CYCLIC DECOMPOSITIONS OF COMPLETE GRAPHS INTO SPANNING TREES

DALIBOR FRONČEK

University of Minnesota Duluth
Department of Mathematics and Statistics
University of Minnesota Duluth
1117 University Dr., Duluth, MN 55812, U.S.A.
and
Technical University Ostrava

e-mail: dfroncek@d.umn.edu

Abstract

We examine decompositions of complete graphs with an even number of vertices, K_{2n}, into n isomorphic spanning trees. While methods of such decompositions into symmetric trees have been known, we develop here a more general method based on a new type of vertex labelling, called flexible q-labelling. This labelling is a generalization of labellings introduced by Rosa and Eldergill.

Keywords: graph factorization, graph labelling, spanning trees.

2000 Mathematics Subject Classification: 05C70, 05C78.

1. Introduction

There are many results on decompositions of complete graphs into isomorphic trees of smaller order. Research of this problem was initiated by Ringel [5]. An overview of this and related topics can be found in [1]. Surprisingly enough, almost nothing has been published on factorizations into isomorphic spanning trees. A simple arithmetic condition shows that only complete graphs with an even number of vertices can be factorized into spanning trees.

It is a well known fact that each such graph K_{2n} can be factorized into hamiltonian paths P_{2n}. On the other hand, it is easy to observe that each K_{2n} can be also factorized into double stars; that is, two stars $K_{1,n-1}$ joined by an edge. But what about trees between these two extremal cases? Here we present several classes of trees that allow factorizations of complete graphs with an even number of vertices.

Many factorization methods are based on graph labellings. Roughly speaking, a labelling works as follows. We assign to each vertex different label of the set $\{0, 1, 2, \ldots, 2n-1\}$. Then each edge obtains a label, induced by the labels of its endvertices. These labels are than used to guarantee that each edge appears in exactly one factor. There are many different ways how to induce the edge label. For a review of this topic, see [4]. We will mostly rely on labellings defined by Rosa [6, 7] or generalized from these labellings by Eldergill [2]. We also develop a new generalization of these labellings.

A tree is *symmetric* if there is an automorphism ψ and an edge xy such that $\psi(x) = y$ and $\psi(y) = x$.

2.   Labellings

As we mentioned above, Rosa introduced some important types of vertex labellings that we list now. However, before we do that we need to define a few more notions.

Definition 1. Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. The decomposition is *cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of K_n and isomorphisms $\phi_i : G_0 \to G_i$, $i = 1, 2, \ldots, s$ such that $\phi_i(x_j) = x_{i+j}$ for every $j = 1, 2, \ldots, n$, where the subscripts are taken modulo n. If G has exactly n vertices and none of them is isolated, then G is called a *factor* and the decomposition is called a *G-factorization* of K_n.

Graceful labellings (also called β-labellings) and ρ-labellings are being used for decompositions of complete graphs K_{2n+1} into graphs with n edges. We state here the definition of a ρ-labelling in a slightly modified form which will suit better our further needs.

Definition 2. A *labelling* of a graph G with n edges is an injection λ from the vertex set of G, $V(G)$, into a subset S of the set $\{0, 1, 2, \ldots, 2n\}$.

The length of an edge (x, y) is defined as $\ell(x, y) = \min\{|\lambda(x) - \lambda(y)|, 2n + 1 - |(\lambda(x) - \lambda(y))|\}$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$ and $S \subseteq \{0, 1, \ldots, 2n\}$, then λ is a *ρ-labelling*; if $S \subseteq \{0, 1, \ldots, n\}$ instead, then λ is a *graceful labelling*.

Each graceful labelling is indeed a ρ-labelling as well. One can observe that if a graph G with n edges has a graceful labelling or a ρ-labelling, then K_{2n+1} can be cyclically decomposed into $2n - 1$ copies of G. It is so because K_{2n+1} has exactly $2n + 1$ edges of length i for every $i = 1, 2, \ldots, n$ and each copy of G contains exactly one edge of each length.

