

ON THE  $p$ -DOMINATION NUMBER OF  
CACTUS GRAPHS

MOSTAFA BLIDIA

MUSTAPHA CHELLALI

*Department of Mathematics, University of Blida*  
*B.P. 270, Blida, Algeria*

**e-mail:** mblidia@hotmail.com

**e-mail:** mchellali@hotmail.com

AND

LUTZ VOLKMANN

*Lehrstuhl II für Mathematik, RWTH Aachen*  
*Templergraben 55, D-52056 Aachen, Germany*

**e-mail:** volkm@math2.rwth-aachen.de

**Abstract**

Let  $p$  be a positive integer and  $G = (V, E)$  a graph. A subset  $S$  of  $V$  is a  $p$ -dominating set if every vertex of  $V - S$  is dominated at least  $p$  times. The minimum cardinality of a  $p$ -dominating set of  $G$  is the  $p$ -domination number  $\gamma_p(G)$ . It is proved for a cactus graph  $G$  that  $\gamma_p(G) \leq (|V| + |L_p(G)| + c(G))/2$ , for every positive integer  $p \geq 2$ , where  $L_p(G)$  is the set of vertices of  $G$  of degree at most  $p - 1$  and  $c(G)$  is the number of odd cycles in  $G$ .

**Keywords:**  $p$ -domination number, cactus graphs.

**2000 Mathematics Subject Classification:** 05C69.

## 1. Introduction

Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *order* of  $G$  is  $n(G) = |V(G)|$  and the *degree* of a vertex  $v$ , denoted by  $\deg_G(v)$ , is the number of vertices adjacent to  $v$ . A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A vertex of  $V$  is called a *cut vertex* if removing it from  $G$  increases the number of components of  $G$ . A graph  $G$  is called a *cactus graph* if each edge of  $G$  is contained in at most one cycle. A *unicycle graph* is a graph with exactly one cycle. A tree  $T$  is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively  $p$  and  $q$  leaves attached at each support vertex, is denoted by  $S_{p,q}$ .

For a positive integer  $p$ , a subset  $S$  of  $V(G)$  is a  *$p$ -dominating set* if every vertex not in  $S$  is adjacent to at least  $p$  vertices of  $S$ . The  *$p$ -domination number*  $\gamma_p(G)$  is the minimum cardinality of a  $p$ -dominating set of  $G$ . Note that every graph  $G$  has a  $p$ -dominating set, since  $V(G)$  is such a set. Also the 1-domination number  $\gamma_1(G)$  is the usual *domination number*  $\gamma(G)$ . The concept of  $p$ -domination was introduced by Fink and Jacobson [2, 3]. For more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [4, 5].

We make a straightforward observation.

**Observation 1.** Every  $p$ -dominating set of a graph  $G$  contains any vertex of degree at most  $p - 1$ .

In this paper we present an upper bound for the  $p$ -domination number for cactus graphs in terms of the order, the number of odd cycles and the number of vertices of degrees at most  $p - 1$ .

The following result due to Blidia et al. [1] will be useful for the next. Let  $L_p(G)$  denote the set  $\{x \in V(G) : \deg_G(x) \leq p - 1\}$ .

**Theorem 2** (Blidia, Chellali and Volkmann [1]). *Let  $p$  be a positive integer. If  $G$  is a bipartite graph then*

$$\gamma_p(G) \leq (n + |L_p(G)|)/2.$$

## 2. Main Results

We begin by giving an upper bound for the  $p$ -domination number for connected unicycle graphs.

**Theorem 3.** *Let  $p \geq 2$  be a positive integer. If  $G$  is a connected unicycle graph then*

$$\gamma_p(G) \leq (n + |L_p(G)| + 1)/2$$

*and this bound is sharp.*

**Proof.** Let  $G$  be a connected unicycle graph. If  $G$  is bipartite then the result is valid by Theorem 2. So assume that  $G$  contains an odd cycle denoted by  $C$ . If  $G = C$ , then  $\gamma_p(G) = n$  if  $p \geq 3$  and  $\gamma_p(G) = (n + 1)/2$  if  $p = 2$ , in both cases the result holds. Thus we assume that  $G \neq C$ , that is  $G$  contains at least one leaf.

