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# THE CYCLE-COMPLETE GRAPH RAMSEY NUMBER $R(C_5, K_7)$

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**Abstract**

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer $N$ such that every graph $G$ of order $N$ contains a cycle $C_m$ on $m$ vertices or has independence number $\alpha(G) \geq n$. It has been conjectured by Erdős, Faudree, Rousseau and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). This conjecture holds for $3 \leq n \leq 6$. In this paper we will present a proof for $r(C_5, K_7) = 25$.

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## 1. Introduction

We use [3] for terminology and notation not defined here and consider finite and simple graphs only.

For two graphs $G$ and $H$, the Ramsey number $r(G, H)$ is the smallest integer $N$ such that every 2-colouring of the edges of the complete graph $K_N$ contains a subgraph isomorphic to $G$ in the first colour or a subgraph isomorphic to $H$ in the second colour.

A cycle on $m$ vertices will be denoted by $C_m$ and the independence number of a graph by $\alpha(G)$. The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer $N$ such that for every graph $G$ of order

$N$, $C_m \subset G$ or $\alpha(G) \geq n$. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m-1)(n-1)+1$ for all $m \geq n \geq 3$.

**Question 1** [5]**.** With $n$ given, what is the smallest value of $m$ such that

$$r(C_m, K_n) = (m-1)(n-1)+1 \text{ ?} \tag{1}$$

**Conjecture 1** [5]**.** With the only exception of $r(C_3, K_3) = 6$, formula (1) holds for all $m \geq n \geq 3$.

## 2. Results

The following observation is easily verified.

**Observation 1.** Formula (1) also holds for $n = 1, 2$ and all $m \geq 3$.

Conjecture 1 was confirmed for $n = 3$ in early work on Ramsey theory ([6], [12]), and it has been proved recently for $n = 4$ [14], $n = 5$ [2] and $n = 6$ [13].

Table 1. Exact Values of $r(C_m, K_n)$.

| $m \backslash n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| 3 | 6 | 9 | 14 | 18 | 23 | 28 | 36 |
| 4 | 7 | 10 | 14 | 18 | 22 | 26 | |
| 5 | 9 | 13 | 17 | 21 | 25 | | |
| $\geq 6$ | $2m-1$ | $3m-2$ | $4m-3$ | $5m-4$ | | | |

Bondy and Erdős [1] have proved that formula (1) holds if $m \geq n^2 - 2$. This was improved by Thomason [15] to $m \geq n^2 - n - 1$ for all $n \geq 4$ and further to $m \geq n^2 - 2n$ for all $n \geq 5$ in [13]. Recently, Nikiforov succeeded to show a lower bound which is linear in $n$.

**Theorem 1** [9]**.** *$r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq 4n+2$ and all $n \geq 4$.*

Nikiforov has also posed the following challenging conjecture.

**Conjecture 2.** For every $k$ there exists $n_0 = n_0(k)$ such that for $n > n_0$ and $m > n^{1/k}$,

$$r(C_m, K_n) = (m-1)(n-1)+1.$$

The known numbers for small values of $m$ and $n$ do not contradict this conjecture.

In [8] it has been proved that $r(C_5, K_6) = 21$. In this paper we will compute $r(C_5, K_7)$.

**Theorem 2.** $r(C_5, K_7) = 25$.

Moreover, the fact that $r(C_5, K_6) = 21$ and $r(C_5, K_7) = 25$, justifies the following question.

**Question 2.** Does Formula (1) hold for all $m \geq 5$?

## 3. Preliminary Results

For a vertex $u \in V(G)$ let $N_i(u) = \{v \in V(G)|d(u,v) = i\}$ and $N_i^*(u) = \{v \in V(G)|d(u,v) \geq i\}$. For given $N_i(u)$ and $N_i^*(u)$ let $G_i = G[N_i(u)]$ and $G_i^* = G[N_i^*(u)]$.

**Lemma 1.** *Let $G$ be a $C_5$-free graph. Then the graphs $G_1$ and $G_2$ are $P_4$-free for every vertex $u \in V(G)$.*

***Proof.*** If $G_1 = G[N_1(u)]$ contains a $P_4$, then $u$ is contained in a $C_5$, a contradiction. Hence, $G_1$ is $P_4$-free.