Graceful labelling can be easily modified to produce a factorization of K_{2n} into n copies of a graph G with $2n - 1$ edges. To observe that, we will from now on simplify our notation and unify a vertex with its label by saying "a vertex i" rather than "a vertex x with $\lambda(x) = i$". We will also say that a graph *is* graceful if it has a graceful labelling. To obtain a factorization, we take one copy of a graceful graph H with $n - 1$ edges (notice the change from n to $n - 1$). The vertices belong, according to our definition, to the set $S \subseteq \{0, 1, \ldots, n - 1\}$. Then we take another copy H' of H and label its vertices such that for corresponding pairs of vertices i, i' it holds that $i' = i + n$. Obviously, corresponding edges in H and H', respectively, then have the same lengths. We now pick any pair of vertices $i, i + n$ and join them by an edge of length n. The graph arising this way is our graph G_0 and together with its cyclically obtained counterparts $G_1, G_2, \ldots, G_{n-1}$ it forms a G-factorization of K_{2n}.

The labelling described above was defined by Eldergill [2] together with a similar generalization of the ρ-labelling.

Definition 3. A connected graph G is *symmetric* if it has a *bridge* (x, y) and an automorphism ψ such that $\psi(x) = y$ and $\psi(y) = x$. The isomorphic components of $G - (x, y)$ are called *banks* and denoted H and H', respectively. A labelling of a symmetric graph G with $2n - 1$ edges and banks H and H' is *ρ-symmetric graceful* if H has a ρ-labelling and $\psi(i) = i + n$ for each vertex i of H. A labelling of a symmetric graph G with $2n - 1$ edges is *symmetric graceful* if it is ρ-symmetric graceful and the bank H is moreover graceful.

In fact, a ρ-symmetric graceful labelling is a special form of a q-labelling that was defined earlier by Rosa in [6]. Like a symmetric graceful labelling, a ρ-symmetric graceful labelling of G also guarantees the existence of a G-factorization of K_{2n} as proved in [2]. It was actually proved only for

symmetric trees, however, the assumption that the graph is acyclic is never used in the proof and therefore the assertion holds for symmetric graphs as defined above.

Theorem 4 (Eldergill). *Let G be a symmetric graph with $2n - 1$ edges. Then there exists a cyclic G-decomposition of K_{2n} if and only if G is ρ-symmetric graceful.*

Both symmetric graceful labelling and ρ-symmetric graceful labelling are too restrictive, allowing only factorizations into symmetric graphs. Our goal is to develop techniques that would be more general. We define new labelling that is a generalization of Rosa's q-labelling.

3. Flexible q-Labelling

Let us first look again at cyclic decompositions. What we do there can be alternatively described as follows. Take K_{2n} and G_0 with vertices $i_1, i_2, \ldots, i_k$. Recall that we have identified vertices with their labels, so each i_t is now an element of the set $\{0, 1, \ldots, 2n - 1\}$. If G is a factor of K_{2n}, then $V(G) = \{0, 1, \ldots, 2n - 1\}$. If (r, s) is an edge of G_0, then $(r + j, s + j)$ is an edge of G_j. Obviously, they both have the same length. If G_0 has a ρ-symmetric graceful labelling, than a particular edge (r, s) has exactly one "symmetric copy" in G_0, namely $(r + n, s + n)$. We will call this pair of edges *twin edges*. The only exception is the longest edge $(r, r + n)$ with no twin edge. So none of the first $n - 1$ copies of G_n can contain an edge that belongs to G_0. But if we take the next copy of G_0, G_n, we can see that it is actually again the original graph G_0 in which (r, s) becomes $(r + n, s + n)$. This is due to the symmetry of G_0. However, this is not the only possible way how to "rotate" symmetric graphs to get a decomposition of K_{2n}. One can check that if n is odd, say $2k + 1$, then the copies $G_0, G_2, G_4, \ldots, G_{2n-2}$ also form a decomposition of K_{2n}. It is so because for any given length $m, m < n$, each edge $(r, r + m)$ has its twin in the form $(r + 2k + 1, r + m + 2k + 1)$. Therefore the image of $(r, r + m)$ in a factor G_{2j} is $(r + 2j, r + 2j + m)$ and no edge appears in two factors. For the same reasons, the edge $(r, r + n)$ cannot appear in two factors as n is odd.

