ON \((k, l)\)-KERNEL PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

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Abstract

In the first part of this paper we give necessary and sufficient conditions for some special classes of digraphs to have a \((k, l)\)-kernel. One of them is the duplication of a set of vertices in a digraph. This duplication come into being as the generalization of the duplication of a vertex in a graph (see [4]). Another one is the \(D\)-join of a digraph \(D\) and a sequence \(\alpha\) of nonempty pairwise disjoint digraphs. In the second part we prove theorems, which give necessary and sufficient conditions for special digraphs presented in the first part to be \((k, l)\)-kernel-perfect digraphs. The concept of a \((k, l)\)-kernel-perfect digraph is the generalization of the well-know idea of a kernel perfect digraph, which was considered in [1] and [6].

Keywords: kernel, \((k, l)\)-kernel, kernel-perfect digraph.

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1. Introduction

Let \(D\) denote a finite, directed graph (for short: a digraph) without loops and multiple arcs, where \(V(D)\) is the set of vertices of \(D\) and \(A(D)\) is the set of arcs of \(D\). By \(D[S]\) we denote the subdigraph of \(D\) induced by a nonempty subset \(S \subseteq V(D)\). A vertex \(x \in V(D)\) is a source of a digraph \(D\), if for every \(y \in V(D)\) there is no arc \(\overrightarrow{yx}\) in \(D\). By a path from a vertex \(x_1\) to a vertex \(x_n\) in \(D\) we mean a sequence of distinct vertices
$x_1, x_2, \ldots, x_n$ from $V(D)$ and arcs $\overrightarrow{x_ix_{i+1}} \in A(D)$, for $i = 1, 2, \ldots, n - 1$ and $n \geq 2$ for the simplicity we denote it by $P[x_1, x_2, \ldots, x_n]$. A circuit is a path with $x_1 = x_n$, for $n \geq 3$. By $P_m$ we denote an elementary path on $m$ vertices meant as a digraph with $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. By $d_D(x, y)$ we denote the length of the shortest path from $x$ to $y$ in $D$. For any $X, Y \subseteq V(D)$ and $x \in V(D) \setminus X$ we put $d_D(x, X) = \min_{y \in X} d_D(x, y)$, $d_D(X, x) = \min_{y \in X} d_D(y, x)$ and $d_D(X, Y) = \min_{x \in X, y \in Y} d_D(x, y)$. Let $k, l$ be fixed integers, $k \geq 2$ and $l \geq 1$. We say that a subset $J \subseteq V(D)$ is a $(k, l)$-kernel of $D$ if

(i) for each $x, y \in J$ and $x \neq y$, $d_D(x, y) \geq k$ and
(ii) for each $x \in V(D) \setminus J$, $d_D(x, J) \leq l$.

The concept of a $(k, l)$-kernel was introduced by M. Kwaśniki in [13] and considered in [7, 8, 12] and [14]. If $k = 2$ and $l = 1$, then we obtain the definition of a kernel or in other words a $(2, 1)$-kernel of a digraph. We call a $(k, k - 1)$-kernel a $k$-kernel. If $J$ satisfies the condition (i), then we say that $J$ is $k$-stable in $D$. Moreover, we assume that the subset including exactly one vertex also is $k$-stable in $D$, for $k \geq 2$. We say that $J$ is $l$-dominating in $D$, when the condition (ii) is fulfilled. More precisely with respect to the vertex $x$ we say: $x$ is $l$-dominated by $J$ in $D$ or $J$ $l$-dominates $x$ in $D$.

A digraph whose every induced subdigraph has a $(k, l)$-kernel is called a $(k, l)$-kernel-perfect digraph (for short a $(k, l)$-KP digraph). If $l = k - 1$, then we obtain $k$-kernel perfect digraph. In [11] we can find some results about $k$-kernel perfectness of special digraphs. The last concept is the generalization of a kernel-perfect digraph, which was considered in [1, 2] and [6].

For concepts not defined here, see [5].

### 2. The Existence of $(k, l)$-kernels of the $D$-join

Let $D$ be a digraph with $V(D) = \{x_1, x_2, \ldots, x_n\}$ and $\alpha = (D_i)_{i \in \{1, 2, \ldots, n\}}$ be a sequence of vertex disjoint digraphs. The $D$-join of the digraph $D$ and the sequence $\alpha$ is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^n V(D_i)$ and

$$A(\sigma(\alpha, D)) = \left( \bigcup_{i=1}^n A(D_i) \right) \cup \{ \overrightarrow{uv} : u \in V(D_s), v \in V(D_t), s \neq t \text{ and } \overrightarrow{x_i} \overrightarrow{x_{i+1}} \in A(D) \}.$$
It may be noted that if all digraphs from the sequence \( \sigma \) have the same vertex set, then from the \( D \)-join we obtain the generalized lexicographic product of the digraph \( D \) and the sequence of the digraphs \( D_i \), i.e., \( \sigma(\alpha, D) = D[ D_1, D_2, \ldots, D_n ] \). For the reminder, the generalized lexicographic product \( G[ G_1, G_2, \ldots, G_n ] \) of the graph \( G \) and the sequence of the graphs \( G_i \) was introduced in [3] and its definition was applied to digraphs in [14]. Additionally if all digraphs from the sequence \( \alpha \) are isomorphic to the same digraph \( D' \), then from the \( D \)-join we obtain the lexicographic product \( D[D'] \) of the digraphs \( D \) and \( D' \). The \( D \)-join \( \sigma(\alpha, D) \) is the special case of a digraph, which was considered with reference to kernels by H. Galeana-Sanchez and V. Neumann-Lara in [9].

**Theorem 1.** Let \( D \) be a digraph without circuits of length less than \( k \). Let \( \alpha = (D_i)_{i \in \{1,2,\ldots,n\}} \) be a sequence of vertex disjoint digraphs. A subset \( J^* \subseteq V(\sigma(\alpha, D)) \) is \( k \)-stable in the \( D \)-join \( \sigma(\alpha, D) \) if and only if there exists a \( k \)-stable set \( J \subseteq V(D) \) of the digraph \( D \) such that \( J^* = \bigcup_{i \in \mathcal{I}} J_i \), where \( \mathcal{I} = \{i : x_i \in J\} \), \( J_i \subseteq V(D_i) \) and \( J_i \) is \( k \)-stable in \( D_i \) for every \( i \in \mathcal{I} \).