Suppose that the result does not hold and let  $G$  be the smallest connected unicycle graph such that  $\gamma_p(G) > (n + |L_p(G)| + 1)/2$ . We claim that every vertex on  $C$  has degree exactly  $p$ . Suppose to the contrary that there is a vertex  $x \in C$  such that  $\deg_G(x) \neq p$  and let  $y$  be one of its two neighbors on  $C$ . Let  $G'$  be the spanning graph of  $G$  obtained by removing the edge  $xy$ . Then  $G'$  is tree and so a bipartite graph. We also have  $|L_p(G')| \leq |L_p(G)| + 1$  and  $n(G) = n(G')$ . According to Theorem 2, we deduce that

$$\gamma_p(G) \leq \gamma_p(G') \leq (n(G') + |L_p(G')|)/2 \leq (n(G) + |L_p(G)| + 1)/2,$$

a contradiction with our assumption.

Similarly, we will show that every vertex not on  $C$  and different to a leaf has degree at least  $p$ . Assume to the contrary that there is a vertex  $x \in V(G) - C$  different to a leaf with  $\deg_G(x) \leq p - 1$  and let  $z$  be its neighbor in the unique path from  $x$  to  $C$ . Let  $G_1$  be the connected unicycle subgraph of  $G$  containing  $x$  and obtained by removing all the edges incident to  $x$  excepted the edge  $xz$ , and let  $G_2$  be the component containing  $x$  by removing the edge  $xz$ . Let  $D_1$  and  $D_2$  denote a  $\gamma_p(G_1)$ -set and a  $\gamma_p(G_2)$ -set, respectively. Clearly  $G_1$  contains  $C$  and  $G_2$  is a tree,  $x \in D_1 \cap D_2$ ,  $x \in L_p(G_1) \cap L_p(G_2)$ ,  $|L_p(G_1)| + |L_p(G_2)| = |L_p(G)| + 1$  and  $n(G_1) + n(G_2) = n(G) + 1$ . Furthermore,  $D_1 \cup D_2$  is a  $p$ -dominating set of  $G$ . In addition,  $G_1$  and  $G_2$  have order less than  $G$  and so satisfy the theorem, implying that

$$\begin{aligned} \gamma_p(G) &\leq |D_1 \cup D_2| = \gamma_p(G_1) + \gamma_p(G_2) - 1 \\ &\leq (n(G_1) + |L_p(G_1)| + 1)/2 + (n(G_2) + |L_p(G_2)|)/2 - 1 \\ &\leq (n + |L_p(G)| + 1)/2, \end{aligned}$$

contradicting the assumption.

Suppose now that  $V(G) - C$  contains a support vertex. Let  $a$  be a support vertex of  $G$  of maximum distance from  $C$ . As seen above,  $a$  has degree at least  $p$ . Let  $G' = G - (L_a \cup \{a\})$ . Then  $\gamma_p(G') + |L_a| = \gamma_p(G)$ ,  $n(G') = n(G) - |L_a| - 1$  and  $L_p(G) \geq L_p(G') + |L_a| - 1$ . It follows that

$$\gamma_p(G') + |L_a| = \gamma_p(G) > (n(G) + |L_p(G)| + 1)/2$$

implying that

$$\gamma_p(G') > (n(G) + |L_p(G)| + 1 - 2|L_a|)/2$$

and so

$$\gamma_p(G') > (n(G') + |L_p(G')| + 1)/2$$

contradicting our assumption that  $G$  is the smallest graph that does not satisfy the theorem.

Consequently, every vertex of  $V(G) - C$  must be a leaf and so every vertex on  $C$  is adjacent to exactly  $p - 2$  leaves, which implies that

$$\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$$

a contradiction.

To see that this bound is sharp, consider the graph  $G$  formed by an odd cycle  $C$  where each vertex on  $C$  is adjacent to exactly  $p - 2$  vertices. Then  $\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$ . ■

**Theorem 4.** *Let  $p \geq 2$  be a positive integer. If  $G$  is a connected cactus graph with  $c(G)$  odd cycles then,*

$$\gamma_p(G) \leq (n + |L_p(G)| + c(G))/2,$$

*and this bound is sharp.*

**Proof.** If  $G$  is a bipartite graph, then by Theorem 2 the result holds. If  $G$  is a unicycle graph then by Theorem 3 the result is also valid. So consider a cactus graph  $G$  containing at least two cycles with one of odd length. Assume that the result does not hold and let  $G$  be the smallest cactus graph such that  $\gamma_p(G) > (n(G) + |L_p(G)| + c(G))/2$ . We also assume that among all such graphs,  $G$  is the one having the fewest edges.