Suppose now that $G_2$ contains a $P_4$ with vertices labeled $w_1w_2w_3w_4$. If $N(u) \cap N(w_1) \cap N(w_4) \neq \emptyset$, then there is a $C_5$, a contradiction. Hence we may assume that there are two vertices $u_1, u_2 \in V(G_1)$ such that $u_1w_1, u_2w_4 \in E(G)$. Now consider the vertex $w_2$. If $w_2v \in E(G)$ for a vertex $v \in V(G_1) - \{u_1\}$, then there is a $C_5$, a contradiction. Hence we may assume that $w_2u_1 \in E(G)$. Now consider the vertex $w_3$. Then $w_3$ is always contained in a $C_5$, a contradiction. Hence, $G_2$ is $P_4$-free. ■

The following lemma is an immediate consequence of Lemma 1.

**Lemma 2.** *Let $G$ be a $C_5$-free graph and $u \in V(G)$. Then the components of $G_1$ and $G_2$ are of the form $K_1, K_2, K_3$ or $K_{1,r}$ for $r \geq 2$.*

Using Lemma 2 we obtain the following lemma.

**Lemma 3.** *Let $G$ be a $C_5$-free graph with $\alpha(G) \leq 6$. Then*

(a) $\alpha(G_2) \leq 5$ *and* $|V(G_2)| \leq 15$,

(b) $\alpha(G_3^*) \leq 6 - \alpha(G_1)$ *and* $|V(G_3^*)| \leq 24 - 4\alpha(G_1)$,

(c) *If* $W \subset V(G_2)$, *then* $\alpha(G_2[W]) \geq \lceil \frac{|W|}{3} \rceil$.

Using the assumption that $G$ is $C_5$-free we obtain the following lemmas.

**Lemma 4.** *Let* $G$ *be a* $C_5$*-free graph and* $F \subset G$ *with* $F \cong K_4$. *Then* $d_F(v) \leq 1$ *for all* $v \in V(G) - V(F)$.

**Lemma 5.** *Let* $G$ *be a* $C_5$*-free graph with* $|V(G)| = 25$ *and* $\alpha(G) \leq 6$. *If* $I \subset V(G)$ *is independent with* $|I| = k$, $1 \leq k \leq 5$, *then* $|N(I)| \geq 3k + 1$.

***Proof.*** Suppose there is an independent set $I \subset V(G)$ with $|I| = k$, $1 \leq k \leq 5$, and $|N(I)| \leq 3k$. Let $G' = G - (I \cup N(I))$. Then $|V(G')| \geq 25 - 4k = 4(7-k) - 3$. Since $G$ is $C_5$-free, we conclude by Table 1 and Observation 1 that $\alpha(G') \geq 7 - k$. Let $J$ be an independent set of size $\alpha(G') \geq 7 - k$ in $G'$. Then $I \cup J$ is an independent set of size at least 7 in $G$, a contradiction. ■

The following two lemmas are easily verified using the fact that $G$ is $C_5$-free.

**Lemma 6.** *If* $F_i$ *is a component of* $G_2$ *with* $|V(F_i)| \geq 2$, *then* $|N(F_i) \cap N(u)| = 1$.

**Lemma 7.** *Let* $F_1, F_2$ *be two components of* $G_1$. *If* $|V(F_2)| \geq 2$, *then* $N(F_1) \cap N(F_2) \cap V(G_2) = \emptyset$.

**Lemma 8.** *Let* $F \cong K_2$ *be a component of* $G_2$ *with* $V(F) = \{w_1, w_2\}$ *and* $J = N(w_1) \cap N(w_2) \cap V(G_3)$. *Then* $J$ *is independent.*

***Proof.*** Suppose $J$ is not independent. Then there is an edge in $G_3[J]$, say $xy$. By lemma 6 there is a vertex $v \in N(w_1) \cap N(w_2) \cap N(u)$. But then $C_5 \subseteq G[\{v, w_1, w_2, x, y\}]$, a contradiction. ■

Jayawardene and Rousseau have determined all $C_5$-free graphs $G$ with $\alpha(G) = 3$ and order 11 and 12.