**Proof.** I. Let \( J^* \) be \( k \)-stable in the \( D \)-join \( \sigma(\alpha, D) \). Denote
\[
J = \{x_i \in V(D) : J^* \cap V(D_i) \neq \emptyset\}.
\]
At first we will prove that \( J \) is \( k \)-stable in \( D \). Assume on the contrary that there exist distinct vertices \( x_i, x_j \in J \) such that \( d_D(x_i, x_j) < k \). Since \( x_i, x_j \in J \), then \( J^* \cap V(D_i) \neq \emptyset \) and \( J^* \cap V(D_j) \neq \emptyset \). Additionally the definition of the \( D \)-join and the assumption that \( d_D(x_i, x_j) < k \) implies that \( d_{\sigma(\alpha, D)}(u, v) < k \) for every \( u \in V(D_i) \) and \( v \in V(D_j) \). This means that \( J^* \) is not \( k \)-stable in the digraph \( \sigma(\alpha, D) \), a contradiction with the assumption. So \( J \) is \( k \)-stable in the digraph \( D \). The definition of the set \( J \) implies that we can depict \( J^* \) in the following way: \( J^* = \bigcup_{i \in \mathcal{I}} J_i \), where \( \mathcal{I} = \{i : x_i \in J\} \) and \( J_i \subseteq V(D_i) \). Of course for every \( i \in \mathcal{I} \) we have that \( J_i \) is \( k \)-stable in \( D_i \), since \( J_i \subseteq J^* \) and \( J^* \) is \( k \)-stable in \( \sigma(\alpha, D) \).

II. Let \( J \subseteq V(D) \) be a \( k \)-stable set of the digraph \( D \). Let \( \mathcal{I} \) be a set of indexes of vertices belonging to \( J \) and let \( J_i \) be \( k \)-stable in \( D_i \) for every \( i \in \mathcal{I} \). We prove that \( J^* = \bigcup_{i \in \mathcal{I}} J_i \) is \( k \)-stable in the \( D \)-join \( \sigma(\alpha, D) \). Let \( u, v \in J^*, \ u \neq v \). Assume on the contrary that \( d_{\sigma(\alpha, D)}(u, v) < k \). Consider two cases:

**Case 1.** \( u, v \in J_i \) for some \( i \in \mathcal{I} \). Of course \( d_{D_i}(u, v) \geq k \), since \( J_i \) is \( k \)-stable in \( D_i \). So there exists a path \( P \) from \( u \) to \( v \) in \( \sigma(\alpha, D) \) of length
less than \( k \) such that at least one inner vertex of \( P \) does not belong to \( V(D_i) \). In other words there exists a vertex \( z \in V(D_j) \) for \( i \neq j \) such that \( P = \{u, \ldots, z, \ldots, v\} \). The existence of a circuit \( C = [x_i, \ldots, x_j, \ldots, x_i] \) in the digraph \( D \) of length less than \( k \) follows from the definition of the digraph \( \sigma(\alpha, D) \), a contradiction with the assumption.

**Case 2.** \( u \in J_i \) and \( v \in J_j \), where \( i \neq j \). Since \( d_{\sigma(\alpha,D)}(u,v) < k \), so the definition of the digraph \( \sigma(\alpha, D) \) implies the fact that \( d_D(x_i, x_j) < k \), a contradiction with the assumption that \( x_i, x_j \) belong to a \( k \)-stable set \( J \) of the \( D \)-join.

Taking two above cases into consideration we obtain that for distinct \( u, v \in J^* \), \( d_{\sigma(\alpha,D)}(u,v) \geq k \), hence \( J^* \) is \( k \)-stable in \( \sigma(\alpha, D) \). ■

**Theorem 2.** Let \( J \subseteq V(D), \mathcal{I} = \{i : x_i \in J\} \) and \( J_i \subseteq V(D_i) \) for every \( i \in \mathcal{I} \). If \( J \) is \( l \)-dominating in \( D \) and \( J_i \) is \( l \)-dominating in \( D_i \) for every \( i \in \mathcal{I} \), then \( J^* = \bigcup_{i \in \mathcal{I}} J_i \) is \( l \)-dominating in the \( D \)-join \( \sigma(\alpha, D) \).

**Proof.** Assume that \( J \) is \( l \)-dominating in \( D \), \( \mathcal{I} = \{i : x_i \in J\} \) and \( J_i \) is \( l \)-dominating in \( D_i \) for every \( i \in \mathcal{I} \). Let \( J^* = \bigcup_{i \in \mathcal{I}} J_i \) and \( u \in V(\sigma(\alpha, D)) \setminus J^* \).

We show that \( u \) is \( l \)-dominated by \( J^* \) in \( \sigma(\alpha, D) \). Let \( i \) be a positive integer such that \( u \in V(D_i) \). If \( i \notin \mathcal{I} \), then \( u \) is \( l \)-dominated by \( J_i \subseteq J^* \) in the \( D \)-join. If \( i \in \mathcal{I} \), then we obtain that \( d_D(x_i, J) \leq l \), since \( J \) is \( l \)-dominating in \( D \). This means that there exists a vertex \( x_j \in J \) such that \( d_D(x_i, x_j) \leq l \). We obtain that \( d_{\sigma(\alpha,D)}(u, v) \leq l \) for every \( v \in V(D_j) \) in view of the definition of the digraph \( \sigma(\alpha, D) \). Hence \( d_{\sigma(\alpha,D)}(u, J_j) \leq l \). Since \( J_i \subseteq J^* \), then \( d_{\sigma(\alpha,D)}(u, J^*) \leq l \). So we proved that each \( u \in V(\sigma(\alpha, D)) \setminus J^* \) is \( l \)-dominated by \( J^* \) in \( \sigma(\alpha, D) \), i.e., \( J^* \) is \( l \)-dominating in \( \sigma(\alpha, D) \). ■

**Remark 1.** It is not difficult to observe that the sufficient condition from Theorem 2 is not a necessary condition for the set \( J^* \) to be \( l \)-dominating in \( \sigma(\alpha, D) \). For example, let \( D = P_{l+1}, V(P_{l+1}) = \{x_1, x_2, \ldots, x_{l+1}\} \) and \( D_i = P_2, \) where \( V(D_i) = \{u_1^i, u_2^i\} \) for every \( i = 1, \ldots, l+1 \). \( J^* = \{u_1^1, u_2^{l+1}\} \) is \( l \)-dominating in \( \sigma(\alpha, D) \), but \( J^* \cap V(D_1) \) is not \( l \)-dominating in \( D_1 \).

From Theorem 1 and Theorem 2 we obtain the following corollary.

