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ON THE BASIS NUMBER AND THE MINIMUM CYCLE BASES OF THE WREATH PRODUCT OF SOME GRAPHS I

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Abstract

A construction of a minimum cycle bases for the wreath product of some classes of graphs is presented. Moreover, the basis numbers for the wreath product of the same classes are determined.

Keywords: cycle space, basis number, cycle basis, wreath product.

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1. Introduction

The basis number of a graph is one of the numbers which give rise to a better understanding and interpretations of a geometric properties of a graph (see [19]). Minimum cycle bases (MCBs) of a cycle spaces have a variety of applications in sciences and engineering, for example, in structural flexibility analysis, electrical networks, and in chemical structure storage and retrieval systems (see [9, 10] and [17]).

In general, required cycle bases, and minimum cycle bases are not very well behaved under graph operations. Neither the basis number $b(G)$ of a graph G is monotonic (see [3] and [21]), nor the total length $l(G)$ and the length of the longest cycle in a minimum cycle basis $\lambda(G)$ are minor monotone (see [12]). Hence, there does not seem to be a general way of extending required cycle bases and minimum cycle bases of a certain collection of partial graphs of G to a required cycle basis and to a minimum cycle basis of G, respectively. Global upper bounds $b(G) \leq 2\gamma(G) + 2$ and $l(G) \leq \dim \mathcal{C}(G) + \kappa(T(G))$ where $\gamma(G)$ is the genus of G and $\kappa(T(G))$ is the connectivity of the tree graph of G are proven in [21] and [18], respectively.

In this paper, we investigate the basis number for some classes of graphs and we construct minimum cycle bases for same, also, we give their total lengths and the length of longest cycles.

2. Definitions and Preliminaries

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [6]. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

2..1 Cycle bases

Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the *cycle space* of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph G $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$ (see [7]).

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. The *length,* $|C|$, *of the element* C of the cycle space $\mathcal{C}(G)$ is the number of its edges. The *length* $l(\mathcal{B})$ *of a cycle basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C \in \mathcal{B}} |C|$. $\lambda(G)$ is defined to be the minimum length of the longest element in an arbitrary cycle basis of G. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element C has weight $|C|$, the greedy algorithm can be used to extract a MCB (see [23]). The following results will be used frequently in the sequel.

Theorem 1.1.1 (MacLane)**.** *The Graph G is planar if and only if $b(G) \leq 2$.*

A cycle is *relevant* if it is contained in some MCB (see [22]).

Proposition 1.1.2 (Plotkin)**.** *A cycle C is relevant if and only if it cannot be written as a linear combinations modulo 2 of shorter cycles.*

Chickering, Geiger and Heckerman [8], showed that $\lambda(G)$ is the length of the longest element in a MCB.

2..2 Products

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

(1) The *cartesian product* $G\Box H$ has the vertex set $V(G\Box H) = V(G) \times V(H)$ and the edge set $E(G\Box H) = \{(u_1, v_1)(u_2, v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$.

(2) The *direct product* $G\times H$ is the graph with the vertex set $V(G\times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.

(3) The *strong product* $G\boxtimes H$ is the graph with the vertex set $V(G\boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)$ and $u_2 = v_2\}$.

(4) The *semi-strong product* $G_1 \bullet G_2$ is the graph with the vertex set $V(G\bullet H) = V(G)\times V(H)$ and the edge set $E(G\bullet H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)\}$.

(5) The *lexicographic product* $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G)\times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2)|u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$.

(6) The *wreath product* $G \ltimes H$ has the vertex set $V(G \ltimes H) = V(G) \times V(H)$ and the edge set $E(G \ltimes H) = \{(u_1, v_1)(u_2, v_2)|u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$ (see [1] and [11]).

Many authors studied the basis number and the minimum cycle bases of graph products. The cartesian product of any two graphs was studied by Ali and Marougi [4] and Imrich and Stadler [12].

Theorem 1.2.1 (Ali and Marougi)**.** *If G and H are two connected disjoint graphs, then $b(G\Box H) \leq \max\{b(G) + \triangle(T_H), b(H) + \triangle(T_G)\}$ where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G.*

Theorem 1.2.2 (Imrich and Stadler). *If G and H are triangle free, then* $l(G\Box H) = l(G) + l(H) + 4[|E(G)|(|V(H)| - 1) + |E(H)|(|V(G)| - 1) - (|V(H)| - 1)(|V(G)| - 1)]$ *and* $\lambda(G\Box H) = \max\{4, \lambda(G), \lambda(H)\}$.

Schmeichel [21], Ali [2, 3] and Jaradat [13] gave an upper bound for the basis number of the semi-strong and the direct products of some special graphs. They proved the following results:

Theorem 1.2.3 (Schmeichel). *For each* $n \geq 7$, $b(K_n \bullet P_2) = 4$.

Theorem 1.2.4 (Ali). *For each integers* n, m, $b(K_m \bullet K_n) \leq 9$.

Theorem 1.2.5 (Ali). *For any two cycles C_n and C_m with* $n, m \geq 3$, $b(C_n \times C_m) = 3$.

Theorem 1.2.6 (Jaradat). *For each bipartite graphs G and H,* $b(G \times H) \leq 5 + b(G) + b(H)$.

Theorem 1.2.7 (Jaradat). *For each bipartite graph G and cycle C,* $b(G \times C) \leq 3 + b(G)$.

The strong product was studied by Imrich and Stadler [12] and Jaradat [15]. They gave the following results:

Theorem 1.2.8 (Imrich and Stadler). *For any two graphs G and H,* $l(G\boxtimes H) = l(G) + l(H) + 3[\dim C(G\boxtimes H) - \dim C(G) - \dim C(H)]$ *and* $\lambda(G\boxtimes H) = \max\{3, \lambda(G), \lambda(H)\}$.

Theorem 1.2.9 (Jaradat). *Let G be a bipartite graph and H be a graph. Then* $b(G \boxtimes H) \leq \max\{b(H) + 1, 2\Delta(H) + b(G) - 1, \lfloor \frac{3\Delta(T_G)+1}{2} \rfloor, b(G) + 2\}$.

The results cited above trigger off the following question: Can we construct a minimum cycle basis and find the basis number of the wreath product of graphs? In this paper we will answer this question for a class of graphs. In fact, we construct a minimum cycle basis of the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel and we give their basis numbers. Moreover, we give the total lengths and lengths of longest cycles of the minimum cycle bases of the same.

In the rest of this paper, $f_B(e)$ stand for the number of elements of B containing the edge e where $B \subseteq \mathcal{C}(G)$.

3. The Basis Number of the Wreath Product of Graphs

In this section, we investigate the basis number of the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel. Also, in this section, we shall say $\mathcal{B}$ is a basis of $\mathcal{C}(G)$, rather than saying $\mathcal{B}$ is a cycle basis of G. Let $\{v_1, v_2, \ldots, v_m\}$ be a set of vertices and ab be an edge. Also, let $P_m = v_1v_2 \ldots v_m$. Then the automorphism group of the path P_m consists of two elements the identity, I, and the automorphism α which is defined as follows:

$$\alpha(v_i) = v_{m-j+1}, j = 1, 2, \ldots, m.$$

Therefore, $ab \ltimes P_m$ is decomposable into $ab \Box P_m \cup M_1$ where M_1 is the graph with the edge set $\{(a, v_j)(b, v_{m-j+1}), (a, v_{m-j+1})(b, v_j) | j = 1, 2, \ldots, \lfloor m/2 \rfloor\}$. Now, we define the following sets of cycles (see Figure 1):

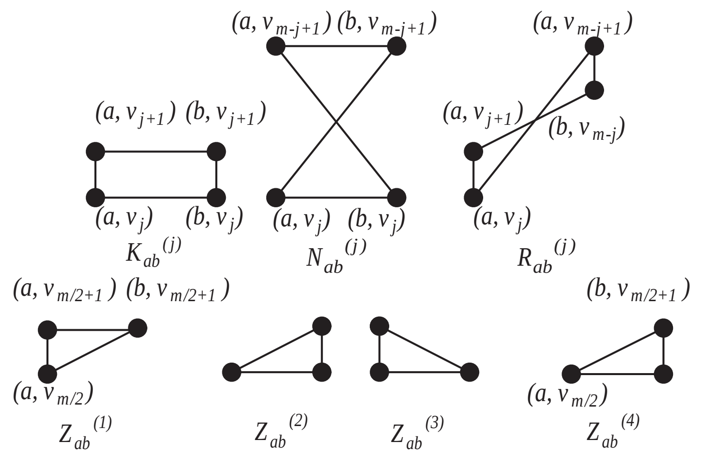

Figure 1. These graphs illustrate the cycles $\mathcal{K}_{ab}^{(j)}$, $\mathcal{N}_{ab}^{(j)}$, $\mathcal{R}_{ab}^{(j)}$, $\mathcal{Z}_{ab}^{(1)}$, $\mathcal{Z}_{ab}^{(2)}$, $\mathcal{Z}_{ab}^{(3)}$ and $\mathcal{Z}_{ab}^{(4)}$ for even m.