**Lemma 9** [8]**.** *Let* $G$ *be a graph with* $C_5 \not\subset G$ *and* $\alpha(G) = 3$.

(a) *If* $|V(G)| = 12$, *then* $3K_4 \subset G$.

(b) *If* $|V(G)| = 11$, *then* $2K_4 \cup K_3 \subset G$.

For a vertex $u \in V(G)$, an independent set $I \subset V(G)$ of type $(n_0, n_1, \ldots, n_{k-1}, n_k^*)$ is an independent set of size $\sum_{i=0}^{k} n_i$, which contains $n_i$ vertices from $G_i$, $1 \leq i \leq k-1$, and $n_k^*$ vertices from $G_k^*$. Furthermore, $n_0 = 1$ $(0)$, if $u$ is (not) contained in $I$.

**Lemma 10.** *Let $G$ be a graph with $C_5 \not\subset G$. Suppose $G_2$ has five components $F_1, F_2, \ldots, F_5$ with $|V(F_i)| = 1$, $1 \leq i \leq p, |V(F_i)| = 2$, $p+1 \leq i \leq q$, $|V(F_i)| = 3$, $q+1 \leq i \leq 5$. Further there are vertices $u_i \in V(G_1)$ such that $G_2[N(u_i)] = F_i$ for $p+1 \leq i \leq q$ and $u_u u_j \in E(G)$ for $p+1 \leq i \leq q$. Suppose $q > p$ and $|V(G_3^*) - (\cup_{i=1}^{p} N(F_i))| \geq q-p+1$. Then there exists an independent set of type $(1,0,5,1)$ or $(1,0,4,2)$.*

***Proof.*** Suppose there is no independent set of type $(1,0,5,1)$. Since $|V(G_3^*) - (\cup_{i=1}^{p} N(F_i))| \geq q-p+1$ there exists $i$ with $p+1 \leq i \leq q$, say $i = p+1$, and two vertices $v_1, v_2 \in V(G_3)$ with $v_1 w_i, v_2 w_i \in E(G)$ for $i = 1, 2$, where $V(F_{p+1}) = \{w_1, w_2\}$. By Lemma 8, $v_1 v_2 \notin E(G)$. Since $G_1[\{u_{p+1}, \ldots, u_q\}]$ is complete and $G$ is $C_5$-free, we have $N(v_i) \cap V(F_j) = \emptyset$ for $i = 1, 2$ and $p+2 \leq j \leq q$. But then $v_1, v_2$ are contained in an independent set $I$ containing $F_i$ for $1 \leq i \leq p$ and a vertex from each $F_i$ for $p+2 \leq i \leq 5$. Hence $I$ is an independent set of type $(1,0,4,2)$, a contradiction. ■

**Lemma 11** [8]. *Let $G$ be a graph with $\delta(G) \geq 4$ and $C_5 \not\subset G$. Then $\alpha(G) \geq \Delta(G)$.*

## 4. Proof of Theorem 2

Let $|V(G)| = 25$. By Lemma 5 and Lemma 11 we may assume that $4 \leq \delta(G) \leq \Delta(G) \leq 6$. We distinguish these three cases.

1. $\Delta(G) = 4$

Then $G$ is 4-regular. Moreover, by Lemma 5, if $d(u,v) = 2$ for two vertices $u, v \in V(G)$, then

$$|N(u) \cap N(v)| = 1. \tag{2}$$

Hence $G$ contains no induced $K_4 - e$ and no induced $C_4$. For the neighbourhood of a vertex $u$ we distinguish the following cases.

*Case* 1. $\alpha(G_1) = 4$
By (2) we conclude that $|V(G_2)| = 3 \cdot 4 = 12$. Since $\alpha(G_2) \leq 6, F_i = G[N_{G_2}(u_i)] \cong K_3$ for some $i$ with $1 \leq i \leq 4$. But then $\alpha(G[N_{G_3}(F_i)]) = 3$. Hence there are three independent vertices in $N_{G_3}(F_i)$ which are contained together with $\{u_1, u_2, u_3, u_4\}$ in an independent set of type $(0, 4, 0, 3)$, a contradiction.