First, let  $u$  be a vertex on an odd cycle  $C$  of  $G$  and assume that  $\deg_G(u) \neq p$ . Let  $G'$  be the spanning graph of  $G$  obtained by removing an

edge of  $C$  incident with  $u$ . Then  $|L_p(G')| \leq |L_p(G)| + 1$  and  $c(G') = c(G) - 1$ . Also  $G'$  satisfies the result and so

$$\begin{aligned} \gamma_p(G) &\leq \gamma_p(G') \leq (n(G') + |L_p(G')| + c(G'))/2 \\ &\leq (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n + |L_p(G)| + c(G))/2, \end{aligned}$$

a contradiction. Thus every vertex in an odd cycle has degree exactly  $p$ .

Now consider a vertex  $v$  different from a leaf and contained in no odd cycle. Then, either  $v$  is a cut vertex or  $v$  is on an even cycle and  $\deg_G(v) = 2$ . Suppose first that  $v$  is a cut vertex with  $\deg_G(v) < p$ . Let  $G_1$  and  $G_2$  be two connected cactus subgraphs of  $G$  with  $V(G) = V(G_1) \cup V(G_2)$  having  $v$  as a unique common vertex. Then,  $c(G) = c(G_1) + c(G_2)$ ,  $n(G) = n(G_1) + n(G_2) - 1$ ,  $|L_p(G)| = |L_p(G_1)| + |L_p(G_2)| - 1$ . Now let  $D_1$  and  $D_2$  denote a  $\gamma_p(G_1)$ -set and a  $\gamma_p(G_2)$ -set, respectively. Then  $v \in D_1 \cup D_2$  and  $D_1 \cup D_2$  is a  $p$ -dominating set of  $G$ . Since  $G_1$  and  $G_2$  satisfy the result,

$$\begin{aligned} \gamma_p(G) &\leq |D_1 \cup D_2| = |D_1| + |D_2| - 1 \\ &\leq (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 - 1 \\ &\leq (n(G) + |L_p(G)| + c(G))/2, \end{aligned}$$

a contradiction. Consequently, every cut vertex contained in no odd cycle has degree at least  $p$ .

Now let  $v$  be a vertex on an even cycle with  $\deg_G(v) = 2$ . Since we have assumed in the beginning of the proof that  $G$  has at least two cycles, we have  $p \geq 3$ . We claim that each neighbor of  $v$  has degree exactly  $p$ . Indeed, let  $u$  be a neighbor of  $v$  and assume that  $\deg_G(u) \neq p$ . Then every  $\gamma_p(G')$ -set  $S$  is a  $p$ -dominating set of  $G$  where  $G'$  is obtained from  $G$  by removing the edge  $vu$ . So

$$\gamma_p(G) \leq |S| \leq (n(G') + |L_p(G')| + c(G'))/2 = (n(G) + |L_p(G)| + c(G))/2,$$

a contradiction. Thus  $\deg_G(u) = p$ .

Now let  $C$  denote an odd cycle of length at least 5 and let  $w$  be a vertex on  $C$ ,  $a$  and  $b$  its neighbors on  $C$ . Delete the edges  $wa, wb$ . The remaining graph has two components for otherwise  $wa$  or  $wb$  would be contained in two cycles. Let  $G_1$  be the component containing  $w$  and  $G_2$  the other component where a new edge is added joining  $a$  and  $b$ . Then both  $G_1$  and  $G_2$  verify the theorem. Also  $\deg_{G_2}(a) = \deg_{G_2}(b) = p$ ,  $|L_p(G_1)| + |L_p(G_2)| \leq |L_p(G)| + 1$

and  $c(G_1) + c(G_2) = c(G) - 1$ . Let  $D_1$  and  $D_2$  be a  $\gamma_p(G_1)$ -set and a  $\gamma_p(G_2)$ -set, respectively. Then  $D_1$  contains  $w$  since  $\deg_{G_1}(w) = p - 2$ . It can be checked that  $D_1 \cup D_2$  is a  $p$ -dominating set of  $G$ . It follows that

$$\begin{aligned} \gamma_p(G) &\leq |D_1 \cup D_2| \\ &\leq (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 \\ &\leq (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n(G) + |L_p(G)| + c(G))/2 \end{aligned}$$

contradicting our assumption. Thus it remains to investigate the case that each odd cycle is a triangle.