$$\mathcal{K}_{ab} = \left\{ \mathcal{K}_{ab}^{(j)} = (a, v_j)(b, v_j)(b, v_{j+1})(a, v_{j+1})(a, v_j) | j = 1, 2, \ldots, m-1 \right\},$$

$$\mathcal{N}_{ab} = \left\{ \mathcal{N}_{ab}^{(j)} = (a, v_j)(b, v_j)(a, v_{m-j+1})(b, v_{m-j+1})(a, v_j) | j = 1, 2, \ldots, \lfloor m/2 \rfloor \right\},$$

$$\mathcal{R}_{ab} = \left\{ \mathcal{R}_{ab}^{(j)} = (a, v_j)(a, v_{j+1})(b, v_{m-j})(b, v_{m-j+1})(a, v_j) | j = 1, 2, \ldots, \lfloor m/2 \rfloor \right\},$$

$$\mathcal{Z}_{ab} = \left\{ \begin{array}{l} \mathcal{Z}_{ab}^{(1)} = (a, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(2)} = (a, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(3)} = (a, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor}), \\ \mathcal{Z}_{ab}^{(4)} = (a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})(a, v_{\lfloor m/2 \rfloor+1}) \end{array} \right\}.$$

Lemma 3.1. *Let m be an odd integer. Then $\mathcal{A}_{ab} = \mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab}$ is a linearly independent subset of $\mathcal{C}(ab \ltimes P_m)$.*

Proof. We prove that $\mathcal{K}_{ab}$ is linearly independent using mathematical induction on m. If $m = 1$, then $\mathcal{K}_{ab}$ consists only of one cycle $\mathcal{K}_{ab}^{(1)}$. Thus, $\mathcal{K}_{ab}$ is linearly independent. Assume that m is greater than 2 and it is true for less than m. Note that $\mathcal{K}_{ab} = (\cup_{j=1}^{m-2} \mathcal{K}_{ab}^{(j)}) \cup \mathcal{K}_{ab}^{(m-1)}$. Since $\mathcal{K}_{ab}^{(m-1)}$ contains the edge $(a, v_m)(b, v_m)$ which is not in any cycle of $\cup_{j=1}^{m-2} \mathcal{K}_{ab}^{(j)}$, as a result $\mathcal{K}_{ab}$ is linearly independent. By a similar way we show that each of $\mathcal{N}_{ab}$ and $\mathcal{R}_{ab}$ are linearly independent. Any linear combination of cycles of $\mathcal{R}_{ab}$ must contain an edge of the form $(b, v_{m-j+1})(a, v_j)$ for some $1 \leq j \leq \lfloor m/2 \rfloor$, which is not in any cycle of $\mathcal{K}_{ab}$. Thus, $\mathcal{K}_{ab} \cup \mathcal{R}_{ab}$ is linearly independent. Similarly, each linear combination of $\mathcal{N}_{ab}$ contains an edge of the form $(b, v_j)(a, v_{m-j+1})$ for some $1 \leq j \leq \lfloor m/2 \rfloor$, which is not in any cycle of $\mathcal{K}_{ab} \cup \mathcal{R}_{ab}$. Therefore, $\mathcal{A}_{ab}$ is linearly independent. The proof is complete. ■

Remark 3.1. For an odd integer m let $e \in ab \ltimes P_m$. Then

(1) if $e = (a, v_j)(a, v_{j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 1$ and $f_{\mathcal{N}_{ab}}(e) = 0$.

(2) If $e = (a, v_j)(a, v_{j+1})$ such that $j \geq \lfloor m/2 \rfloor + 1$, then $f_{\mathcal{K}_{ab}}(e) = 1$ and $f_{\mathcal{R}_{ab}}(e) = f_{\mathcal{N}_{ab}}(e) = 0$.

(3) If $e = (b, v_j)(b, v_{j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{K}_{ab}}(e) = 1$ and $f_{\mathcal{R}_{ab}}(e) = f_{\mathcal{N}_{ab}}(e) = 0$.

(4) If $e = (b, v_j)(b, v_{j+1})$ such that $j \geq \lfloor m/2 \rfloor + 1$, then $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 1$ and $f_{\mathcal{N}_{ab}}(e) = 0$.

(5) If $e = (a, v_j)(b, v_j)$ such that $j \neq \lfloor m/2 \rfloor + 1$, then $f_{\mathcal{K}_{ab}}(e) \leq 2$, $f_{\mathcal{R}_{ab}}(e) = 0$ and $f_{\mathcal{N}_{ab}}(e) = 1$.

(6) If $e = (a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor+1})$, then $f_{\mathcal{K}_{ab}}(e) = 2$, $f_{\mathcal{R}_{ab}}(e) = 1$ and $f_{\mathcal{N}_{ab}}(e) = 0$.

(7) If $e = (a, v_j)(b, v_{m-j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{K}_{ab}}(e) = 0$, $f_{\mathcal{R}_{ab}}(e) \leq 2$ and $f_{\mathcal{N}_{ab}}(e) = 1$.

(8) If $e = (b, v_j)(a, v_{m-j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{K}_{ab}}(e) = f_{\mathcal{R}_{ab}}(e) = 0$ and $f_{\mathcal{N}_{ab}}(e) = 1$.

Lemma 3.2. *Let m be an even integer. Then $\mathcal{T}_{ab} = \mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}, \mathcal{Z}_{ab}^{(3)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ is a linearly independent subset of $\mathcal{C}(ab \ltimes P_m)$.*

Proof. Using the same argument as in Lemma 3.1, we have that $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ is linearly independent. Since $\mathcal{Z}_{ab}^{(1)}$ contains $(a, v_{\lfloor m/2 \rfloor})(a, v_{\lfloor m/2 \rfloor+1})$ which is not in any cycle of $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$, $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ is linearly independent. Also, $\mathcal{Z}_{ab}^{(2)}$ contains $(b, v_{\lfloor m/2 \rfloor})(b, v_{\lfloor m/2 \rfloor+1})$ which is not in any cycle of $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$. Hence, $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$ is linearly independent. Similarly, $\mathcal{Z}_{ab}^{(3)}$ contains the edge $(a, v_{\lfloor m/2 \rfloor+1})(b, v_{\lfloor m/2 \rfloor})$ which is not in any cycle of $\mathcal{K}_{ab} \cup \mathcal{N}_{ab} \cup \mathcal{R}_{ab} \cup \{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\} - \{\mathcal{K}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{N}_{ab}^{(\lfloor m/2 \rfloor)}, \mathcal{R}_{ab}^{(\lfloor m/2 \rfloor)}\}$. Therefore, $\mathcal{T}_{ab}$ is linearly independent. The proof is complete. ■

Throughout this paper we consider $\mathcal{B}_{ab} = \begin{cases} \mathcal{A}_{ab}, & \text{if } m \text{ is odd,} \\ \mathcal{T}_{ab}, & \text{if } m \text{ is even.} \end{cases}$

Remark 3.2. For any integer m let $e \in ab \ltimes P_m$. Then from Lemma 3.1 and Lemma 3.2 and as in Remark 3.1 we have that

(1) if $e = (a, v_j)(a, v_{j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{B}_{ab}}(e) \leq 2$.

(2) If $e = (a, v_j)(a, v_{j+1})$ such that $j \geq \lfloor m/2 \rfloor + 1$, then $f_{\mathcal{B}_{ab}}(e) = 1$.

(3) If $e=(b,v_j)(b,v_{j+1})$ such that $j\le\lfloor m/2\rfloor$, then $f_{\mathcal{B}_{ab}}(e)=1$.
(4) If $e=(b,v_j)(b,v_{j+1})$ such that $j\ge\lfloor m/2\rfloor+1$, then $f_{\mathcal{B}_{ab}}(e)\le 2$.
(5) If $e=(a,v_j)(b,v_j)$ such that $j\ne 1,m$, then $f_{\mathcal{B}_{ab}}(e)\le 3$.
(6) If $e=(a,v_1)(b,v_1)$ or $(a,v_m)(b,v_m)$, then $f_{\mathcal{B}_{ab}}(e)\le 2$.
(7) If $e=(a,v_j)(b,v_{m-j+1})$ such that $j\le\lfloor m/2\rfloor$, then $f_{\mathcal{B}_{ab}}(e)\le 3$.
(8) If $e=(b,v_j)(a,v_{m-j+1})$ such that $j\le\lfloor m/2\rfloor$, then $f_{\mathcal{B}_{ab}}(e)=1$.