*Case* 2. $\alpha(G_1) = 3$
Let $E(G_1) = \{u_1u_2\}$ and $F_i = G[N_{G_2}(u_i)]$ with $V(F_i) = \{w_{i1}, w_{i2}\}$ for $i = 1, 2$. Suppose $F_i = G[N_{G_2}(u_i)] \cong K_3$ for some $i$ with $3 \leq i \leq 4$, say $i = 3$. Then $|N_{G_3}(F_3)| = 3$ and $\alpha(G[N_{G_3}(F_3)]) = 3$. By (2) $d_{F_1 \cup F_2}(v) \leq 1$ for all vertices $v \in N_{G_3}(F_3)$. Hence we may assume that $N_{G_3}(w_{11}) \cap N_{G_3}(F_3) = \emptyset$. But then $\{u_2, u_3, u_4, w_{11}\} \cup N_{G_3}(F_3)$ is an independent set of type $(0, 3, 1, 3)$, a contradiction.

Suppose now $\alpha(G[N_{G_2}(u_i)]) \geq 2$ for $3 \leq i \leq 4$. Since $\alpha(G_2) \leq 6$, we conclude $w_{11}w_{12}, w_{21}w_{22} \in E(G)$. Let $N_{G_3}(w_{ij}) = \{x_{ij1}x_{ij2}\}$ for $1 \leq i, j \leq 2$. Then there are three independent vertices in $N_{G_3}(w_{ij})$ for $ij = 12, 21, 22$. These three vertices are contained together with $w_{11}$ and $u_2, u_3, u_4$ in an independent set of type $(0, 3, 1, 3)$, a contradiction.

For the remaining part we may assume that $|E(G[N(v)])| \geq 2$ for every vertex $v \in V(G)$.

*Case* 3. $\alpha(G_1) = 2$
Let $E(G_1) = \{u_1u_2, u_3u_4\}$. Then $N_{G_2}(u_i) = V(F_i) = \{w_{i1}, w_{i2}\}$ with $w_{i1}w_{i2} \in E(G)$ for $1 \leq i \leq 4$. As above we conclude that there are three independent vertices in $N_{G_3}(w_{ij})$ for $ij = 32, 41, 42$ which are contained together with $u_2, u_4, w_{11}$ and $w_{31}$ in an independent set of type $(0, 2, 2, 3)$, a contradiction.

*Case* 4. $\alpha(G_1) = 2$
Let $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$. We may assume that $G[N(v)] \cong K_3 \cup K_1$ for every vertex $v \in V(G)$. Choose an edge $uw$ with $N(u) = \{w, u_1, u_2, u_3\}$ and $N(w) = \{u, w_1, w_2, w_3\}$ such that $G[\{u_1, u_2, u_3\}] \cong K_3 \cong G[\{w_1, w_2, w_3\}]$. Then there exist vertices $x_i$ and $y_i$ for $1 \leq i \leq 3$ such that $u_ix_i, w_iy_i \in E(G)$. Let $V(G_1) = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ and $V(G_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Hence $\alpha(G_2) \leq 5$. If $\alpha(G_2) = 5$, then there is an independent set of type $(1, 1, 5)$, a contradiction. Hence we may assume $\alpha(G_2) \leq 4$. Since $E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$ contains only independent edges, we may assume that $|E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})| = 2$ (else consider $u_1, x_1$ instead of $u, w$).

We may assume that $x_2, y_2$ and $x_3, y_3$ are contained in a $K_4$. Hence $|V(G_3)| = 2 \cdot 2 + 2 \cdot 3 = 10$. Therefore, $V(G_4^*) \neq \emptyset$ and so there is an independent set of type $(1, 1, 4, 0, 1)$ (with respect to the edge $uw$), a contradiction.

2. $\Delta(G) = 5$

*Case* 1. $\alpha(G_1) = 5$
Since $\alpha(G_1) = 5$ we conclude that $\alpha(G_3^*) \leq 1$ and thus $|V(G_2)| \geq 25 - (1 + 5 + 4) = 15$. By Lemma 3 we have $|V(G_2)| \leq 15$. Therefore, $G_2 \cong 5K_3$ and $G_3 \cong K_4$. Hence by Lemma 4 every vertex of $G_3$ is contained in an independent set of type $(1, 0, 5, 1)$, a contradiction.