The reason why an edge $(r, r + m)$ does not appear in two factors is actually not that its twin is at distance n, but because the distance is odd. To be more precise, it is so because while all images of $(r, r + m)$ have

the form $(r + 2j, r + m + 2j)$ and none of them can be equal to an edge $(r + 2t + 1, r + m + 2t + 1)$ which would possibly collide with $(r + 2k + 1, r + m + 2k + 1) = (r + n, r + m + n)$. We use this observation to define a new type of labelling which is a generalization of the q-labelling first defined by Rosa [6].

Definition 5. Let G be a graph with $2n - 1$ edges and $\lambda : V(G) \to \{0, 1, \ldots, 2n - 1\}$ be a vertex labelling. λ is called a *flexible q-labelling* if

(i) there is exactly one edge of length n,

(ii) for each $m, 1 \leq m \leq n - 1$ there are exactly two edges of length m, and

(iii) if $(r, r + m)$ with $1 \leq m \leq n - 1$ is an edge of G, then the other edge of length m in G is $(r + 2s + 1, r + m + 2s + 1)$ for some $s, 0 \leq s \leq n - 1$, where the labels are taken modulo $2n$.

We will often need to distinguish the vertices of an edge of a given length m. We will say that r is the *original vertex* or simply the *origin* of an edge $(r, s) = (r, r + m)$ whenever $1 \leq m \leq n - 1$. The other vertex s will be called the *terminal vertex*. Using this terminology we can say that in a graph with a flexible q-labelling the two edges with the same length m have always origins of opposite parity.

We now show that if n is odd, then the existence of a flexible q-labelling of G implies the existence of a G-decomposition of K_{2n}. This decomposition consists of subgraphs $G_0, G_2, \ldots, G_{2n-2}$ that are obtained from G_0 by cyclic shifts described in Definition 1. Such a decomposition will be called 2-cyclic.

Theorem 6. *Let $n = 2k + 1$ and G be a graph with $2n - 1$ edges and at most $2n$ vertices that allows a flexible q-labelling. Then there exists a 2-cyclic G-decomposition of K_{2n} into n copies of G.*

Proof. Take a subgraph G_0 of K_{2n} that has a flexible q-labelling and define its copies $G_2, G_4, \ldots, G_{2n-2}$ as in Definition 1. Since there are n copies of G_0 and each has $2n - 1$ edges, we only need to show that no edge belongs to two different copies of G. Suppose it is not the case. We can assume without loss of generality that an edge $(0, m)$ of length m belongs to both G_0 and $G_{2j}, 1 \leq j \leq n - 1$. First suppose that $m < n$. Then G_0 contains one more edge of length m, in particular the edge $(2s + 1, m + 2s + 1)$ for some $s, 0 \leq s \leq n - 1$. According to the definition of the 2-cyclic decomposition, G_{2j} contains exactly two edges of length m. One of them is

the edge $(2j, m + 2j)$ that can be identical to $(0, m)$ only if $j = 0$, which is impossible. Because G_{2j} has a flexible q-labelling, the other edge of length m must be $(2j+2s+1, m+2j+2s+1)$ for some $s, 0 \leq s \leq n-1$. Obviously, this edge can never be identical to $(0, m)$. Therefore $(2j+2s+1, m+2j+2s+1)$ is identical to $(2s + 1, m + 2s + 1)$. This is possible only if $j = 0$ which is again a contradiction.