Lemma 3.3. *If $m\ge 3$, then $b(ab\ltimes P_m)\ge 3$.*

Proof. *Case* 1. m is odd. Let H_1 be the subgraph of $ab\ltimes P_m$ induced by the following set of vertices $\{(a,v_{\lfloor m/2\rfloor}),(a,v_{\lfloor m/2\rfloor+2}),(b,v_{\lfloor m/2\rfloor+1}),(b,v_{\lfloor m/2\rfloor}),(b,v_{\lfloor m/2\rfloor+2}),(a,v_{\lfloor m/2\rfloor+1})\}$. Then H_1 is isomorphic to $K_{3,3}$ and so by MacLane's Theorem $b(ab\ltimes P_m)\ge 3$.

Case 2. m is even. Let H_2 be the subgraph with vertex set $\{(a,v_{\lfloor m/2\rfloor-1}),(a,v_{\lfloor m/2\rfloor}),(a,v_{\lfloor m/2\rfloor+1}),(a,v_{\lfloor m/2\rfloor+2}),(b,v_{\lfloor m/2\rfloor-1}),(b,v_{\lfloor m/2\rfloor}),(b,v_{\lfloor m/2\rfloor+1}),(b,v_{\lfloor m/2\rfloor+2})\}$ and edge set consists of the following nine paths: $P_1=(a,v_{\lfloor m/2\rfloor+1})(a,v_{\lfloor m/2\rfloor+2})(b,v_{\lfloor m/2\rfloor+2})$, $P_2=(a,v_{\lfloor m/2\rfloor+1})\,(a,v_{\lfloor m/2\rfloor})$, $P_3=(a,v_{\lfloor m/2\rfloor+1})(b,v_{\lfloor m/2\rfloor})$, $P_4=(a,v_{\lfloor m/2\rfloor})(a,v_{\lfloor m/2\rfloor-1})$, $P_5=(a,v_{\lfloor m/2\rfloor})(b,v_{\lfloor m/2\rfloor+1})$, $P_6=(a,v_{\lfloor m/2\rfloor-1})(b,v_{\lfloor m/2\rfloor-1})(b,v_{\lfloor m/2\rfloor})$, $P_7=(a,v_{\lfloor m/2\rfloor-1})(b,v_{\lfloor m/2\rfloor+2})$, $P_8=(b,\,v_{\lfloor m/2\rfloor})(b,v_{\lfloor m/2\rfloor+1})$, $P_9=(b,v_{\lfloor m/2\rfloor+1})(b,v_{\lfloor m/2\rfloor+2})$. Then H_2 is homeomorphic to $K_{3,3}$. Thus, by MacLane's Theorem $b(ab\ltimes P_m)\ge 3$. The proof is complete. ■

Let $P_n=a_1a_2\ldots a_n$. Then the graph $P_n\ltimes P_m$ is decomposable into $P_n\square P_m\cup M_2$ where M_2 is the graph consisting of the following edge set $\cup_{i=1}^{n-1}\{(a_i,v_j)\,(a_{i+1},\,v_{n-j+1}),\,(a_i,\,v_{n-j+1})\,(a_{i+1},\,v_j)\,|\,j=1,2,\ldots,\lfloor m/2\rfloor\}$. Hence, $|E(P_n\ltimes P_m)|=n(m-1)+m(n-1)+2(n-1)\lfloor m/2\rfloor$. Therefore, $\dim\mathcal{C}(P_n\ltimes P_m)=n(m-1)+m(n-1)+2(n-1)\lfloor m/2\rfloor-nm+1=mn-n-m+2(n-1)\lfloor m/2\rfloor+1$. The following lemma will be used frequently in the sequel.

Lemma 3.4 (Jaradat, Alzoubi and Rawashdeh)**.** *Let A and B be two linearly independent sets of cycles such that $E(A)\cap E(B)$ is an edge set of a forest. Then $A\cup B$ is linearly independent.*

Theorem 3.5. *Let P_n and P_m be two paths of order $n,m\ge 2$. Then $b(P_n\ltimes P_m)\le 3$. Moreover, the equality holds if $n\ge 2$ and $m\ge 3$.*

Proof. By Lemma 3.3 we have that $b(P_n \ltimes P_m) \geq 3$ for any $n \geq 2, m \geq 3$. To prove that $b(P_n \ltimes P_m) \leq 3$, it suffices to exhibit a 3-fold cycle basis. Define $\mathcal{B}(P_n \ltimes P_m) = \cup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}}$. We now show that $\mathcal{B}(P_n \ltimes P_m)$ is linearly independent using mathematical induction on n. If $n = 2$, then $\mathcal{B}(P_n \ltimes P_m) = \mathcal{B}_{a_1 a_2}$ which is linearly independent by Lemma 3.1 and 3.2. Assume $n \geq 3$ and it is true for less than or equal to $n-2$. Note that $\mathcal{B}(P_n \ltimes P_m) = \left(\cup_{i=1}^{n-2} \mathcal{B}_{a_i a_{i+1}}\right) \cup \left(\mathcal{B}_{a_{n-1} a_n}\right)$ and $E\left(\cup_{i=1}^{n-2} \mathcal{B}_{a_i a_{i+1}}\right) \cap E\left(\mathcal{B}_{a_{n-1} a_n}\right) = E(a_{n-1} \Box P_m)$ which is an edge set of a path. Thus, by Lemma 3.4, $\mathcal{B}(P_n \ltimes P_m)$ is linearly independent. Since

$$|\mathcal{B}_{a_i a_{i+1}}| = |\mathcal{A}_{a_i a_{i+1}}| = (m-1) + 2\lfloor m/2 \rfloor$$

if m is odd, and

$$\begin{aligned} |\mathcal{B}_{a_i a_{i+1}}| &= |\mathcal{T}_{a_i a_{i+1}}| = (m-2) + 2(\lfloor m/2 \rfloor - 1) + 3 \\ &= (m-1) + 2\lfloor m/2 \rfloor, \end{aligned}$$

if m is even, we obtain

$$\begin{aligned} \mathcal{B}(P_n \ltimes P_m) &= \sum_{i=1}^{n-1} |\mathcal{B}_{a_i a_{i+1}}| \\ &= (n-1)\left((m-1) + 2\lfloor m/2 \rfloor\right) \\ &= mn - m - n + 2(n-1)\lfloor m/2 \rfloor + 1 \\ &= \dim \mathcal{C}(P_n \ltimes P_m). \end{aligned}$$

Thus, $\mathcal{B}(P_n \ltimes P_m)$ is a basis for $\mathcal{C}(P_n \ltimes P_m)$. We now show that $\mathcal{B}(P_n \ltimes P_m)$ is a 3-fold basis. Let $e \in E(P_n \ltimes P_m)$. Then

(1) If $e = (a_i, v_j)(a_i, v_{j+1})$ such that $j \leq \lfloor m/2 \rfloor$, then $f_{\mathcal{B}(P_n \ltimes P_m)}(e) = f_{\mathcal{B}_{a_{i-1} a_i}}(e) + f_{\mathcal{B}_{a_i a_{i+1}}}(e) = 1 + 2 = 3$.

(2) If $e = (a_i, v_j)(a_i, v_{j+1})$ such that $j \geq \lfloor m/2 \rfloor + 1$, then $f_{\mathcal{B}(P_n \ltimes P_m)}(e) = f_{\mathcal{B}_{a_{i-1} a_i}}(e) + f_{\mathcal{B}_{a_i a_{i+1}}}(e) = 2 + 1 = 3$.

(3) If e is not of the above form, then e belongs only to cycles of $\mathcal{B}_{a_i a_{i+1}}$ for some $1 \leq i \leq n-1$ and so $f_{\mathcal{B}(P_n \ltimes P_m)}(e) = f_{\mathcal{B}_{a_i a_{i+1}}}(e) \leq 3$.

The proof is complete. ■

Now we turn our attention to deal with $C_n \ltimes P_m$. Let $C_n = a_1 a_2 \ldots a_n a_1$. Note that $C_n \ltimes P_m$ is decomposable into $P_n \ltimes P_m \cup M_3$ where M_3 is the subgraph consists of the following edges: $\{(a_1, v_j)(a_n, v_j) | j = 1, 2, \ldots, m\} \cup$

$\{(a_1, v_j)(a_n, v_{m-j+1}), (a_1, v_{m-j+1})(a_n, v_j) | j = 1, 2, \ldots, \lfloor m/2 \rfloor\}$. Therefore, $|E(C_n \ltimes P_m)| = |E(P_n \ltimes P_m)| + m + 2\lfloor m/2 \rfloor$ and so $\dim \mathcal{C}(C_n \ltimes P_m) = \mathcal{C}(P_n \ltimes P_m) + m + 2\lfloor m/2 \rfloor = mn - n + 2n\lfloor m/2 \rfloor + 1$.
Let G and H be two graphs and $e = (a, u)(b, v) \in E(G \ltimes H)$. Then the *projection* of e in $G, P_G(e)$, is defined to be the edge ab if $a \neq b$ and to be the vertex a if $a = b$ (see [12]).