*Case* 2. $\alpha(G_1) = 4$
Then $|V(G_3^*)| \leq 8$ by Lemma 3 and thus $|V(G_2)| \geq 11$.

*Case* 2.1. $E(G_1) = \{u_1u_2\}$
Let $U_1 = \{u_1, u_2\}$ and $U_2 = \{u_3, u_4, u_5\}$. By Lemma 5 we conclude that $|N_{G_2}(U_2)| \geq 9$. Since $\alpha(G_2[N(U_1)]) \geq 2$, we get $\alpha(G_2[N(U_1)]) = 2$ and $\alpha(G_2[N(U_2)]) = 3$ by Lemma 3 and Lemma 7. Moreover, $G_2[N(U_2)] \cong 3K_3$ by Lemma 6.

Let $J = \{u_3, u_4, u_5\}$ and $G' = G - (J \cup N(J))$. Then $|V(G')| = 12$ and $\alpha(G') \geq 3$ by Table 1. Since $I \cup J$ is an independent set in $G$ with $|I \cup J| \leq \alpha(G) \leq 6$ for every independent set $I$ of $G'$, we conclude $\alpha(G') = 3$. Hence $3K_4 \subset G'$ by Lemma 9. Therefore, $G_2[N(u_1)] = \{F_1, F_2\} \cong \{K_3, K_3\}$ and we follow the arguments of Case 1 above.

*Case* 2.2. $E(G_1) = \{u_1u_2, u_1u_3\}$
Let $U_1 = \{u_1, u_2, u_3\}$ and $U_2 = \{u_4, u_5\}$. Similarily as in the previous case we conclude that $\alpha(G_2[N(U_1)]) = 3$, $\alpha(G_2[N(U_2)]) = 2$ and $G_2[N(U_2)] \cong 2K_3$. Let $F_1, F_2, F_3$ be the three components of $G_2[N(U_1)]$ with $F_i = G_2[N(u_i)]$ for $i = 1, 2, 3$. Let $J = \{u_1, u_4, u_5\}$ and $G' = G - (J \cup N(J))$. Then $11 \leq |V(G')| \leq 12$ and thus $3K_4 \subset G'$ or $2K_4 \cup K_3 \subset G'$ by Lemma 9. Since $N_{G_3}(F_1)$ and $F_2, F_3$ are independent, $N_{G_3}(F_i) \cong K_3$ for $i = 2$ or 3. But then $H_i = N_{G_3}(F_i) \cong K_4$ is contained in a $K_4 \subset G'$ for $i = 2$ or 3, a contradiction.

*Case* 2.3. $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4\}$

*Case* 2.4. $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5\}$
For both cases let $J = \{u_2, u_3, u_4\}$ and $G' = G - (J \cup N(J))$. By Lemma 5 we need $|J \cup N(J)| \geq 13$. Since $4 \leq \alpha(G_2) \leq 5$ we conclude that

$G[N_{G_2}(u_i)] \cong K_3$ for some $i, 2 \le i \le 4$. Now we can follow the proof of Case 4 (by considering $u_i$ with $d(u_i) = 5$ instead of $u$).

*Case* 3. $\alpha(G_1) = 3$

*Case* 3.1. $E(G_1) = \{u_1u_2, u_3u_4\}$

As in previous cases we conclude that $\alpha(G_2) = 5$ and $G_2[N(u_5)] \cong K_3$. Suppose $G_2$ is isomorphic to one of $\{K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}, K_{n_5}\}$ with $2 \le n_1 \le n_2 \le 3, 2 \le n_3 \le n_4 \le 3, n_5 = 3$. If $d(w_1) = 4 = d(w_3)$, let $J = \{u, w_1, w_3\}$ and $G' = G - (J \cup N(J))$. Then $|V(G')| = 11$ and thus $2K_4 \cup K_3 \subset G'$ by Lemma 9. Since $F_2, F_4$ and $F_5$ are independent and $|V(F_i)| \ge 2$ for $i = 2, 4, 5$, there exist $F_i, i = 2, 4$ or $5$, such that $F_i$ is contained in a $K_4 \subset G' - \{u_i\}$, a contradiction. Suppose $G_2$ is isomorphic to one of $\{K_1, K_1, K_{n_3}, K_{n_4}, K_{n_5}\}$ with $2 \le n_3 \le n_4 \le 3, n_5 = 3$. Then $|V(G_3)| \ge 25 - 6 - (5 + n_3 + n_4) = 14 - n_3 - n_4$. Hence $|V(G_3) - (N(F_1) \cup N(F_2))| \ge 14 - n_3 - n_4 - 6 = 8 - n_3 - n_4$. Now by Lemma 10 there is an independent set of type $(1, 0, 5, 1)$ or $(1, 0, 4, 2)$, a contradiction.