Let  $C = uvw$  be a triangle of  $G$ . If  $p = 2$  then as claimed before  $G = C_3$  and the theorem is valid. So assume that  $p \geq 3$ . Let  $G_u, G_v$  and  $G_w$  be the three components of  $G$  containing  $u, v, w$ , respectively, by removing the edges  $uv, uw$  and  $vw$ . Suppose that each component contains at most one vertex of degree at least  $p$  and let  $j$  the number of vertices of degree at least  $p$  in the three components. Then  $j \leq 3$  and  $|L_p(G)| = n - 3 - j$ . In this case,  $G_u$  is either a star of center vertex  $u$  with  $p - 2$  leaves, or star of order at least 4 where  $u$  is a leaf if  $p = 3$ , or a double star  $S_{p-3, p-1}$  with  $u$  as a support vertex if  $p \geq 4$ , or a graph formed by a cycle  $C_4$  where  $u \in V(C_4)$  and is adjacent to  $p - 4$  leaves (if  $p \geq 4$ ), its neighbors on the cycle have degree 2 and the remaining vertex of the cycle is adjacent to  $p - 2$  leaves. Likewise  $G_v$  and  $G_w$ . If each component is a tree then  $G$  is a unicycle and the result follows by Theorem 3. So we assume that  $G_u$  is a component containing the cycle  $C_4$ . Now it is a routine matter to check that

$$\gamma_p(G) = n - (j + 1) \leq (n(G) + |L_p(G)| + c(G))/2 = n - 1 - j/2,$$

a contradiction.

Thus we may assume, without loss of generality, that  $G_u$  contains at least two vertices of degree at least  $p$ . Let  $G'$  be the component containing  $v, w$  by removing the edges  $uv, uw$ . Let  $G_0$  be the graph constructed from  $G'$  by attaching  $v$  and  $w$  to the support vertices say  $a, b$  of a double star  $S_{p-2, p-2}$  (so  $v, w, a, b$  induce a cycle  $C_4$ ) and let  $D_u$  and  $D_0$  a  $\gamma_p(G_u)$ -set and a  $\gamma_p(G_0)$ -set, respectively. Then, without loss of generality,  $D_0$  contains  $v, w, a$  all the leaves adjacent to  $a$  and  $b$ . Also  $D_u$  contains  $u$  since it has degree at most  $p - 2$ . Obviously  $D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))$  is a  $p$ -dominating set of  $G$ . It is easy to check that  $G_u$  contains at least  $2p - 1$  vertices. Thus  $G_0$  has order less than  $G$  since we have added  $2p - 2$  vertices and so both

$G_u, G_0$  verify the result. On the other hand,  $n(G) = n(G_u) + n(G_0) - 2p + 2$ ,  $L_p(G) = L_p(G_u) - 1 + L_p(G_0) - 2p + 4$ ,  $c(G) = c(G_u) + c(G_0) + 1$ . Consequently

$$\begin{aligned} \gamma_p(G) &\leq |D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))| = \gamma_p(G_u) + \gamma_p(G_0) - 2p + 3 \\ &\leq (n(G_u) + |L_p(G_u)| + c(G_u))/2 \\ &\quad + (n(G_0) + |L_p(G_0)| + c(G_0))/2 - 2p + 3 \\ &\leq (n(G) + |L_p(G)| + c(G))/2, \end{aligned}$$

a contradiction with our assumption.

That this bound is sharp may be seen by considering the graph  $G_k$  formed by  $k \geq 1$  triangles where each vertex of the triangle is attached to  $p - 2$  leaves, and identifying a vertex of every triangle with a vertex of a path  $P_k$ . Then  $n(G_k) = (3p - 3)k$ ,  $|L_p(G_k)| = 3(p - 2)k$ ,  $c(G_k) = k$  and  $\gamma_p(G) = (n(G_k) + |L_p(G_k)| + c(G_k))/2 = (3p - 4)k$ . ■

## References

- [1] M. Blidia, M. Chellali and L. Volkmann, *Some bounds on the  $p$ -domination number in trees*, submitted for publication.
- [2] J.F. Fink and M.S. Jacobson,  *$n$ -domination in graphs*, in: Y. Alavi and A.J. Schwenk, eds, *Graph Theory with Applications to Algorithms and Computer Science* (Wiley, New York, 1985) 283–300.
- [3] J.F. Fink and M.S. Jacobson, *On  $n$ -domination,  $n$ -dependence and forbidden subgraphs*, in: *Graph Theory with Applications to Algorithms and Computer Science* (Wiley, New York, 1985) 301–312.
- [4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), *Domination in Graphs: Advanced Topics* (Marcel Dekker, New York, 1998).

Received 24 March 2004

Revised 26 August 2004