Lemma 3.6. *$C_n \Box v_i$ is relevant in $C_n \ltimes P_m$ for each $i = 1, 2, \ldots, n$.*

Proof. For simplicity assume that $e_i = a_i a_{i+1}$ for each $i = 1, 2, \ldots, n-1$ and $e_n = a_n a_1$. Let O be any cycle of $C_n \ltimes P_m$ of length less than n. Since

$$E(e_i \ltimes P_m) \cap E(e_j \ltimes P_m) = \begin{cases} a_j \Box P_m, & \text{if } j - i = 1, n-1, \\ \phi, & \text{if } j - i \neq 1, n-1, \end{cases}$$

(assuming $i < j$), as a result O consists of edges of successive graphs of $\{e_r \ltimes P_m\}_{r=1}^n$, say $e_{l+1} \ltimes P_m, e_{l+2} \ltimes P_m, \ldots, e_{l+k} \ltimes P_m$ for some $l,\ k < n$. Since $e_i \ltimes P_m = (a_i \Box P_m) \cup (a_{i+1} \Box P_m) \cup X_i$ where X_i is a bipartite graph with independent sets of vertices $a_i \times V(P_m)$ and $a_{i+1} \times V(P_m)$, as a result O contains even number of edges with projection e_{l+s} for each $s = 1, 2, \ldots, k$. Thus, if

$$C_n \Box v_1 = \sum_{i=1}^{f} O_i \pmod 2.$$

where O_i is a cycle of length less than n. Then the number of edge of the ring sum $O_1 \oplus O_2 \oplus \cdots \oplus O_f$ with projection e_i in C_n is even for each $i = 1, 2, \ldots, n$. In contrast, the number of edges of $C_n \Box v_1$ with projection e_i in C_n is 1 for each $i = 1, 2, \ldots, n$. Thus, $C_n \Box v_1$ is relevant. The proof is complete. ■

Theorem 3.7. *Let C_n be a cycle and P_m be a path. Then $b(C_n \ltimes P_m) = 3$.*

Proof. By Lemma 3.3, to prove that $b(C_n \ltimes P_m) \geq 3$, it suffices to show that $b(C_n \ltimes P_2) \geq 3$. Let H be the spanning subgraph of $C_n \ltimes P_2$ with the edge set consists of the following paths: $(a_1, v_1)(a_1, v_2), (a_2, v_1)(a_2, v_2)$, $(a_1, v_1)\ (a_2, v_2)$, $(a_1, v_2)\ (a_2, v_1)$, $(a_1, v_2)\ (a_n, v_1)$, $(a_1, v_1)\ (a_n, v_2)$, (a_n, v_1) (a_n, v_2), $(a_2, v_2)\ (a_3, v_1)\ (a_4, v_1) \ldots (a_n, v_1)$ and $(a_2, v_1)\ (a_3, v_2)\ (a_4, v_2) \ldots$ (a_n, v_2). Then, H is homeomorphic to $K_{3,3}$ and so $b(C_n \ltimes P_2) \geq 3$. Define $\mathcal{B}(C_n \ltimes P_m) = \mathcal{B}(P_n \ltimes P_m) \cup \mathcal{B}_{a_n a_1} \cup \{C_n \Box v_1\}$ where $\mathcal{B}(P_n \ltimes P_m)$ is as defined in

Theorem 3.5. Now, since $E(P_n \ltimes P_m) \cap E(\mathcal{B}_{a_n a_1}) = E(a_1 \Box P_m) \cup E(a_n \Box P_m)$ which is an edge set of a forest, by Lemma 3.4 $\mathcal{B}(P_n \ltimes P_m) \cup \mathcal{B}_{a_n a_1}$ is linearly independent. Since each cycle of $\mathcal{B}(C_n \ltimes P_m) - \{C_n \Box v_1\}$ is of length less than n and since $C_n \Box v_1$ is relevant in $C_n \ltimes P_m$ (Lemma 3.6), $\mathcal{B}(C_n \ltimes P_m)$ is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_n \ltimes P_m)| &= mn - m - n + (n-1)2\lfloor m/2 \rfloor + 1 + (m-1) + 2\lfloor m/2 \rfloor + 1 \\ &= mn - n + 2n\lfloor m/2 \rfloor + 1 \\ &= \dim \mathcal{C}(C_n \ltimes P_m), \end{aligned}$$

$\mathcal{B}(C_n \ltimes P_m)$ is a basis of $\mathcal{C}(C_n \ltimes P_m)$. It is an easy task to show that $\mathcal{B}(C_n \ltimes P_m)$ is a 3-fold basis. The proof is complete. ■

Now consider S_m to be the star with the vertex set $\{v_1, v_2, \ldots, v_m\}$ and $d_{S_m}(v_1) = m - 1$. Note that the automorphism group of S_m is isomorphic to the symmetric group on the set $\{v_2, v_3, \ldots, v_m\}$. Therefore, for any $\gamma \in \mathrm{Aut}(G), \gamma(v_1) = v_1$. Moreover, for any two vertices v_i, v_j such that $2 \leq i, j \leq m$ there is an automorphism α such that $\alpha(v_i) = v_j$. Hence, the graph $ab \ltimes S_m$ is decomposable into $(a \Box S_m) \cup (b \Box S_m) \cup \{(a, v_1)(b, v_1)\} \cup ab[N_{m-1}]$ where N_{m-1} is the null graph with the vertex set $\{v_2, v_3, \ldots, v_m\}$. Let

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_l)(a, v_{j+1})(b, v_{l+1})(a, v_j) : 2 \leq j, l \leq m\}.$$

Then $\mathcal{H}_{ab}$ is the Schemeichel's 4-fold basis of $\mathcal{C}(ab[N_{m-1}])$ (see Theorem 2.4 in [21]). Moreover,

(1) if $e = (a, v_2)(b, v_m)$ or $e = (a, v_m)(b, v_2)$ or $e = (a, v_2)(b, v_2)$ or $e = (a, v_m)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$.
(2) If $e = (a, v_2)(b, v_l)$ or $(a, v_j)(b, v_2)$ or $(a, v_m)(b, v_l)$ or $(a, v_j)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) \leq 2$.
(3) If $e \in E(e[N_{m-1}])$ and is not of the above form, then $f_{\mathcal{H}_{ab}}(e) \leq 4$. Now, define the following sets of cycles (see Figure 2):

$$\mathcal{G}_{ab} = \left\{ \mathcal{G}_{ab}^{(l)} = (a, v_1)(a, v_l)(b, v_2)(a, v_{l+1})(a, v_1) | 2 \leq l \leq m-1 \right\},$$

and

$$\mathcal{S}_{ab} = \mathcal{K}_{ab}^{(1)} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\}.$$

Lemma 3.8. *$\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$ is a linearly independent subset of cycles of $\mathcal{C}(ab \ltimes S_m)$.*

Proof. $\mathcal{G}_{ab}$ is a basis for the cycle subspace of $\mathcal{C}(ab \ltimes S_m)$ corresponding to the planar subgraph of $ab \ltimes S_m$ obtained by pasting all the cycles of $\mathcal{G}_{ab}$, which are 4-cycles, at the common edges of the successive cycles. Similarly, $\mathcal{G}_{ba}$ is a basis for the cycle subspace of $\mathcal{C}(ab \ltimes S_m)$. Since $E(\mathcal{G}_{ab}) \cap E(\mathcal{G}_{ba}) = \{(a, v_1)(a, v_2), (a, v_2)(b, v_2), (b, v_2)(b, v_1)\}$ which is an edge set of a path and since $\mathcal{S}_{ab}$ contains $(b, v_1)(a, v_1)$ which occurs in no cycle of $\mathcal{G}_{ab} \cup \mathcal{G}_{ba}$, $\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$ is a linearly independent set. The proof is complete. ■

Lemma 3.9. *$\mathcal{L}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$ is a linearly independent set of cycles.*

Proof. The proof of this lemma follows by noting that every linear combination of cycles of $\mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$ contains at least one edge of the set $E(a\Box S_m) \cup E(b\Box S_m)$ which occurs in no cycle of $\mathcal{H}_{ab}$. The proof is complete. ■

Remark 3.3. Let $e \in E(P_n \ltimes S_m)$. Then

(1) if $e = (a, v_1)(a, v_l)$ or $(b, v_1)(b, v_l)$, then $f_{\mathcal{L}_{ab}}(e) \leq 2$.

(2) If $e = (a, v_2)(b, v_2)$, then $f_{\mathcal{L}_{ab}}(e) = 4$.

(3) If $e = (a, v_m)(b, v_m)$ or $(a, v_1)(b, v_1)$, then $f_{\mathcal{L}_{ab}}(e) = 1$.

(4) If $e = (a, v_2)(b, v_l)$ or $(a, v_j)(b, v_2)$ such that $m > j, l \geq 2$, then $f_{\mathcal{L}_{ab}}(e) \leq 4$.

(6) If $e = (a, v_m)(b, v_l)$ or $(a, v_j)(b, v_m)$ such that $j, l \geq 2$, then $f_{\mathcal{L}_{ab}}(e) \leq 2$.