Finally suppose that $G_2$ is isomorphic to $\{K_1, K_1, K_1, K_1, K_3\}$. Let $w_1, w_2, w_3, w_4 \in V(G_2)$ be four independent vertices with $N_{G_2}(u_1) = N_{G_2}(u_2) = \{w_1, w_2\}$ and $N_{G_2}(u_3) = N_{G_2}(u_4) = \{w_3, w_4\}$. If there is a vertex $v \in V(G_3^*)$ with $v \notin N(w_i)$ for $1 \le i \le 4$, then $v$ is contained in an independent set of type $(1, 0, 5, 1)$ by Lemma 4, a contradiction.

Hence we may assume that $V(G_3^*) = V(G_3) \subset N(w_1) \cup N(w_2) \cup N(w_3) \cup N(w_4)$. Furthermore, $d_{G_3}(w_i) = 3$ for $1 \le i \le 4$. Let $H_i = G[N_{G_3}(w_i)]$ for $1 \le i \le 4$. Since $|V(G_3)| = 12$ we have $V(H_i) \cap V(H_j) = \emptyset$ for $1 \le i < j \le 4$. Moreover, there are no edges between $V(H_i)$ and $V(H_{i+1})$ for $i = 1, 3$, since $G$ contains no $C_5$. Suppose $\alpha(G_2[H_i \cup H_{i+1}]) \ge 3$, then there are three independent vertices in $V(H_i) \cup V(H_{i+1})$, which are contained together with $u_i, u_5, w_{4-i}, w_{5-i}$ in an independent set of type $(0, 2, 2, 3)$, a contradiction.

Hence we may assume that $G_2[H_i \cup H_{i+1}] \cong 2K_3$ for $i = 1, 3$. Now any two vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$ are contained in an independent set of size four in $G_3$ by Lemma 4. Hence $4 \le \alpha(G_3) \le 3$ by Lemma 3, a contradiction.

*Case* 3.2. $E(G_1) = \{u_1u_2, u_1u_3, u_4u_5\}$

See Case 4.

*Case* 3.3. $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$

We first conclude that $\alpha(G_2) = 5$. Hence by Lemma 5 we get $G_2[N(u_i)] \cong K_3$ for $i = 4, 5$. We have $1 \le d_{G_2}(u_1) \le d_{G_2}(u_2) \le d_{G_2}(u_3) \le 2$. Let $F_i = G_2[N(u_i)]$ for $1 \le i \le 3$ and $J = \{u_1, u_4, u_5\}$. If $d_{G_2}(u_1) = 1$, then $G' = G[V(G) - (J \cup N(J))]$ has $|V(G')| = 12$. So $3K_4 \subset G'$ by Lemma 9. Since $N_{G_3}(F_1), V(F_2)$ and $V(F_3)$ are independent, $N_{G_3}(F_1)$ is contained in a $K_4 \subset G - F_1$. Hence there is a $C_5$, a contradiction.

If $d_{G_2}(u_i) = 2$ for $1 \le i \le 3$, then $|V(G')| = 11$. So $2K_4 \cup K_3 \subset G'$ by Lemma 9. Thus $F_i \cong K_2$ is contained in a $K_4 \subset G - u_i$ for some $i, 2 \le i \le 3$. Hence there is a $C_5$, a contradiction.

*Case* 4. $\alpha(G_1) = 2$

Then $G_1 = K_3 \cup K_2$. Let $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3, u_4u_5\}$.