The case when $m = n$ is essentially similar. Since G_{2j} contains just one edge of length $n = 2k + 1$, namely $(2j, 2j + 2k + 1)$, it is obvious that this can be identical to $(0, m) = (0, 2k + 1)$ only if $j = 0$. This is impossible and the proof is complete. ∎

One can notice that while for short edges the proof does not depend on the parity of n, the argument for the longest edge of length n does not hold if n is even. In this case we need more complicated labelling, which can be found in [3].

Now we present two infinite classes of asymmetric trees with flexible q-labellings.

Construction 7. Let $n = 2k+1$. We construct a graph G which consists of two stars, $K_{1,k}$ and $K_{1,2k-1}$, joined by an extra edge into a double star, and a path P_{k+1} joined by an edge to one of the vertices of degree 1 of the smaller star or to the central vertex of the other one. The first star, $K_{1,k}$, has the central vertex 1 and the leaves (i.e., vertices of degree 1) $2, 4, \ldots, 2k$. There are edges of all odd lengths $1, 3, 5, \ldots, 2k - 3, 2k - 1 = n - 2$, namely edges $(1, 2), (1, 4), \ldots, (1, 2k)$. The other star, $K_{1,2k-1}$, has the central vertex 0 and odd leaves $3, 5, \ldots, 2k - 1$. There are edges of odd lengths $3, 5, \ldots, 2k - 3, 2k - 1 = n - 2$, namely edges $(0, 3), (0, 5), \ldots, (0, 2k - 1)$. The only missing odd length is 1, which we obtain by joining the central vertices 0 and 1. Thus we have all odd edges twice with odd and even origin for each length as required. Moreover, there are edges $(0, 2k + 2), (0, 2k + 4), \ldots, (0, 4k)$ of even lengths $2k, 2k - 2, 2k - 4, \ldots, 4, 2$, respectively, with all origins having even labels.

The remaining even edges with odd origins we obtain by adding the path P_{k+1} induced on vertices $4k+1, 2k+1, 4k-1, 2k+3, 4k-3, \ldots, 3k-1, 3k+1$ when k is even and on vertices $4k + 1, 2k + 1, 4k - 1, 2k + 3, 4k - 3, \ldots, 3k - 2, 3k + 2, 3k$ when k is odd. Their respective lengths are $2k, 2k - 2, 2k - 4, \ldots, 4, 2$. Now we just need to add the single edge of length $n = 2k + 1$, joining P_{k+1} to the other component. We can choose this edge to be any one of the edges $(0, 2k + 1), (2, 2k + 3), \ldots, (2k, 4k + 1)$.

In Figure 1 we present an example for $n = 11$ with the longest edge $(0, 11)$. The dashed edges are the ones joining the subgraphs $K_{1,k}, K_{1,2k-1}$, and P_{k+1}.
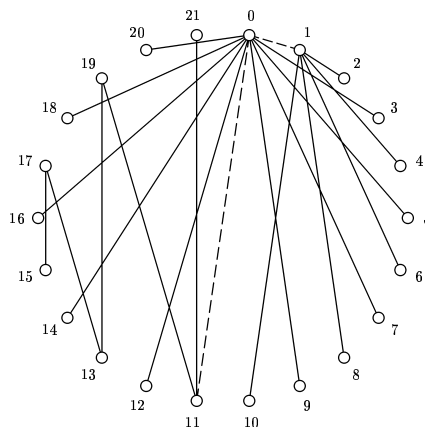

Figure 1

Construction 8. Let $n = 2k + 1$. This time we construct a graph G which consists of three stars, $K_{1,k-2}, K_{1,k+1}$, and $K_{1,2k}$, joined by two extra edges. First we construct the star $K_{1,2k}$ with the central vertex 0. There are k even leaves $4, 6, 8, \ldots, 2k - 2, 2k, 4k$ inducing edges with all even lengths $4, 6, 8, \ldots, 2k-2, 2k$, and 2, respectively. Furthermore there are k odd leaves $1, 2k + 3, 2k + 5, \ldots, 4k - 1$ inducing edges with all odd lengths except for $2k + 1 = n$, namely $1, 2k - 1, 2k - 3, \ldots, 5, 3$. Notice that the edge of length 1 has an even origin 0 while all other ones have odd origins.