**Corollary 1.** Let \( D \) be a digraph without circuits of length less than \( k \) and let \( \alpha = (D_i)_{i \in \{1, 2, \ldots, n\}} \) be a sequence of vertex disjoint digraphs. If \( J \subseteq V(D) \) is a \((k, l)\)-kernel of \( D \), \( \mathcal{I} = \{i : x_i \in J\} \) and \( J_i \) is a \((k, l)\)-kernel of \( D_i \) for every \( i \in \mathcal{I} \), then \( J^* = \bigcup_{i \in \mathcal{I}} J_i \) is a \((k, l)\)-kernel of the \( D \)-join \( \sigma(\alpha, D) \).
Theorem 3. Let $l \leq k - 1$. Let $D$ be a digraph without circuits of length less than $k$ and $\alpha = (D_i)_{i \in \{1, 2, \ldots, n\}}$ be a sequence of vertex disjoint digraphs. If $J^*$ is a $(k, l)$-kernel of the $D$-join $\sigma(\alpha, D)$, then there exists a $k$-kernel $J \subseteq V(D)$ of the digraph $D$ such that $J^* = \bigcup_{i \in I} J_i$, where $I = \{i : x_i \in J\}$, $J_i \subseteq V(D_i)$ and $J_i$ is a $k$-kernel of $D_i$ for every $i \in I$.

Proof. Let $J^*$ be a $(k, l)$-kernel of $\sigma(\alpha, D)$, where $l \leq k - 1$. From Theorem 1 we get that $J^* = \bigcup_{i \in I} J_i$, where $J \subseteq V(D)$ is $k$-stable in $D$ and $J_i \subseteq V(D_i)$ is $k$-stable in $D_i$ for every $i$ such that $i \in I$. We will show that $J$ is $l$-dominating in $D$. Let $x_p \in V(D) \setminus J$. Hence $p \notin I$ and $V(D_p) \cap J^* = \emptyset$. This means that if $u \in V(D_p)$, then $u \in V(\sigma(\alpha, D)) \setminus J^*$. Since $J^*$ is a $(k, l)$-kernel of $\sigma(\alpha, D)$, hence $d_{\sigma(\alpha, D)}(u, J^*) \leq l$. So there exists $v \in J^*$ such that $d_{\sigma(\alpha, D)}(u, v) \leq l$. Hence $v \in V(D_i)$, where $t \in I$, i.e., $x_t \in J$ and $d_D(x_p, x_t) \leq l$ in view of the definition of the $D$-join, so $x_p$ is $l$-dominated by $J$ in $D$.

Now we will prove that $J_i$ is $l$-dominating in $D_i$ for every $i \in I$. Assume on the contrary that there exists an integer $i$ such that $J_i$ is not $l$-dominating in the digraph $D_i$. This means that the existence of a vertex $u \in J_i$ such that $d_{D_i}(u, J_i) > l$ is assured. Because of the assumption that $J^*$ is $l$-dominating in the digraph $\sigma(\alpha, D)$, there must exist a vertex $v \in J^* \setminus V(D_i)$ such that $d_{\sigma(\alpha, D)}(u, v) \leq l$. From the definition of the $D$-join we obtain the inequality $d_{\sigma(\alpha, D)}(V(D_i), v) \leq l$ and finally $d_{\sigma(\alpha, D)}(J_i, v) \leq l \leq k - 1$, a contradiction with the assumption that $J^*$ is a $(k, l)$-kernel of the $D$-join $\sigma(\alpha, D)$. This means that $J_i$ is $l$-dominating in $D_i$ for every $i \in I$.

So every $(k, l)$-kernel $J^*$ of the $D$-join $\sigma(\alpha, D)$, where $l \leq k - 1$ can be described in the form $J^* = \bigcup_{i \in I} J_i$, where $J$ is a $(k, l)$-kernel of $D$ and $J_i$ is a $(k, l)$-kernel of $D_i$ for every $i \in I$.

From Corollary 1 and Theorem 3 we obtain the next corollary.

Corollary 2. Let $D$ be a digraph without circuits of length less than $k$ and let $\alpha = (D_i)_{i \in \{1, 2, \ldots, n\}}$ be a sequence of vertex disjoint digraphs. The subset $J^*$ is a $k$-kernel of the $D$-join $\sigma(\alpha, D)$ if and only if there exists a $k$-kernel $J \subseteq V(D)$ of the digraph $D$ such that $J^* = \bigcup_{i \in I} J_i$, where $I = \{i : x_i \in J\}$, $J_i \subseteq V(D_i)$ and $J_i$ is a $k$-kernel of $D_i$ for every $i \in I$. 
3. The Existence of a \((k, l)\)-kernel of the Duplication

In [11] was given the definition of the duplication of a subset of vertices of a graph as the generalization of the duplication of a vertex of a graph introduced in [4]. This definition can be apply to digraphs in the following way. Let \(X\) be a proper subset of the vertex set of a digraph \(D\) and let \(H\) be a digraph isomorphic to \(D[X]\). A vertex belonging to \(V(H)\) and corresponding to a vertex \(x \in X\) will be denoted by \(x'\). The duplication of the subset \(X\), \(X \subset V(D)\) is the digraph \(D^X\) such that \(V(D^X) = V(D) \cup V(H)\) and \(A(D^X) = A(D) \cup A(H) \cup A_0 \cup A_1\), where

\[
A_0 = \left\{ \overrightarrow{xy} : x' \in V(H), y \in V(D) \text{ and } \overrightarrow{xy} \in A(D) \right\} \quad \text{and} \\
A_1 = \left\{ \overrightarrow{yx'} : x' \in V(H), y \in V(D) \text{ and } \overrightarrow{yx} \in A(D) \right\}.
\]

Denote \(X' = V(H)\). A vertex \(x' \in X'\) (resp. a subset \(S' \subseteq X'\)) will be called the copy of the vertex \(x \in X\) (resp. the copy of the subset \(S \subseteq X\)). We will call the vertex \(x\) as the original of the vertex \(x'\) and the subset \(S\) the original of the subset \(S'\). We will prove a necessary and sufficient condition for the duplication \(D^X\) to have a \((k, l)\)-kernel. To this end some lemmas will be given. The next one follows directly from the definition of \(D^X\).

**Lemma 1.** Let \(D^X\) be the duplication of a subset \(X\), \(X \subset V(D)\). Let \(x, y \in V(D)\), \(x', y' \in X'\) and \(w, z \in V(D) \setminus X\). Then

1. \(d_D(x, y) = d_D(x, y) = d_{D^X}(x', y') = d_{D^X}(x, y') = d_{D^X}(x', y)\),
2. \(d_D(w, z) = d_{D^X}(w, z)\),
3. \(d_D(w, x) = d_{D^X}(w, x) = d_{D^X}(w, x')\),
4. \(d_D(x, w) = d_{D^X}(x, w) = d_{D^X}(x', w)\).

The next corollary follows from Lemma 1.