(7) If e is not of the above form, then $f_{\mathcal{L}_{ab}}(e) = f_{\mathcal{H}_{ab}}(e) \leq 4$.

The graph $P_n \ltimes S_m$ is decomposable into $(\cup_{i=1}^{n}(a_i\Box S_m)) \cup P_n[N_{m-1}] \cup (P_n\Box v_1)$. Thus $|E(P_n \ltimes S_m)| = m^2(n-1) - m(n-2) + n - 2$. And so $\dim \mathcal{C}(P_n \ltimes S_m) = m^2(n-1) - 2m(n-1) + n - 1$.

Lemma 3.10. *If $n \geq 4$ and $(m^2+1)(n-1) - m(5n-2) + 3 \leq 0$, then $m < 6$.*

Proof. $(m^2+1)(n-1) \leq m(5n-2) - 3$. Hence $(m^2+1)(n-1) \leq 5m(n-1) + 3(m-1)$ which implies that $(m^2+1) \leq 5m + 3(m-1)/(n-1)$ and so $m \leq 5 + 3(m-1)/m(n-1) - 1/m$. But $3(m-1)/m(n-1) - 1/m < 1$. Therefore, $m < 6$. ■

Theorem 3.11. *For any path P_n of order $n \geq 2$ and star S_m, we have that $b(P_n \ltimes S_m) \leq 4$. Moreover, the equality holds if $n \geq 4$ and $m \geq 6$.*

Proof. Defined $\mathcal{B}(P_n \ltimes S_m) = \cup_{i=1}^{n-1} \mathcal{L}_{a_i a_{i+1}}$. Then by Lemma 3.9 and using the same arguments as in Theorem 3.5 we have that $\mathcal{B}(P_n \ltimes S_m)$ is linearly independent. Now,

$$|\mathcal{L}_{a_i a_{i+1}}| = m^2 - 2m + 1.$$

Thus,

$$\mathcal{B}(P_n \ltimes S_m) = \sum_{i=1}^{n-1} (m^2 - 2m + 1) = \dim \mathcal{C}(P_n \ltimes S_m).$$

Therefore, $\mathcal{B}(P_n \ltimes S_m)$ is a basis for $\mathcal{C}(P_n \ltimes S_m)$. Now we show that $\mathcal{B}(P_n \ltimes S_m)$ is a 4-fold basis. Let $e \in E(P_n \ltimes S_m)$. Then

(1) if $e = (a_i, v_1)(a_i, v_l)$, then $f_{\mathcal{B}(P_n \ltimes S_m)}(e) = f_{\mathcal{L}_{a_{i-1} a_i}}(e) + f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 2 + 2 = 4$.

(2) If $e = (a_i, v_j)(a_{i+1}, v_l)$ such that $j, l \geq 2$, then $f_{\mathcal{B}(P_n \ltimes S_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}}(e) \leq 4$.

(3) If $e = (a_i, v_1)(a_{i+1}, v_1)$, then $f_{\mathcal{B}(P_n \ltimes S_m)}(e) = f_{\mathcal{L}_{a_i a_{i+1}}}(e) = 1$.

Now, we show that $b(P_n \ltimes S_m) \geq 4$ for each $n \geq 4$ and $m \geq 6$. Suppose that $\mathcal{B}$ is a 3-fold basis of $\mathcal{C}(P_n \ltimes S_m)$ for each $n \geq 4$ and $m \geq 6$. Since the girth of $P_n \ltimes S_m$ is 4, as a result

$$4 \dim \mathcal{C}(P_n \ltimes S_m) \leq 3|E(P_n \ltimes S_m)|$$

and so

$$\begin{aligned} 4(nm^2 - 2mn - m^2 + 2m + n - 1) &\leq 3(nm^2 - mn - m^2 + 2m + n - 2), \\ nm^2 - 5mn - m^2 + 2m + n + 2 &\leq 0, \\ m^2(n-1) - m(5n-2) + (n-1) + 3 &\leq 0, \\ (m^2+1)(n-1) - m(5n-2) + 3 &\leq 0. \end{aligned}$$

By Lemma 3.10, for $n \geq 4$, we have that $m < 6$. This is a contradiction. The proof is complete. ∎

Now, $C_n \ltimes S_m$ is decomposable into $P_n \ltimes S_m \cup a_1 a_m[N_{m-1}] \cup \{(a_1, v_1)(a_m, v_1)\}$ where N_{m-1} is the null graph with the vertex set $\{v_2, v_3, \ldots, v_m\}$. Thus, $|E(C_n \ltimes S_m)| = |E(P_n \ltimes S_m)| + (m-1)^2 + 1$. Hence, $\dim \mathcal{C}(C_n \ltimes S_m) = \dim \mathcal{C}(P_n \ltimes S_m) + (m-1)^2 + 1 = n(m-1)^2 + 1$. By applying the same arguments as in Lemma 3.6 in $C_n \ltimes S_m$, we have the following result:

Lemma 3.12. *$C_n \square v_i$ is relevant in $C_n \ltimes S_m$ for each $i = 1, 2, \ldots, n$.*

Theorem 3.13. *For any cycle C_n and star S_m, we have that $b(C_n \ltimes S_m) \leq 4$. Moreover, the equality holds if $n \geq 4$ and $m \geq 5$.*

Proof. Define $\mathcal{B}(C_n \ltimes S_m) = \mathcal{B}(P_n \ltimes S_m) \cup \mathcal{L}_{a_n a_1} \cup \{C_n \Box v_1\}$. By using the same arguments as in Theorem 3.7, we show that $\mathcal{B}(C_n \ltimes S_m)$ is a 4-fold basis of $\mathcal{C}(C_n \ltimes S_m)$. On the other hand to show that $b(C_n \ltimes S_m) \geq 4$, we suppose that $\mathcal{B}$ is a 3-fold basis of the space $\mathcal{C}(C_n \ltimes S_m)$ for each $n \geq 4$ and $m \geq 5$, then we argue more or less as in Theorem 3.12 by taking into account that if $n \geq 4$ and $nm^2 - 5mn + n + 4 \leq 0$, then $m < 5$. The proof is complete. ■

Now, consider W_m to be the wheel graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and $d_{W_m}(v_1) = m - 1$. Note that for $m \geq 5$, $\mathrm{Aut}(W_m)$ is isomorphic to $\mathrm{Aut}(S_m)$. Hence, $ab \ltimes W_m$ is decomposable into $ab \ltimes S_m \cup (a \Box C) \cup (b \Box C)$ where $C = v_2 v_3 \ldots v_m v_2$. For each $k = 2, 3, \ldots, m$, define,

$$\mathcal{P}_{ab}^{(k)} = \{\mathcal{P}_{ab}^{(k,j)} = (b, v_k)(a, v_j)(a, v_{j+1})(b, v_k) | 2 \leq j \leq m - 1\},$$

$$\mathcal{Q}_a = \{(a, v_2)(a, v_3) \ldots (a, v_m)(a, v_2)\}.$$

Analogously, we define $\mathcal{Q}_b$ (see Figure 2).

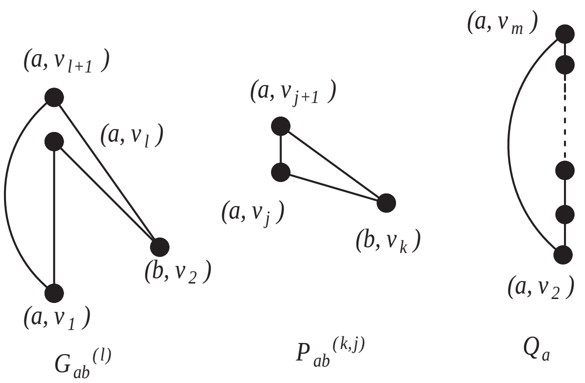

Figure 2. These graphs illustrate the cycles $\mathcal{G}_{ab}^{(l)}$, $\mathcal{P}_{ab}^{(k,j)}$ and $\mathcal{Q}_a$.