Now we construct the star $K_{1,k-2}$ with the central vertex $4k + 1$. This star has the leaves $2k + 4, 2k + 6, \ldots, 4k - 2$ and the edges have lengths $2k - 3, 2k - 5, \ldots, 5, 3$. Their origins are even. It remains to find edges of odd lengths $1, 2k - 1$ and $2k + 1$ and edges of all even lengths. This will be attained by constructing the star $K_{1,k+1}$ with the central vertex $2k+1$. There will be the leaves $3, 5, \ldots, 2k - 1$ giving edges of lengths $2k - 2, 2k - 4, \ldots, 2$. All origins are now odd so the even edges satisfy our requirements. Notice that one even length, $2k$, with an odd origin is still missing. The first of the two additional leaves is 2, producing an edge of length $2k - 1$ and even origin 2. The remaining leaf is $2k + 2$, producing the edge of length 1 with an odd origin. Now we join the stars $K_{1,k-2}$ and $K_{1,k+1}$, using the last even edge of length $2k$. We actually have two choices here, either $(2k+1, 4k+1)$, or $(4k + 1, 2k - 1)$. In each case the origin ($2k + 1$ or $4k + 1$, respectively)

is odd. Finally we join one vertex of $K_{1,2k}$ by the edge of length $2k + 1 = n$ with its counterpart in the other component to complete the tree. We can for instance choose the edge $(0, 2k + 1)$.

In Figure 2 we present an example for $n = 11$. The edge of lenth $2k = 10$ is $(11, 21)$. The dashed edges are the ones joining the subgraphs $K_{1,k-2}$, $K_{1,k+1}$, and $K_{1,2k}$.
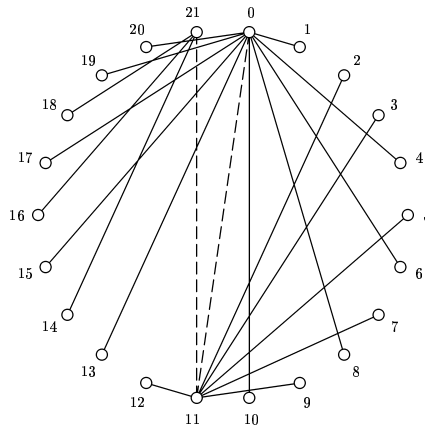

Figure 2

4.   Concluding Remarks

We have defined a new type of labelling that allows 2-cyclic decompositions of complete graphs into certain spanning trees that are not symmetric. However, our experience shows that there are infinite classes of trees for which 2-cyclic factorizations of K_{2n} do not exist although there is no reason to a priori assume that *no* factorizations into such trees exist. Therefore, more powerful tools are needed. We believe that maybe some methods used in design theory could be useful.

Acknowledgement

The author would like to thank Petr Kovář for drawing the figures.

References

[1] J. Bosák, Decompositions of Graphs (Kluwer, Dordrecht, 1990).

[2] P. Eldergill, Decompositions of the complete graph with an even number of vertices (M.Sc. thesis, McMaster University Hamilton, 1997).

[3] D. Fronček and M. Kubesa, *Factorizations of complete graphs into spanning trees*, Congressus Numerantium **154** (2002) 125–134.

[4] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic Journal of Combinatorics **DS6** (2001).

[5] G. Ringel, *Problem* 25, in: Theory of Graphs and its Applications, (Proc. Symp. Smolenice 1963) ed., M. Fiedler (Academia, Prague, 1964) 162.

[6] A. Rosa, Cyclic decompositions of complete graphs (Ph.D. thesis, Slovak Academy of Science, Bratislava, 1965).

[7] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris (1967) 349–355.