**Corollary 3.** Let \(D^X\) be the duplication of a subset \(X\), where \(X \subset V(D)\). If \(x, y \in V(D)\), then \(d_D(x, y) = d_{D^X}(x, y)\).
Lemma 2. Let $X \subseteq V(D)$. If $J^* \subseteq V(D^X)$ is $k$-stable in the duplication $D^X$, then $(J^* \cap V(D)) \cup S$ is a $k$-stable set of $D$, where $S$ is the original of the set $J^* \cap X'$.

Proof. Assume that $J^* \subseteq V(D^X)$ is $k$-stable in the duplication $D^X$ and $S$ is the original of $J^* \cap X'$, i.e., $J^* \cap X' = S'$. Put $J = J^* \cap V(D)$. Of course $J, S'$ and $S$ are $k$-stable in $D^X$, so $J$ and $S$ are $k$-stable in $D$. To show that $J \cup S$ is $k$-stable in the digraph $D$ it is enough to prove that $d_D(J, S) \geq k$ and $d_D(S, J) \geq k$. Let $x \in J \setminus S$ and $y \in S \setminus J$. From Lemma 1 we obtain that $d_D(x, y) = d_{D^X}(x, y')$ and $d_D(y, x) = d_{D^X}(y', x)$, where $y' \in S' \setminus (J \cap X)'$ is the copy of the vertex $y$. Since $J^*$ is $k$-stable in the duplication $D^X$, then $d_{D^X}(x, y') \geq k$ and $d_{D^X}(y', x) \geq k$. Hence $d_D(x, y) \geq k$ and $d_D(y, x) \geq k$, which means that $d_D(J, S) \geq k$ and $d_D(S, J) \geq k$. Thus the theorem is proved.

Theorem 4. Let $D$ be a digraph and $X \subseteq V(D)$. If $J^*$ is a $(k, l)$-kernel of the duplication $D^X$ and $J^* \subseteq V(D^X)$, then $(J^* \cap V(D)) \cup S$ is a $(k, l)$-kernel of the digraph $D$, where $S$ is the original of $J^* \cap X'$.

Proof. Assume that $J^* \subseteq V(D^X)$ is a $(k, l)$-kernel of $D^X$. Lemma 2 implies that $J^* \cap V(D) \cup S$ is $k$-stable in $D$. We show that $(J^* \cap V(D)) \cup S$ is $l$-dominating in the digraph $D$. Let $x \in V(D) \setminus (J^* \cup S)$. Since $J^*$ is $l$-dominating in $D^X$, hence $d_{D^X}(x, J^*) \leq l$. This means that there exists $y \in J^*$ such that $d_{D^X}(x, y) \leq l$. Consider two cases.

Case 1. Let $x \in X$. If $y \in J^* \cap V(D)$, then $d_D(x, y) = d_{D^X}(x, y) \leq l$ in view of Corollary 3. If $y \in J^* \cap X'$, then from the condition (1) of Lemma 1 we obtain that $d_D(x, z) = d_{D^X}(x, y) \leq l$, where $z \in S$ is the original of the vertex $y$.

Case 2. Let $x \in V(D) \setminus X$. If $y \in J^* \cap V(D)$, then Corollary 3 implies that $d_D(x, y) = d_{D^X}(x, y) \leq l$. If $y \in J^* \cap X'$, then from the condition (3) of Lemma 1 we obtain $d_D(x, z) = d_{D^X}(x, y) \leq l$, where $z \in S$ is the original of the vertex $y$.

Finally $d_D(x, (J^* \cap V(D)) \cup S) \leq l$, which means that $(J^* \cap V(D)) \cup S$ is $l$-dominating in $D$ and completes the proof.

Lemma 3. Let $D$ be a digraph, in which there exists a subset $X \subseteq V(D)$ such that $D$ has no circuit of length less than $k$ including vertices from $X$. Let $D^X$ be the duplication of $X$. If $J$ is $k$-stable in $D$ and $(J \cap X)'$ is the copy of $J \cap X$ in $D^X$, then $J \cup (J \cap X)'$ is $k$-stable in $D^X$. 


Proof. Assume that \( D \) is a digraph, in which there exists a subset \( X \subset V(D) \) such that \( D \) has no circuit of length less than \( k \) including vertices from \( X \). Let \( J \) be an arbitrary subset of vertices of the digraph \( D \) and let \( (J \cap X)' \) be the copy of \( J \cap X \) in the duplication \( D^X \). Assume that \( J \cup (J \cap X)' \) is not \( k \)-stable in \( D^X \). We will show that \( J \) is not a \( k \)-stable set of \( D \). Consider two cases.

Case 1. If \( J \cap X = \emptyset \), then \( J \cup (J \cap X)' = J \). From the assumption the set \( J \) is not \( k \)-stable in \( D^X \), so \( J \) is not \( k \)-stable in \( D \).

Case 2. If \( J \cap X \neq \emptyset \), then there exist two distinct vertices \( x, y \in J \cup (J \cap X)' \) such that \( d_D(x, y) < k \). If \( x, y \in J \), then the inequality \( d_D(x, y) = d_{D^X}(x, y) < k \) follows from Corollary 3. If \( x, y \in (J \cap X)' \), then from the condition (1) of Lemma 1 we obtain that \( d_D(z, w) = d_{D^X}(x, y) < k \), where \( z, w \in J \cap X \) are the copies of vertices \( x, y \), respectively. If \( x \in J \) and \( y \in (J \cap X)' \), resp. \( y \in J \) and \( x \in (J \cap X)' \), then in view of Lemma 1 we obtain that \( d_D(x, w) = d_{D^X}(x, y) < k \) (resp. \( d_D(z, y) = d_{D^X}(x, y) < k \)), where \( w \in J \cap X \) is the original of the vertex \( y \) (resp. \( z \in J \cap X \) is the original of the vertex \( x \)). Of course \( w \neq x \) (resp. \( z \neq y \)). Otherwise, there exists a circuit of length less than \( k \) including a vertex from \( X \), a contradiction with the assumption.

To recapitulate, we proved that \( J \) is not a \( k \)-stable in \( D \). □

Theorem 5. Let \( D \) be a digraph, in which there exists a subset \( X \subset V(D) \) such that \( D \) has no circuit of length less than \( k \) including vertices from \( X \). Let \( D^X \) be the duplication of \( X \). If \( J \) is a \((k, l)\)-kernel of \( D \) and \( (J \cap X)' \) is the copy of \( J \cap X \) in \( D^X \), then \( J \cup (J \cap X)' \) is a \((k, l)\)-kernel of \( D^X \).