Lemma 3.14. $(\cup_{k=2}^{m} \mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$ *is linearly independent.*

Proof. Since $\mathcal{P}_{ab}^{(k,j)}$ contains an edge of the form $(a, v_j)(a, v_{j+1})$ which is not in any other cycle of $\mathcal{P}_{ab}^{(k)}$, as a result $\mathcal{P}_{ab}^{(k)}$ is linearly independent for each

$k = 2, 3, \ldots, m$. Now by the inductive step, we assume that $\cup_{k=2}^{m-1}\mathcal{P}_{ab}^{(k)}$ is linearly independent. Note that $E(\cup_{k=2}^{m-1}\mathcal{P}_{ab}^{(k)}) \cap E(\mathcal{P}_{ab}^{(m)}) = E(a\Box v_2 v_3 \ldots v_m)$ which is an edge set of a path. Thus, $\cup_{k=2}^{m}\mathcal{P}_{ab}^{(k)}$ is linearly independent. Now, each cycle $\mathcal{P}_{ba}^{(m,j)}$ contains an edge of the form $(b, v_j)(b, v_{j+1})$ which occurs in no other cycles of $(\cup_{k=2}^{m}\mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$. Thus, $(\cup_{k=2}^{m}\mathcal{P}_{ab}^{(k)}) \cup \mathcal{P}_{ba}^{(m)}$ is linearly independent. The proof is complete. ■

Lemma 3.15. *If $n \geq 2$ and $m^2(n-1) - 4m(n-1) - 2m + 3n - 1 \leq 0$, then $m < 6$.*

Proof. $m^2(n-1) \leq 4m(n-1) + 2m - 3n + 1$. Thus, $m \leq 4 + 2/(n-1)$ $-3n/m(n-1) + 1/m(n-1)$ which implies that $m \leq 4 + 2 - 2/m(n-1)$. Hence, $m < 6$. ■

Lemma 3.16. *If $n \geq 2$ and $m^2(n-1) - 4m(n-1) - 2m + 2 \leq 0$, then $m < 6$.*

Proof. As in Lemma 3.15 we have that $m \leq 4 + 2/(n-1) - 2/m(n-1)$ which implies that $m \leq 4 + 2 - 2/m(n-1) < 6$. ■

Lemma 3.17. *If $n \geq 2$ and $m^2(n-1) - 7m(n-1) - 5m + 3n + 2 \leq 0$, then $m < 12$.*

Proof. As in Lemma 3.15, we have that $m \leq 7 + 5/(n-1) - 3n/m(n-1)$ $-2/m(n-1) < 12$.

Note that $P_n \ltimes W_m$ is decomposable into $P_n \ltimes S_m \cup (\cup_{i=1}^{n}(a_i \Box C))$ where $C = v_2 v_3 \ldots v_m v_2$. Thus, $|E(P_n \ltimes W_m)| = |E(P_n \ltimes S_m)| + (m-1)n$. Hence, $\dim \mathcal{C}(P_n \ltimes W_m) = (n-1)m^2 + 2m - mn - 1$. ■

Theorem 3.18. *For each wheel W_m of order $m \geq 5$ and path P_n of order $n \geq 2$, we have that $b(P_n \ltimes W_m) \leq 4$. Moreover, the equality holds if $n \geq 2$ and $m \geq 12$.*

Proof. Define $\mathcal{B}(P_n \ltimes W_m) = \mathcal{B}(P_n \ltimes S_m) \cup (\cup_{i=1}^{n-1}\mathcal{P}_{a_i a_{i+1}}^{(m)}) \cup \mathcal{P}_{a_n a_{n-1}}^{(m)} \cup (\cup_{i=1}^{n}\mathcal{Q}_{a_i})$ where $\mathcal{B}(P_n \ltimes S_m)$ is defined as in Theorem 3.11. By Lemma 3.14 each one of $\mathcal{P}_{a_i a_{i+1}}^{(m)}$ and $\mathcal{P}_{a_n a_{n-1}}^{(m)}$ is linearly independent. Since $E(\mathcal{P}_{a_i a_{i+1}}^{(m)}) \cap E(\mathcal{P}_{a_l a_{l+1}}^{(m)}) = \varnothing$ whenever $i \neq l$, $\cup_{i=1}^{n-1}\mathcal{P}_{a_i a_{i+1}}^{(m)}$ is linearly independent. Now, each linear combination of cycles of $\mathcal{P}_{a_n a_{n-1}}^{(m)}$ contains at least one edge of

$E(a_n\square v_1v_2\dots v_m)$ which is not in any cycle of $\cup_{i=1}^{n-1}\mathcal{P}^{(m)}_{a_ia_{i+1}}$. Thus $(\cup_{i=1}^{n-1}\mathcal{P}^{(m)}_{a_ia_{i+1}})\cup\mathcal{P}^{(m)}_{a_na_{n-1}}$ is linearly independent. $E(\mathcal{Q}_{a_i})\cap E(\mathcal{Q}_{a_j})=\varnothing$ whenever $i\neq j$, also $\mathcal{Q}_{a_i}$ is the only cycle of $\mathcal{B}(P_n\ltimes W_m)$ containing $(a_i,v_m)(a_i,v_2)$ for each i. Therefore, $(\cup_{i=1}^{n-1}\mathcal{P}^{(m)}_{a_ia_{i+1}})\cup\mathcal{P}^{(m)}_{a_na_{n-1}}\cup(\cup_{i=1}^{n}\mathcal{Q}_{a_i})$ is linearly independent. Any linear combination of cycles of $(\cup_{i=1}^{n-1}\mathcal{P}^{(m)}_{a_ia_{i+1}})\cup\mathcal{P}^{(m)}_{a_na_{n-1}}\cup(\cup_{i=1}^{n}\mathcal{Q}_{a_i})$ contains at least one edge of the set $\cup_{i=1}^{n}E(a_i\square v_2v_3,\dots,v_mv_2)$ which is not in any cycle of $\mathcal{B}(P_n\ltimes S_m)$. Thus, $\mathcal{B}(P_n\ltimes W_m)$ is linearly independent. Since

$$\begin{aligned}|\mathcal{B}(P_n\ltimes W_m)| &= |\mathcal{B}(P_n\ltimes S_m)|+\sum_{i=1}^{n-1}|\mathcal{P}^{(m)}_{a_ia_{i+1}}|+|\mathcal{P}^{(m)}_{a_na_{n-1}}|+\sum_{i=1}^{n}|\mathcal{Q}_{a_i}|\\ &= m^2n-2mn-m^2+2m+(n-1)+(n-1)(m-2)\\ &\quad +(m-2)+n\\ &= (n-1)m^2+2m-mn-1\\ &= \dim\mathcal{C}(P_m\ltimes W_n),\end{aligned}$$

$\mathcal{B}(P_m\ltimes W_n)$ is a basis for $\mathcal{C}(P_m\ltimes W_n)$. Now we show that $\mathcal{B}(P_m\ltimes W_n)$ is a 4-fold basis. Let $e\in E(P_n\ltimes W_m)$. Then

(1) if $e=(a_i,v_1)(a_i,v_l)$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{L}_{a_{i-1}a_i}}(e)+f_{\mathcal{L}_{a_ia_{i+1}}}(e)\leq 2+2=4$.

(2) If $e=(a_i,v_j)(a_{i+1},v_l)$ such that $m>j,l\geq 2$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{L}_{a_ia_{i+1}}}(e)\leq 4$.

(3) If $e=(a_i,v_m)(a_{i+1},v_l)$ or $(a_i,v_l)(a_{i+1},v_m)$ such that $m>l\geq 2$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{L}_{a_ia_{i+1}}}+f_{\mathcal{P}^{(m)}_{a_ia_{i+1}}\cup\mathcal{P}^{(m)}_{a_na_{n-1}}}(e)\leq 2+2=4$.

(4) If $e=(a_i,v_1)(a_{i+1},v_1)$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{S}_{a_ia_{i+1}}}(e)=1$.

(5) If $e=(a_i,v_2)(a_i,v_m)$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{Q}_{a_i}}(e)=1$.

(6) If $e=(a_i,v_m)(a_{i+1},v_m)$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{H}_{a_ia_{i+1}}}(e)+f_{\mathcal{P}^{(m)}_{a_ia_{i+1}}}(e)+f_{\mathcal{P}^{(m)}_{a_na_{n-1}}}(e)\leq 1+1+1=3$.

(7) If $e=(a_i,v_j)(a_i,v_{j+1})$ such that $j\geq 2$ and $i\leq n-1$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{P}^{(m)}_{a_ia_{i+1}}}(e)+f_{\mathcal{Q}_{a_i}}(e)\leq 1+1=2$.

(8) If $e=(a_n,v_j)(a_n,v_{j+1})$ such that $j\geq 2$, then $f_{\mathcal{B}(P_n\ltimes W_m)}(e)=f_{\mathcal{P}^{(m)}_{a_na_{n-1}}}(e)+f_{\mathcal{Q}_{a_n}}(e)\leq 1+1=2$.

(9) If $e = (a_i, v_2)(a_{i+1}, v_m)$ or $(a_i, v_m)(a_{i+1}, v_2)$, then $f_{\mathcal{B}(P_n \ltimes W_m)}(e) = f_{\mathcal{L}_{a_n a_{n-1}}}(e) + f_{\mathcal{P}_{a_i a_{i+1}} \cup \mathcal{P}^{(m)}_{a_n a_{n-1}}}(e) \leq 2 + 1 = 2$.