Suppose first $N_{G_2}(u_4) = N_{G_2}(u_5) = \{w_4, w_5\}$ for two vertices $w_4, w_5 \in V(G_2)$. Let $F_i = G_2[N(u_i)]$ for $1 \le i \le 3$ and set $F_i = \{w_i\}$ for $i = 4, 5$. Now let $H_i = G_3[N(F_i)]$ for $1 \le i \le 5$, $J = \{u, w_4, w_5\}$ and $G' = G - (J \cup N(J))$. Then $11 \le |V(G')| \le 12$ by Lemma 5. Suppose $|V(G')| = 12$. Then $3K_4 \subset G'$ by Lemma 9. Thus $G[F_i \cup H_i] \cong K_4$ for $1 \le i \le 3$. Since there is no $C_5$, we have $|F_i| = 1$ and $|H_i| = 3$ for $1 \le i \le 3$. We may assume $|V(H_4)| = 2$ and $|V(H_5)| = 3$. Thus $H_i \cong K_3$ for $i = 1, 2, 3$ and 5 and $H_4 \cong K_2$. Since $E(H_4, H_5) = \emptyset$, there is always an independent set with four vertices, one from $H_2, H_3, H_4$ and $H_5$. Together with $w_1, u_2$ and $u_4$ this gives an independent set of type $(0, 2, 1, 4)$, a contradiction. Suppose now $|V(G')| = 11$. Then $2K_4 \cup K_3 \subset G'$ by Lemma 9. We can follow the arguments above and may assume that $|V(F_3) \cup V(H_3)| = K_3$. Again we can find an independent set of type $(0, 2, 1, 4)$ as above, a contradiction.

Suppose next $F_i = G[N_{G_2}(u_i)]$ for $i = 4, 5$ with $|V(F_i)| \ge 2$ for two independent components $F_4$ and $F_5$. Furthermore, $F_i = G[N_{G_2}(u_i)]$ for $i = 1, 2, 3$, since $\alpha(G_2) = 5$. We have $1 \le |V(F_1)| \le |V(F_2)| \le |V(F_3)| \le 2$. If $|V(F_i)| = 1$ (i.e., $F_i = \{w_i\}$) for some $i$ with $1 \le i \le 3$, then $d_{G_3}(w_i) = 3$, else we would be in a previous case.

Suppose there are two vertices $w_1 \in V(F_1)$ and $w_2 \in V(H_2)$ with $d(w_i) = 4$, $1 \le i \le 2$. Let $J = \{u, w_1, w_2\}$ and $G' = G - (J \cup N(J))$. Then $|V(G')| = 11$ and $2K_4 \cup K_3 \subset G'$ by Lemma 9. Thus $F_i$ is contained in a $K_4 \subset G' - \{u_i\}$ for some $i, 4 \le i \le 5$. But then there is a $C_5$, a contradiction. Hence we may assume that $V(F_i) = \{w_{i1}, w_{i2}\}$ for $i = 2, 3$ and $d_{G_3}(w_{ij}) = 3$ for $i = 2, 3$ and $1 \le j \le 2$. But then $|V(G)| \ge 1 + 5 + (1 + 2 \cdot 2 + 2 \cdot 2) + 4 \cdot 3 = 27 > 25$, a contradiction.

3. $\Delta(G) = 6$

*Case* 1. $\alpha(G_1) = 6$
Since $\alpha(G_1) = 6$ we conclude that $V(G_3^*) = \emptyset$ and thus $15 \geq |V(G_2)| = 25 - 7 = 18$ by Lemma 3, a contradiction.

*Case* 2. $\alpha(G_1) = 5$
Then $E(G_1) = \{u_1u_2, u_1u_3, \ldots, u_1u_r\}, 2 \leq r \leq 6$. Since $\alpha(G_1) = 5$ we conclude by Lemma 3 (b) that $|V(G_3^*)| \leq 4$ and thus $|V(G_2)| \geq 25 - 7 - 4 = 14$. Then $\alpha(G_2) \geq 5$ by Lemma 3 (c). Thus $\alpha(G_2^*) = 5$ and $G_2 \cong 5K_3$ or $G_2 \cong 4K_3 \cup K_2$. By Lemma 6 we conclude $|V(G_1)| \leq 5 < 6$, a contradiction.

*Case* 3. $\alpha(G_1) \leq 4$
Using Lemma 2 and Lemma 7 we can show that $\alpha(G_2) \geq 6$ and thus there is an independent set of type $(1, 0, 6)$, a contradiction. ■

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