Proof. Assume that \( J \) is a \((k, l)\)-kernel of \( D \) and \( (J \cap X)' \) is the copy of \( J \cap X \) in \( D^X \). We will show that \( J \cup (J \cap X)' \) is a \((k, l)\)-kernel of \( D^X \). If \( J \cap X = \emptyset \), then \( J \cup (J \cap X)' = \emptyset \). Hence \( J \cup (J \cap X)' = J \). Since \( J \) is a \((k, l)\)-kernel of the digraph \( D \), then \( d_D(x, y) \geq k \) and \( d_D(z, J) \leq l \) for every \( x, y \in J \) and \( z \in V(D) \setminus J \). So from Lemma 1 it follows that \( d_{D^X}(x, y) \geq k \), \( d_{D^X}(z, J) \leq l \) and \( d_{D^X}(z', J) \leq l \), where \( z' \) is the copy of a vertex \( z \), if \( z \in X \setminus J \). Hence \( J \cup (J \cap X)' \) is a \((k, l)\)-kernel of the duplication \( D^X \) in the case when \( J \cap X = \emptyset \). Thus assume that \( J \cap X \neq \emptyset \). From Lemma 3 we get that \( J \cup (J \cap X)' \) is a \( k \)-stable in \( D^X \). So we need only prove that this set is \( l \)-dominating in the digraph \( D^X \). Since \( V(D^X) \setminus (J \cup (J \cap X)') = (V(D) \setminus J) \cup (X' \setminus (J \cap X)') \), so let us consider two cases.
Case 1. If \( x \in V(D) \setminus J \), then \( x \) is \( l \)-dominated by \( J \) in the digraph \( D \), because \( J \) is a \((k, l)\)-kernel of \( D \). Thus \( x \) is \( l \)-dominated by \( J \) in the duplication \( D^X \).

Case 2. If \( x \in X' \setminus (J \cap X)' \), then its original \( y \in X \setminus J \) is \( l \)-dominated by \( J \) in \( D \). Therefore there exists a path from the vertex \( y \) to some vertex \( z \in J \) in \( D \) of length not greater than \( l \), i.e., \( d_D(y, z) \leq l \). If \( z \in J \cap X \), then the condition (1) of Lemma 1 implies that \( d_D(x, z') = d_D(y, z) \leq l \), where \( z' \in (J \cap X)' \). This means that \( d_{DX}(x, (J \cap X)') \leq l \). If \( z \in J \cap (V(D) \setminus X) \), then from the condition (3) of Lemma 1 we obtain that \( d_{DX}(x, z) \leq l \). So \( d_{DX}(x, J) \leq l \).

Therefore \( x \) is \( l \)-dominated by \( J \cup (J \cap X)' \) in the duplication \( D^X \). Because of the fact that \( J \cup (J \cap X)' \) is \( k \)-stable in \( D^X \) we obtain that \( J \cup (J \cap X)' \) is a \((k, l)\)-kernel of the duplication \( D^X \).

The next corollary follows from Theorem 4 and Theorem 5.

**Corollary 4.** Let \( D \) be a digraph, in which there exists a subset \( X \subset V(D) \) such that \( D \) has no circuit of length less than \( k \) including vertices from \( X \). Then the duplication \( D^X \) possesses a \((k, l)\)-kernel if and only if the digraph \( D \) has a \((k, l)\)-kernel.

### 4. The Existence of a \( k \)-kernel of the Digraph \( D(a, P_m) \)

Let \( D \) be an arbitrary digraph and \( P_m \) be a path meant as a digraph for \( m \geq 2 \), where \( V(P_m) = \{x_1, x_2, \ldots, x_m\} \) and \( V(D) \cap V(P_m) = \emptyset \). If \( a = \overrightarrow{pq} \) is an arc of the digraph \( D \), then \( D(a, P_m) \) is a digraph such that \( V(D(a, P_m)) = V(D) \cup V(P_m) \) and \( A(D(a, P_m)) = A(D) \cup A(P_m) \cup \{\overrightarrow{px_1}, \overrightarrow{x_mq}\} \).

The following theorem gives a necessary and sufficient condition for the existence of a \( k \)-kernel of \( D(a, P_m) \).

**Theorem 6.** Let \( D \) be a digraph without circuits of length less than \( k \). Let \( a = \overrightarrow{pq} \in A(D) \) and \( n \geq 1 \). \( J^* \) is a \( k \)-kernel of the digraph \( D(a, P_{nk}) \) if and only if there exists a \( k \)-kernel \( J \) of \( D \) such that \( J^* = J \cup J' \), where \( J' = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\} \subset V(P_{nk}) \) and \( s = d_D(q, J) \).

**Proof.** 1. Let \( a = \overrightarrow{pq} \in A(D) \) and let \( J^* \) be a \( k \)-kernel of the digraph \( D(a, P_{nk}) \). We will prove that \( J^* \cap V(P_{nk}) = J' \) and \( J^* \cap V(D) \) is a \( k \)-kernel.
of $D$. Put $J = J^* \cap V(D)$. Let $s = d_D(q, J)$. It is not difficult to observe that $J^* \cap V(P_{nk}) = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\}$, i.e., $J^* \cap V(P_{nk}) = J'$. Otherwise, $J^*$ is not $k$-stable or $(k-1)$-dominating in $D(a, P_{nk})$.

Of course $J$ and $J^* \cap V(P_{nk})$ are $k$-stable in $D(a, P_{nk})$, so $J$ is $k$-stable in $D$. So it remains to show that $J$ is $(k-1)$-dominating in $D$. Let $x \in V(D) \setminus J^*$. Since $J^*$ is a $k$-kernel of $D(a, P_{nk})$, hence $d_{D(a,P_{nk})}(x, J^*) \leq k - 1$. It is enough to prove that if $x$ is $(k-1)$-dominated by $J'$ in $D(a, P_{nk})$, then it is $(k-1)$-dominated by $J^* \cap V(D)$ in $D$. Let $x$ be $(k-1)$-dominated in $D(a, P_{nk})$ by a vertex belonging to $J'$. Hence $d_{D(a,P_{nk})}(x, x_{1+s}) \leq k - 1$. At the same time $d_{D(a,P_{nk})}(x, x_{1+s}) = d_D(x, p) + d_{D(a,P_{nk})}(p, x_{1+s}) = d_D(x, p) + s + 1$. Thus $d_D(x, p) \leq k - s - 2$. On the other hand from the assumption we have that $d_D(q, J) = s$. So we get that

$$d_D(x, J) \leq d_D(x, p) + d_D(p, q) + d_D(q, J) = d_D(x, p) + 1 + s \leq k - 1,$$

which means that $x$ is $(k-1)$-dominated by $J$ in $D$. Finally, $J$ is a $k$-kernel of $D$, what completes this part of the proof.