On the other hand, to show that $b(P_n \ltimes W_m) \geq 4$ for any $n \geq 2$ and $m \geq 12$, we have to exclude any possibility for the cycle space $\mathcal{C}(P_n \ltimes W_m)$ to have a 3-fold basis for any $n \geq 2$ and $m \geq 12$. To this end, suppose that $\mathcal{B}$ is a 3-fold basis of the cycle space $\mathcal{C}(P_n \ltimes W_m)$ for any $n \geq 2$ and $m \geq 12$. First, suppose that $\mathcal{B}$ consists only of 3-cycles. Then $|\mathcal{B}| \leq 3(m-1)n$ because any 3-cycle must contain an edge of $E(a_i \square (v_2 v_3 \dots v_m v_2))$, for $i = 1, 2, \dots, n$ and each edge is of fold at most 3. This is equivalent to the inequality $m^2(n-1) - mn + 2m - 1 \leq 3(m-1)n$ which implies that $m^2(n-1) - 4m(n-1) - 2m + 3n - 1 \leq 0$ and so by Lemma 3.15, $m < 6$. This is a contradiction. Now, suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4. Then $4|\mathcal{B}| \leq 3|E(P_n \ltimes W_m)|$ because the length of each cycle of $\mathcal{B}$ greater than or equal to 4 and each edge is of fold at most 3. Thus, $4(m^2(n-1) - mn + 2m - 1) \leq 3(m^2(n-1) + 2m - 2)$ which is equivalent to $m^2(n-1) - 4m(n-1) - 2m + 2 \leq 0$ and so by Lemma 3.16, $m < 6$. This is a contradiction. Finally, Suppose that $\mathcal{B}$ consists of r 3-cycles and f cycles of length greater than or equal to 4. Then $f \leq \lfloor (3(m^2(n-1) + 2m - 2) - 3r)/4 \rfloor$ because the length of each cycle of r is 3 and each cycle of f is at least 4 and the fold of each edge is at most 3. Hence, $|\mathcal{B}| = r + f \leq r + \lfloor (3(m^2(n-1) + 2m - 2) - 3r)/4 \rfloor$ which implies that $4(m^2(n-1) - mn + 2m - 1) \leq r + 3(m^2(n-1) + 2m - 2)$. Thus, $4(m^2(n-1) - mn + 2m - 1) \leq 3(m-1)n + 3(m^2(n-1) + 2m - 2)$. By simplifying the inequality we have that $m^2(n-1) - 7m(n-1) - 5m + 3n + 2 \leq 0$. Thus, by Lemma 3.17 $m < 12$. This is a contradiction. The proof is complete. ■

Now, $C_n \ltimes W_m$ is decomposable into $P_n \ltimes W_m \cup a_1 a_m[N_{m-1}] \cup \{(a_1, v_1)(a_m, v_1)\}$ where N_{m-1} is the null graph with the vertex set $\{v_2, v_3, \dots, v_m\}$. Thus, $|E(C_n \ltimes W_m)| = |E(P_n \ltimes W_m)| + (m-1)^2 + 1$. Hence, $\dim \mathcal{C}(C_n \ltimes W_m) = \dim \mathcal{C}(P_n \ltimes W_m) + (m-1)^2 + 1 = nm^2 - mn + 1$. By employing the same ideas as in Lemma 3.6, we have the following result.

Lemma 3.19. *$C_n \square v_i$ is relevant in $C_n \ltimes W_m$.*

Theorem 3.20. *For each cycle C_n of order n and wheel W_m of order $m \geq 5$, we have that $b(C_n \ltimes W_m) \leq 4$. Moreover, the equality holds if and only if $n \geq 3$ and $m \geq 7$.*

Proof. Define $\mathcal{B}(C_n \ltimes W_m) = \mathcal{B}(P_n \ltimes W_m) \cup \mathcal{L}_{a_n a_1} \cup \{C_n \Box v_1\}$. By noting that $E(\mathcal{L}_{a_n a_1}) \cap E(\mathcal{B}(P_n \ltimes W_m)) = (a_1 \Box S_m) \cup (a_n \Box S_m)$ which is an edge set of a forest, we have that $\mathcal{B}(P_n \ltimes W_m) - \{C_n \Box v_1\}$ is linearly independent. By Lemma 3.18, $\mathcal{B}(P_n \ltimes W_m)$ is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_n \ltimes W_m)| &= |\mathcal{B}(P_n \ltimes W_m)| + |\mathcal{L}_{a_n a_1}| + 1 \\ &= nm^2 - mn + 1 \\ &= \dim \mathcal{C}(C_m \ltimes W_n), \end{aligned}$$

$\mathcal{B}(C_m \ltimes W_n)$ is a basis for $\mathcal{C}(C_m \ltimes W_n)$. Now we can easily show that $\mathcal{B}(C_m \ltimes W_n)$ is a 4-fold basis. To show that $\mathcal{C}(C_m \ltimes W_n)$ has no 3-fold basis we argue more or less as in the last paragraph of Theorem 3.18. The proof is complete. ■

4. The Minimum Cycle Bases of the Wreath Product of Graphs

In this section, we present minimum cycle bases (MCBs) for the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel. Moreover, we give the length of their maximum cycle.

Theorem 4.1. *$\mathcal{B}(P_n \ltimes P_m)$ is a minimum cycle basis of $P_n \ltimes P_m$.*

Proof. Recall that a MCB is obtained by a greedy algorithm, that is, an algorithm that selects independent cycles starting with the shortest ones from the set of all cycles. We consider two cases:

Case 1. m is odd. Then the girth of $P_n \ltimes P_m$ is 4. Since each cycle of $\mathcal{B}(P_n \ltimes P_m)$ is of length 4, as a result $\mathcal{B}(P_n \ltimes P_m)$ is a MCB.

Case 2. m is even. Note that the only 3-cycles of $P_n \ltimes P_m$ are $\cup_{i=1}^{n-1} \mathcal{Z}_{a_i a_{i+1}}$ and only three cycles of the four cycles of $\mathcal{Z}_{a_i a_{i+1}}$ are linearly independent for each $i = 1, 2, \ldots, n-1$. Thus, $\{\mathcal{Z}^{(1)}_{a_i a_{i+1}}, \mathcal{Z}^{(2)}_{a_i a_{i+1}}, \mathcal{Z}^{(3)}_{a_i a_{i+1}} | i = 1, 2, \ldots n-1\}$ is a set consisting of the largest number of 3-cycles linearly independent of $\mathcal{C}(P_n \ltimes P_m)$. Since $\{\mathcal{Z}^{(1)}_{a_i a_{i+1}}, \mathcal{Z}^{(2)}_{a_i a_{i+1}}, \mathcal{Z}^{(3)}_{a_i a_{i+1}} | i = 1, 2, \ldots n-1\} \subseteq \mathcal{B}(P_n \ltimes P_m)$ and $\mathcal{B}(P_n \ltimes P_m) - \{\mathcal{Z}^{(1)}_{a_i a_{i+1}}, \mathcal{Z}^{(2)}_{a_i a_{i+1}}, \mathcal{Z}^{(3)}_{a_i a_{i+1}} | i = 1, 2, \ldots n-1\}$ are 4-cycles, $\mathcal{B}(P_n \ltimes P_m)$ is MCB. The proof is complete. ■

Corollary 4.2.
$l(P_n \ltimes P_m) = \begin{cases} 4\left(mn - m - n + 2(n-1)\lfloor m/2 \rfloor + 1\right), & \textit{if } n \textit{ is odd,} \\ 4mn - 4m - 7n + 8(n-1)\lfloor m/2 \rfloor + 7, & \textit{if } n \textit{ is even.} \end{cases}$
$\lambda(P_n \ltimes P_m) = 4.$

Theorem 4.3. *For each $n \geq 4$, $\mathcal{B}(C_n \ltimes P_m)$ is a minimum cycle basis of $C_n \ltimes P_m$.*

Proof. By Lemma 3.6 and following, word by word, the same arguments as in the proof of Theorem 4.1 by taking into account that in Case 2 the set $\{\mathcal{Z}^{(1)}_{a_i a_{i+1}}, \mathcal{Z}^{(2)}_{a_i a_{i+1}}, \mathcal{Z}^{(3)}_{a_i a_{i+1}} | i = 1, 2, \ldots n-1\} \cup \{\mathcal{Z}^{(1)}_{a_n a_1}, \mathcal{Z}^{(2)}_{a_n a_1}, \mathcal{Z}^{(3)}_{a_n a_1}\}$ is consisting of the largest number of 3-cycles linearly independent of $\mathcal{C}(C_n \ltimes P_m)$, we have the result. The proof is complete. ∎

Corollary 4.4.
For $n \geq 4$, $l(C_n \ltimes P_m) = \begin{cases} 4mn - 3n + 8n\lfloor m/2 \rfloor, & \textit{if } n \textit{ is odd,} \\ 4mn - 6n + 8n\lfloor m/2 \rfloor, & \textit{if } n \textit{ is even.} \end{cases}$
and $\lambda(C_n \times P_m) = n.$

By noting that each of $P_n \ltimes S_m$ and $C_r \ltimes S_m$ has no 3-cycle for each $r \geq 4$ and by Theorems 3.11 and 3.13 and Lemma 3.12, we have the following result.

