k_1 + k_2 + k_3 + k_4) - 12.$$

Note that the number of edges $q = k_1 + k_2 + k_3 + k_4$ is divisible by 3.

In this case, we could not find an infinite family of stars with property (1). If the above equation gives a polynomial of degree $m \geq 2$ of one variable, one can find at most m integer-valued roots. For example, consider a family of stars with branches of length $k_1 = (k^2+3k+4)/2$, $k_2 = k_3 = 3(2k^2-k+2)$, and $k_4 = 52k^2 - 18k + 55$, where $k \geq 0$. Then equation (14) can be rewritten as follows: $k(k-1)(k-2)(11k+3) = 0$. Therefore, this family contains only three stars having property (1).

2. $\Delta = 5$. In this case, the Diophantine equation is

$$(15)\qquad 3(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2) = \left(\sum_{i=1}^{5} k_i\right)^2 - 3\sum_{i=1}^{5} k_i - 30,$$

where $q = \sum_{i=1}^{5} k_i$ is divisible by 6.

Now we construct three infinite families of such trees.

Let S_k, $k \geq 0$, be a star with 5 branches of length $k_1 = 1$, $k_2 = 2$, $k_3 = k_4 = 2k^2 - k + 5$, and $k_5 = 2k^2 + 2k + 5$. The number of edges of S_k is equal to $q = 6(k^2 + 3)$.

Stars S_k^*, $k \geq 0$, of the second family have branches of lengths $k_1^* = 1$, $k_2^* = 2$, $k_3^* = 2k^2 - 2k + 5$ and $k_4^* = k_5^* = 2k^2 + k + 5$. The stars S_k^* and S_k have the same number of edges.

Proposition 2. *For every $k \geq 0$,*

$$W(S_k) = W(L^2(S_k)) = 28k^6 + 252k^4 - 2k^3 + 764k^2 + 760,$$

$$W(S_k^*) = W(L^2(S_k^*)) = 28k^6 + 252k^4 + 2k^3 + 764k^2 + 760.$$

In order to prove the coincidence of the Wiener indices, it is sufficient to verify equality (15). Proposition 1 and Theorem 1 can be used for calculating W for stars and their quadratic line graphs.

One can see that coefficients of the Wiener index of stars from the both families differ only in sign. The Wiener index of the star S_k and lengths of its branches can be regarded as abstract polynomials in k: $k_i = k_i(k)$ and $W(S_k) = W(k)$, *etc.* There is a simple relation between branches' lengths of stars from the considered families. Namely, $k_i^*(k) = k_i(-k)$, for all $1 \leq i \leq 5$. This implies that $W^*(k) = W(-k)$.

Table 2. Infinite families of stars having property (1) for $\Delta = 5$.

q	k_1	k_2	k_3	k_4	k_5	W	k_1	k_2	k_3	k_4	k_5	W	q	k_1	k_2	k_3	k_4	k_5	W
18	1	2	5	5	5	760	1	2	5	5	5	760	24	1	2	6	6	9	1802
24	1	2	6	6	9	1802	1	2	5	8	8	1806	36	1	2	8	11	14	6076
42	1	2	11	11	17	9624	1	2	9	15	15	9656	60	1	2	14	20	23	28080
72	1	2	20	20	29	48406	1	2	17	26	26	48514	96	1	2	24	33	36	114866
114	1	2	33	33	45	192056	1	2	29	41	41	192312	144	1	2	38	50	53	387406
168	1	2	50	50	65	614610	1	2	45	60	60	615110	204	1	2	56	71	74	1101072
234	1	2	71	71	89	1660792	1	2	65	83	83	1661656	276	1	2	78	96	99	2726276
312	1	2	96	96	117	3936734	1	2	89	110	110	3938106	360	1	2	104	125	128	6049270
402	1	2	125	125	149	8420856	1	2	117	141	141	8422904	456	1	2	134	158	161	12293106
504	1	2	158	158	185	16594906	1	2	149	176	176	16597822	564	1	2	168	195	198	23258756

The third family contains stars S'_k, $k \geq 0$, with $q = 6(k^2 + k + 4)$ edges and branches of length $k_1 = 1$, $k_2 = 2$, $k_3 = 2k^2 + 6$, $k_4 = 2k^2 + 3k + 6$, and $k_5 = 2k^2 + 3k + 9$.

Proposition 3. *For every $k \geq 0$,*

$$W(S'_k) = W(L^2(S'_k)) = 28k^6 + 84k^5 + 420k^4 + 702k^3 + 1691k^2 + 1349k + 1802.$$

Note that this family contains only asymmetrical trees except the initial star S'_0. Numerical data for the smallest stars of the above infinite families are presented in Table 2. The star S'_0 with branches of length 1, 2, 6, 6, 9 belongs also to the first considered family.

3. $\Delta = 6$. Lengths of star's branches must satisfy the following equation

$$5(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2 + k_6^2) = \left(\sum_{i=1}^{6} k_i\right)^2 - 5\sum_{i=1}^{6} k_i - 90,$$

where $q = \sum_{i=1}^{6} k_i$ is divisible by 10.

An infinite family of such trees is formed by stars with the following lengths of branches: $k_1 = 3$, $k_2 = 4k^2 + 33$, $k_3 = k_4 = 4k^2 - k + 36$, and $k_5 = k_6 = 4k^2 + k + 36$. The number of edges of stars are equal to $q = 20(k^2 + 9)$.

Proposition 4. *For every $k \geq 0$,*

$$W(S_k) = W(L^2(S_k)) = \frac{1}{3}(2080\,k^6 + 56256\,k^4 + 506972\,k^2 + 1522332).$$

The first stars of this family are shown in Table 3.

It would be interesting to find an infinite family of stars with $\Delta = 4$ branches and infinite families of asymmetric stars for $\Delta = 4, 6$ having property (1).

Table 3. Infinite family of stars for $\Delta = 6$.

q	k_1	k_2	k_3	k_4	k_5	k_6	W
180	3	33	36	36	36	36	507444
200	3	37	39	39	41	41	695880
260	3	49	50	50	54	54	1527812
360	3	69	69	69	75	75	4052712
500	3	97	96	96	104	104	10851700
680	3	133	131	131	141	141	27285544
900	3	177	174	174	186	186	63241860
1160	3	229	225	225	239	239	135381512
1460	3	289	284	284	300	300	269884212
1800	3	357	351	351	369	369	505693320

In conclusion, we reformulate a hypothesis from [13]: a homeomorphic irreducible tree (that is, without vertices of degree two) does not satisfy property (1).

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