II. Let $J$ be a $k$-kernel of $D$ and $J' = \{x_{1+s}, x_{1+k+s}, \ldots, x_{1+(n-1)k+s}\} \subset V(P_{nk})$, where $s = d_D(q, J)$. We prove that $J \cup J'$ is a $k$-kernel of $D(a, P_{nk})$. Since $J$ is a $k$-kernel of $D$, then every $x \in V(D) \setminus J$ is $(k-1)$-dominated by $J$ in $D$, which means that $x$ is $(k-1)$-dominated by $J \cup J'$ in $D(a, P_{nk})$. To show that $J \cup J'$ is $(k-1)$-dominating in $D(a, P_{nk})$, it is enough to prove that vertices from $V(P_{nk})$ not belonging to $J \cup J'$ are $(k-1)$-dominated by $J \cup J'$ in the digraph $D(a, P_{nk})$. Let $x_i \in V(P_{nk}) \setminus J'$. If $1 \leq i \leq 1 + (n-1)k + s$, then $d_{P_{nk}}(x_i, J') \leq k - 1$. Hence $d_{D(a, P_{nk})}(x_i, J \cup J') \leq k - 1$. If $2 + (n-1)k + s \leq i \leq nk$, then

$$d_{D(a, P_{nk})}(x_i, J) = d_{P_{nk}}(x_i, q) + d_D(q, J) = nk + 1 - i + s \\ \leq nk + 1 - (2 + (n-1)k + s) + s = k - 1.$$

So $J \cup J'$ is $(k-1)$-dominating in $D(a, P_{nk})$. Moreover, the definition of the digraph $D(a, P_{nk})$ implies that $J$ and $J'$ are $k$-stable in $D(a, P_{nk})$. To prove that $J \cup J'$ is $k$-stable in $D(a, P_{nk})$ it is enough to show that $d_{D(a,P_{nk})}(J', J) \geq k$ and $d_{D(a,P_{nk})}(J, J') \geq k$. Since $d_D(q, J) = s$, then

$$d_{D(a, P_{nk})}(x_{1+(n-1)k+s}, J) = d_{P_{nk}}(x_{1+(n-1)k+s}, q) + d_D(q, J) = (k - s) + s = k.$$
Hence \( d_{D(a,P_{nk})}(J', J) \geq k \). So we need only to prove that \( d_{D(a,P_{nk})}(J, J') \geq k \). Assume on the contrary that \( d_{D(a,P_{nk})}(J, J') < k \). Hence there exists a vertex \( y \in J \) such that there is a path \( [y, \ldots, p, \ldots, x_{1+s}] \) of length less than \( k \) in \( D \). This means that there exists a path \( [y, \ldots, p] \) of length less than \( k - s - 1 \) in the digraph \( D \). At the same time, since \( s = d_D(q, J) \), then there exists \( z \in J \) such that \( d_D(q, z) = s \). So we can conclude that if \( y \neq z \), then \( J \) is not \( k \)-stable in \( D \) or if \( y = z \), then there is a circuit \( [y, \ldots, p, q, \ldots, z = y] \) in \( D \) of length less than \( k \), a contradiction with the assumptions. Finally \( d_{D(a,P_{nk})}(J, J') \geq k \). The facts proved above imply that \( J \cup J' \) is a \( k \)-kernel of \( D(a, P_{nk}) \), which completes the part II of the proof. Thus theorem is proved.

Theorem 6 implies the next corollary.

**Corollary 5.** Let \( D \) be a digraph without circuits of length less than \( k \). The digraph \( D(a, P_{nk}) \) has a \( k \)-kernel for an arbitrary \( a \in A(D) \) and \( n \geq 1 \) if and only if the digraph \( D \) possesses a \( k \)-kernel.

### 5. \( (k, l) \)-kernel Perfect Digraphs

This section includes necessary and sufficient conditions for special classes of digraphs considered above to be \( (k, l) \)-kernel perfect digraphs. The definition of a \( (k, l) \)-KP digraph implies the next propositions.

**Proposition 1.** If \( D \) is a \( (k, l) \)-KP digraph, then every induced subdigraph of \( D \) is a \( (k, l) \)-KP digraph.

**Proposition 2.** The disjoint union of \( D_1 \) and \( D_2 \) is a \( (k, l) \)-KP digraph if and only if digraphs \( D_1 \) and \( D_2 \) are \( (k, l) \)-KP digraphs.

**Theorem 7.** Let \( D \) be a digraph, in which there exists \( X \subset V(D) \) such that \( D \) has no circuit of length less than \( k \) including vertices from \( X \). Then the duplication \( D^X \) is a \( (k, l) \)-KP digraph if and only if \( D \) is a \( (k, l) \)-KP digraph.

**Proof.** I. If the duplication \( D^X \) is a \( (k, l) \)-KP digraph, then the induced subdigraph \( D^X[V(D)] \) is a \( (k, l) \)-KP digraph and it is isomorphic to \( D \). So \( D \) is a \( (k, l) \)-KP digraph.

II. Let \( D \) be a \( (k, l) \)-KP digraph, in which there exists \( X \subset V(D) \) such that \( D \) has no circuit of length less than \( k \) including vertices from \( X \).
We will prove that $D^X$ is a $(k,l)$-KP digraph. Let $Y \subseteq V(D^X)$. We show that $D^X[Y]$ has a $(k,l)$-kernel. If $Y \subseteq V(D)$ or $Y \subseteq X'$, where $X'$ is the copy of $X$ in the duplication $D^X$, then the induced subdigraph $D^X[Y]$ possesses a $(k,l)$-kernel, because it is isomorphic to some induced subdigraph of the digraph $D$. Now assume that $Y \cap V(D) \neq \emptyset$, $Y \cap X' \neq \emptyset$ and denote $Y_D = Y \cap V(D)$, $Z' = Y \cap X'$. Of course $Y = Y_D \cup Z'$. Let $Z$ denotes the original of $Y \cap X'$.

Since $D$ is a $(k,l)$-KP digraph, then the induced subdigraph $D[Y_D \cup Z]$ has a $(k,l)$-kernel, say $J$. Let $K = J \cap Z$ and let $K'$ be the copy of $K$, i.e., $K' = (J \cap Z)'$. If $K = \emptyset$, then we assume that $K' = \emptyset$. We show that $J^* = (J \cap Y_D) \cup K'$ is a $(k,l)$-kernel of $D^X[Y]$. First, we prove that $J^*$ is $l$-dominating in $D^X[Y]$. Let $x \in V(D^X[Y]) \setminus J^*$. Since

$$V(D^X[Y]) \setminus J^* = (Y_D \cup Z') \setminus J^* = (Y_D \setminus J^*) \cup (Z' \setminus J^*)$$

then consider two cases.