Theorem 4.5. *For each $r \geq 4$, $\mathcal{B}(P_n \ltimes S_m)$ and $\mathcal{B}(C_r \ltimes S_m)$ are minimum cycle bases.*

Corollary 4.6. *For each $r \geq 4$, $l(P_n \ltimes S_m) = 4(m^2 n - 2mn - m^2 + 2m + n - 1)$, $l(C_r \ltimes S_m) = 4(m^2 r - 2mr + r + 1), \lambda(P_n \ltimes P_m) = 4$ and $\lambda(C_r \ltimes S_m) = r$.*

The proof of the following result is a straightforward.

Lemma 4.7. *Let H be a subgraph of the graph G. Let A and B be a cycle basis and a minimum cycle basis of H and G, respectively. If $A \subseteq B$, then A is a minimum cycle basis of H.*

In the following result $\mathcal{B}_{a_i \square W_m}$ denotes to the cycle basis of the wheel $a_i \square W_m$ consisting of 3-cycles.

Theorem 4.8. $\mathcal{B}^*(P_n \ltimes W_m) = (\cup_{i=1}^{n-1} \cup_{j=2}^{m} \mathcal{P}^{(j)}_{a_i a_{i+1}}) \cup (\cup_{i=1}^{n-1} \mathcal{P}^{(m)}_{a_{i+1} a_i}) \cup (\cup_{i=1}^{n} \mathcal{B}_{a_i \square W_m}) \cup (\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}})$ *and* $\mathcal{B}^*(C_n \ltimes Wm) = \mathcal{B}^*(P_n \ltimes W_m) \cup$

$(\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)} \cup \mathcal{S}_{a_na_1} \cup \{C_n \times v_1\}$ *are minimum bases of* $P_n \ltimes W_m$ *and* $C_n \ltimes W_m$*, respectively.*

Proof. By Lemma 4.7, it is enough to show that $\mathcal{B}^*(C_n \ltimes Wm)$ is a minimum cycle basis of $C_n \ltimes W_m$ and $\mathcal{B}^*(P_n \ltimes W_m)$ is a cycle basis of $P_n \ltimes W_m$. By Lemma 3.14, each one of the two sets $(\cup_{j=2}^{m}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup \mathcal{P}_{a_{i+1}a_i}^{(m)}$ and $(\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)}$ is linearly independent. Note that

$$E\left(\left(\cup_{j=2}^{m}\mathcal{P}_{a_ka_{k+1}}^{(j)}\right) \cup \mathcal{P}_{a_{k+1}a_k}^{(m)}\right) \cap E\left(\cup_{i=1}^{k-1}\left(\left(\cup_{j=2}^{m}\mathcal{P}_{a_ia_{i+1}}^{(j)}\right) \cup \mathcal{P}_{a_{i+1}a_i}^{(m)}\right)\right)$$

$$= E(a_k\Box v_2v_3\ldots v_m)$$

which is an edge set of path for each $k = 1, 2, \ldots, n-1$ and

$$E\left(\left(\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}\right) \cup \mathcal{P}_{a_1a_n}^{(m)}\right) \cap E\left(\cup_{i=1}^{n-1}\left(\left(\cup_{j=2}^{m}\mathcal{P}_{a_ia_{i+1}}^{(j)}\right) \cup \mathcal{P}_{a_{i+1}a_i}^{(m)}\right)\right)$$

$$= E(a_1\Box v_2v_3\ldots v_m) \cup E(a_n\Box v_2v_3\ldots v_m)$$

which is an edge set of a forest. Thus,$(\cup_{i=1}^{n-1}\cup_{j=2}^{m}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup (\cup_{i=1}^{n-1}\mathcal{P}_{a_{i+1}a_i}^{(m)}) \cup (\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)}$ is linearly independent set. Now, for each $i = 1, 2, \ldots, n$, $\mathcal{B}_{a_i\times W_m}$ is a cycle basis of $a_i\Box W_m$. Since $E(\mathcal{B}_{a_i\Box W_m}) \cap E(\mathcal{B}_{a_j\Box W_m}) = \phi$ whenever $i \neq j$, $\cup_{i=1}^{n}\mathcal{B}_{a_i\Box W_m}$ is linearly independent. Now any linear combination of $\cup_{i=1}^{n}\mathcal{B}_{a_i\Box W_m}$ contains an edge of $\cup_{i=1}^{n}E(a_i\Box(W-S))$ which is not in any cycle of $(\cup_{i=1}^{n-1}\cup_{j=2}^{m}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup (\cup_{i=1}^{n-1}\mathcal{P}_{a_{i+1}a_i}^{(m)}) \cup (\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)}$ where S is the star graph which is obtained from W_m by deleting the edges of the cycle $v_2v_3\ldots v_mv_2$, as a result $(\cup_{j=1}^{n}\cup_{i=1}^{m-1}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1}\mathcal{P}_{a_na_1}^{(j)}) \cup (\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)} \cup (\cup_{i=1}^{n}\mathcal{B}_{a_i\Box W_m})$ is linearly independent. Now, $(\cup_{i=1}^{n-1}\mathcal{S}_{a_ia_{i+1}}) \cup \mathcal{S}_{a_na_1}$ is a cycle basis of the planar graph $P_n\Box v_1v_2$ which obtained by pasting all the cycle of $(\cup_{i=1}^{n-1}\mathcal{S}_{a_ia_{i+1}}) \cup \mathcal{S}_{a_na_1}$, which are 4-cycles, at the common edges of the successive cycles. Note that any linear combinations of cycles of $(\cup_{i=1}^{n-1}\mathcal{S}_{a_ia_{i+1}}) \cup \mathcal{S}_{a_na_1}$ contains an edge of $E(P_n\Box v_1)$ which is not in any cycle of $(\cup_{j=1}^{n}\cup_{i=1}^{m-1}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1}\mathcal{P}_{a_na_1}^{(j)}) \cup (\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)} \cup (\cup_{i=1}^{n}\mathcal{B}_{a_i\Box W})$, thus $(\cup_{j=1}^{n}\cup_{i=1}^{m-1}\mathcal{P}_{a_ia_{i+1}}^{(j)}) \cup (\cup_{i=1}^{m-1}\mathcal{P}_{a_na_1}^{(j)}) \cup (\cup_{j=2}^{m}\mathcal{P}_{a_na_1}^{(j)}) \cup \mathcal{P}_{a_1a_n}^{(m)} \cup (\cup_{i=1}^{n}\mathcal{B}_{a_i\Box W}) \cup (\cup_{i=1}^{n-1}\mathcal{S}_{a_ia_{i+1}}) \cup \mathcal{S}_{a_na_1}$ is linearly independent. Now By Lemma 3.19, $C_n\Box v_1$ is relevant. Thus, $\mathcal{B}^*(C_m \ltimes W_n)$ is a linearly independent. Since

$$\begin{aligned} |\mathcal{B}^*(C_n \ltimes W_m)| &= (m-1)(m-2)(n-1) + (n-1)(m-2) + (m-1)n \\ &\quad +(n-1) + (m-2)(m-1) + (m-2) + 1 + 1 \\ &= m^2n - mn + 1 = \dim \mathcal{C}(C_m \ltimes W_n) \end{aligned}$$

$\mathcal{B}^*(C_n \ltimes W_m)$ is a cycle basis of $C_n \ltimes W_m$. Since each cycle of $\mathcal{B}^*(C_m \ltimes W_n) - \{(\cup_{i=1}^{n-1} \mathcal{S}_{a_i a_{i+1}}) \cup \mathcal{S}_{a_n a_1} \cup (C_n \Box v_1)\}$ is of length three and since the smallest cycle contains any edge of $(a_i, v_1)(a_{i+1}, v_1)$, $(a_1, v_1)(a_n, v_1)$ is of length 4 and by Lemma 3.19, we have that each cycle of $\mathcal{B}^*(C_n \ltimes W_m)$ is relevant in $C_n \ltimes W_m$. Therefore, $\mathcal{B}^*(C_m \ltimes W_n)$ is a minimum cycle basis $C_m \ltimes W_n$. Since $\mathcal{B}^*(P_m \ltimes W_n) \subset \mathcal{B}^*(C_m \ltimes W_n)$ and $|\mathcal{B}^*(P_m \ltimes W_n)| = m^2n - mn - m^2 + 2m - 1 = \dim \mathcal{C}(P_m \ltimes W_n)$, we have that $\mathcal{B}^*(P_m \ltimes W_n)$ is a cycle basis of $P_m \ltimes W_n$. The proof is complete. ■

Corollary 4.9. *$l(P_n \ltimes W_m) = 3m^2n - 3mn - 3m^2 + 6m - 3$, $l(C_n \ltimes W_m) = 3m^2n - 3mn + n$, $\lambda(P_n \ltimes W_m) = 4$ and $\lambda(C_n \ltimes W_m) = n$.*

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