**Case 1.** If $x \in Y_D \setminus J^*$, then $d_{D[Y_D \cup Z]}(x,J) \leq l$, because $J$ is $l$-dominating in $D[Y_D \cup Z]$. This means that there exists a path $P = [x,\ldots,y]$ of length not greater than $l$ in the digraph $D[Y_D \cup Z]$, where $y \in J$. Replacing all vertices of the path $P$ belonging to $Z$ with their copies from $Z'$ we get the path $P'$ from the vertex $x$ to some vertex from $J^*$ of length not greater than $l$ in $D^X[Y]$, hence $d_{D^X[Y]}(x,J^*) \leq l$.

**Case 2.** If $x \in Z' \setminus J^* = Z' \setminus K'$ and $y \in Z$ is the original of $x$, then $d_{D[Y_D \cup Z]}(y,J) \leq l$, since $J$ is a $(k,l)$-kernel of $D[Y_D \cup Z]$. Arguing like in Case 1 we obtain that $d_{D^X[Y]}(x,J^*) \leq l$.

So we proved that for every $x \in V(D^X[Y]) \setminus J^*$, $d_{D^X[Y]}(x,J^*) \leq l$, which means that $J^*$ is $l$-dominating in $D^X[Y]$.

Now we will show the $k$-stability of $J^*$ in the digraph $D^X[Y]$. Of course $J \cap Y_D$ and $K$ are $k$-stable in $D^X[Y_D \cup Z]$ in view of the $k$-stability of $J$ in $D[Y_D \cup Z]$ and the definition of $D^X$. Assume on the contrary that $J \cap Y_D$ (resp. $K'$) is not $k$-stable in $D^X[Y]$. This means that there exists a path $P = [x,\ldots,y]$ in $D^X[Y]$ of length less than $k$, where $x, y \in J \cap Y_D$ (resp. $x, y \in K'$). Exchanging all vertices of the path $P$ belonging to $Z'$ for their originals from $Z$ we obtain a path $P'$ from $x$ to $y$ (resp. from $w$ to $z$, where $w, z$ are the originals of vertices $x, y$ and $w, y \in K$) in the digraph $D[Y_D \cup Z]$ of length less than $k$, a contradiction with the fact given above that $J \cap Y_D$ and $K$ are $k$-stable in $D[Y_D \cup Z]$. This means that $J \cap Y_D$ and $K'$ are
$k$-stable in $D^X[Y]$. Since $J^* = (J \cap Y_D) \cup K'$, we need only show that $d_{D^X[Y]}(J \cap Y_D, K') \geq k$ and $d_{D^X[Y]}(K', J \cap Y_D) \geq k$. Let $x \in J \cap Y_D$ and $y' \in K'$. If $x \in X \cap J \cap Y_D$, then there exists its copy $x'$. Since vertices $x', y'$ are not necessary distinct, consider two cases.

Case (a). Let $x \in X \cap J \cap Y_D$ and $x' \neq y'$ or $x \notin X$. If $d_{D^X[Y]}(x, y') < k$, then there is a path $P = [x_1, \ldots, y']$ of length less than $k$ in $D^X[Y]$. Replacing all vertices of the path $P$ belonging to $Z'$ with their originals from $Z$ we get the path $P'$ from the vertex $x \in J \cap Y_D$ to the vertex $y \in K = J \cap Z$ of length less than $k$ in $D[Y_D \cup Z]$, a contradiction with the assumption that $J$ is a $(k,l)$-kernel of $D[Y_D \cup Z]$. Hence $d_{D^X[Y]}(x, y') \geq k$. Analogously it can be proved that $d_{D^X[Y]}(y', x) \geq k$.

Case (b). Let $x \in X \cap J \cap Y_D$ and $x' = y'$. This means that $d_{D^X[Y]}(x, y') \geq k$ and $d_{D^X[Y]}(y', x) \geq k$. Otherwise, there exists a circuit in $D$ of length less than $k$ including vertices from $X$, a contradiction with the assumption.

So $J^*$ is $k$-stable in $D^X[Y]$ and finally $J^*$ is a $(k,l)$-kernel of $D^X[Y]$. This means that the duplication $D^X$ is a $(k,l)$-KP digraph.

The definition of the $D$-join implies the next result.

**Proposition 3.** Every induced subdigraph of the $D$-join $\sigma(\alpha, D)$ is:

1. the $D_0$-join $\sigma(\alpha_0, D_0)$, where $D_0$ is an induced subdigraph of $D$ with the vertex set $V(D_0) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ and $\alpha_0$ is a sequence of digraphs $\{D_{i_1}, D_{i_2}, \ldots, D_{i_m}\}$ or
2. an induced subdigraph of $D_i$ for some $1 \leq i \leq n$ or
3. the disjoint union of digraphs from items (1) or (2).

**Theorem 8.** Let $D$ be a digraph without circuits of length less than $k$ and $V(D) = \{x_1, x_2, \ldots, x_n\}$. Let $\alpha = (D_i)_{i \in \{1,2,\ldots,n\}}$ be a sequence of vertex disjoint digraphs. The $D$-join $\sigma(\alpha, D)$ is a $(k,l)$-KP digraph if and only if the digraph $D$ and the digraphs $D_1, D_2, \ldots, D_n$ are $(k,l)$-KP digraphs.

**Proof.** I. If the digraph $\sigma(\alpha, D)$ is a $(k,l)$-KP digraph, then a subdigraph of the digraph $\sigma(\alpha, D)$ induced by $V(D_i)$ is a $(k,l)$-KP digraph for $i = 1, 2, \ldots, n$. The definition of the $D$-join implies that the induced subdigraph $\sigma(\alpha, D)[V(D_i)]$ is isomorphic to $D_i$. Hence digraph $D_i$ is a $(k,l)$-KP digraph for $i = 1, 2, \ldots, n$. Now consider a subset $X$ of the vertex set of $\sigma(\alpha, D)$
including exactly one vertex from \( V(D_i) \) for every \( i = 1, 2, \ldots, n \). From the definition of the \( D \)-join we obtain that the induced subdigraph \( \sigma(\alpha, D)[X] \) is isomorphic to the digraph \( D \). So the digraph \( D \) is a \((k, l)\)-KP digraph.

II. Let \( D \) and \( D_1, D_2, \ldots, D_n \) be \((k, l)\)-KP digraphs. Corollary 1 implies that the \( D \)-join \( \sigma(\alpha_0, D_0) \), where \( D_0 \) is an induced subdigraph of the digraph \( D \) with the vertex set \( V(D_0) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \) and \( \alpha_0 \) is a sequence of induced subdigraphs of digraphs \( \{D_{i_1}, D_{i_2}, \ldots, D_{i_m}\} \), has a \((k, l)\)-kernel. So from Proposition 2 and Proposition 3 we get that the digraph \( \sigma(\alpha, D) \) is a \((k, l)\)-KP digraph.

For \( k = 2 \) and \( l = 1 \) Theorem 8 is similar to result given in [9].

We give the necessary and sufficient condition for the digraph \( D(a, P_m) \) to be a \( k \)-KP digraph. But first we prove some useful lemmas.

Let \( D \) be a digraph and \( P_m \) be a path meant as a digraph for \( m \geq 2 \), where \( V(P_m) = \{x_1, x_2, \ldots, x_m\} \) and \( V(D) \cap V(P_m) = \emptyset \). If \( x \) is a vertex of the digraph \( D \), then symbols \( D(x^+, P_m) \) and \( D(x^-, P_m) \) will denote digraphs such that \( V(D(x^+, P_m)) = V(D(x^-, P_m)) = V(D) \cup V(P_m) \), and \( A(D(x^+, P_m)) = A(D) \cup A(P_m) \cup \{xx_1^{-}\} \) \( A(D(x^-, P_m)) = A(D) \cup A(P_m) \cup \{x_m^{-x}\} \).

From the definition of digraphs \( D(x^+, P_m) \) and \( D(x^-, P_m) \) we get immediately the following proposition.

**Proposition 4.** Every induced subdigraph of the digraph \( D(x^+, P_m) \) (resp. \( D(x^-, P_m) \)), where \( x \in V(D) \), is:

1. a digraph in the form \( D_0(x^+, P_s) \) (resp. \( D_0(x^-, P_s) \)), where \( D_0 \) is an induced subdigraph of the digraph \( D \) and \( 2 \leq s \leq m \) or
2. an induced subdigraph of the digraph \( D \) or
3. induced subdigraph of the path \( P_m \) or
4. the disjoint sum of digraphs from items (1), (2) or (3).

Since every \( k \)-kernel \( J \) of the digraph \( D \) can be easily extended to a \( k \)-kernel of the digraph \( D(x^-, P_m) \) by adding to \( J \) some vertices from \( V(P_m) \), then on basis of Proposition 4 and Proposition 2 we can formulate the following result.

**Proposition 5.** A digraph \( D \) is a \( k \)-KP digraph if and only if \( D(x^-, P_m) \) is a \( k \)-KP digraph, for every \( x \in V(D) \), where \( m \geq 2 \).
Theorem 9. Let $D_1$, $D_2$ and $D$ be digraphs such that $V(D_1) \cap V(D_2) = \{x\}$ and $D = D_1 \cup D_2$, where $x$ is a source of digraphs $D_1$ and $D_2$. The digraph $D$ is a $k$-KP digraph if and only if $D_1$ and $D_2$ are $k$-KP digraphs.

Proof. The necessary condition follows from Proposition 1. Assume that $D_i$ is a $k$-KP digraph for $i = 1, 2$. We will show that $D$ is a $k$-KP digraph. Let $X \subseteq V(D)$. 

If $X \subseteq V(D_1)$ or $X \subseteq V(D_2)$, then an induced subdigraph $D[X]$ has a $k$-kernel, since digraphs $D_1$ and $D_2$ are $k$-KP digraphs.

If $x \in V(D) \setminus X$ and $X \cap V(D_i) \neq \emptyset$ for $i = 1, 2$, then

$$d_{D[X]} (X \cap V(D_1), X \cap V(D_2)) \geq k,$$

since $x$ is a source of digraphs $D_1$ and $D_2$. This means that $J_1 \cup J_2$, where $J_i$ is a $k$-kernel of $D_i[X \cap V(D_i)]$, for $i = 1, 2$, is a $k$-kernel of the digraph $D[X]$.

So assume that $x \in X$ and $X \cap V(D_i) \neq \emptyset$ for $i = 1, 2$. Let $J_i$ be a $k$-kernel of the subdigraph of $D[X]$ induced by $X \cap V(D_i) \setminus \{x\}$ for $i = 1, 2$. The existence of a $k$-kernel $J_i$ follows from the assumption that $D_i$ is a $k$-KP digraph.

If $d_{D[X]} (x, J_1 \cup J_2) \leq k - 1$, then $J_1 \cup J_2$ is a $(k - 1)$-dominating in the digraph $D[X]$. Of course $J_1 \cup J_2$ is a $k$-stable in $D[X]$, since $x$ is a source of digraphs $D_1$ and $D_2$. So $J_1 \cup J_2$ is a $k$-kernel of the digraph $D[X]$.

If $d_{D[X]} (x, J_1 \cup J_2) \geq k$, then $J_1 \cup J_2 \cup \{x\}$ is $k$-stable and $(k - 1)$-dominating in $D[X]$. This means that $J_1 \cup J_2 \cup \{x\}$ is a $k$-kernel of $D[X]$. Hence $D$ is a $k$-KP digraph.

For $k = 2$ Theorem 9 is a special case of a result given by H. Jacob in [10].

Theorem 10 [10]. Let $D_1$, $D_2$ and $D$ be digraphs such that $V(D_1) \cap V(D_2) = \{x\}$ and $D = D_1 \cup D_2$. Then $D$ is a KP digraph if and only if $D_1$ and $D_2$ are KP digraphs.

Assuming that $x$ is a source of the digraph $D$, from Theorem 9 we obtain the next corollary.

Corollary 6. If $x \in V(D)$ is a source of $D$, then $D(x^+, P_m)$ is a $k$-KP digraph if and only if $D$ is a $k$-KP digraph.

The definition of the digraph $D(a, P_m)$ implies the following proposition.
Proposition 6. Every induced subdigraph of the digraph $D(a, P_m)$, where $a \in A(D)$ and $a = \overrightarrow{pq}$ is:

(1) a digraph in the form $D_0(a, P_m)$, where $D_0$ is an induced subdigraph of $D$
(2) an induced subdigraph of $D$
(3) an induced subdigraph of $P_m$
(4) an induced subdigraph of $D(p^+, P_m)$ or an induced subdigraph of $D(q^-, P_m)$
(5) the disjoint sum of digraphs from items (1), (2), (3) or (4).

Taking Proposition 5, Proposition 6 and Corollary 5, Corollary 6 into consideration we get the next theorem.

Theorem 11. Let $D$ be a digraph without circuits of length less than $k$ for $k \geq 2$. If $a \in A(D)$ and the initial vertex of the arc $a$ is a source of $D$, then the digraph $D$ is a $k$-KP digraph if and only if the digraph $D(a, P_{nk})$ is a $k$-KP digraph, for $n \geq 1$.

References


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