ON THE DOMINATION NUMBER OF PRISMS
OF GRAPHS

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Abstract

For a permutation $\pi$ of the vertex set of a graph $G$, the graph $\pi G$ is
obtained from two disjoint copies $G_1$ and $G_2$ of $G$ by joining each $v$ in
$G_1$ to $\pi(v)$ in $G_2$. Hence if $\pi = 1$, then $\pi G = K_2 \times G$, the prism
of $G$. Clearly, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$. We study graphs for
which $\gamma(K_2 \times G) = 2\gamma(G)$, those for which $\gamma(\pi G) = 2\gamma(G)$ for at least one
permutation $\pi$ of $V(G)$ and those for which $\gamma(\pi G) = 2\gamma(G)$ for each
permutation $\pi$ of $V(G)$.

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1. Introduction

We generally follow the notation and terminology of [4]. Specifically, in
a graph $G = (V(G), E(G))$, if $S, T \subseteq V(G)$, then the set of all edges of $G$
with one endvertex in $S$ and the other endvertex in $T$ is denoted by $E(S,T)$. Further, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the open and closed neighbourhoods, respectively, of a vertex $v$ of $G$. The closed neighbourhood of a set $S \subseteq V(G)$, denoted by $N[S]$, is the set $\bigcup_{s \in S} N[s]$ and the open neighbourhood $N(S)$ of $S$ is $\bigcup_{s \in S} N(s)$. We will also need the set $N\{S\} = N(S) - S$. If $s \in S$, then the private neighbourhood of $s$ relative to $S$, denoted by $pn(s,S)$, is the set $N[s] - N[S - \{s\}]$. A vertex in $pn(s,S)$ is called a private neighbour, abbreviated $pn$, of $s$ relative to $S$.

As usual $\gamma(G)$ denotes the domination number of $G$. The set $S \subseteq V(G)$ is a $\gamma$-set if it is a dominating set with $|S| = \gamma(G)$. For $A, B \subseteq V(G)$, we abbreviate “$A$ dominates $B$” to “$A \triangleright B$”; if $B = V(G)$ we write $A \triangleright G$ and if $B = \{b\}$ we write $A \triangleright b$. The subgraph induced by $B$ is denoted by $\langle B \rangle$. A universal vertex is one that is adjacent to all other vertices of the graph. The double star $S(k,l)$ is the graph obtained by joining the central vertices of the stars $K_{1,k}$ and $K_{1,l}$.

For a permutation $\pi$ of the vertex set of a graph $G$, the graph $\pi G$ is obtained from two disjoint copies $G_1$ and $G_2$ of $G$ by joining each $v \in G_1$ to $\pi(v)$ in $G_2$, that is, $V(\pi G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{\{v, \pi(v)\} : v \in V(G_1), \pi(v) \in V(G_2)\}$. Hence if $\pi = 1$, then $\pi G = K_2 \times G$; this graph is sometimes referred to as the prism of (or over) $G$. Thus we may also think of $\pi G$, where $\pi$ is an arbitrary permutation of $V(G)$, as a type of prism of $G$. If $C_5$ has vertex sequence $0,1,2,3,4$ and $\pi$ is the permutation $i \rightarrow 2i \pmod 5$, then $\pi C_5$ is the Petersen graph; other Petersen-type graphs are obtained similarly. Clearly, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$.

We are mostly interested in graphs where the domination number of (some of) the prisms is equal to twice the domination number of the graph. We investigate graphs for which $\gamma(K_2 \times G) = 2\gamma(G)$, called prism doublers, those for which $\gamma(\pi G) = 2\gamma(G)$ for some but not all permutations $\pi$ of $V(G)$, called partial doublers, and those for which $\gamma(\pi G) = 2\gamma(G)$ for each permutation $\pi$ of $V(G)$, called universal doublers. As we shall see, the double star $S(2,2)$ is an example of a graph which satisfies $\gamma(\pi G) = 2\gamma(G)$ for $\pi = 1$ but not for all permutations $\pi$ of $V(G)$, i.e., a prism doubler but not a universal doubler, $P_5$ is an example of a graph that satisfies $\gamma(\pi G) = 2\gamma(G)$ for at least one permutation $\pi$ of $V(G)$ but not for $\pi = 1$, i.e., a partial doubler, and $C_6$ is an example of a universal doubler. In addition, the graph $G$ obtained from $C_4$ by joining one of its vertices to two new vertices is an example of a graph that satisfies $\gamma(\pi_k G) = \gamma(G) + k$ for some permutation $\pi_k$ of $V(G)$, $0 \leq k \leq \gamma(G)$.  

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For any vertex $v$ of $G$, we denote the corresponding vertex in the subgraph $G_i$, $i = 1, 2$, of $\pi G$ by $v_i$. Similarly, any set $X \subseteq V(G)$ will be denoted by $X_i$ when considered in the subgraph $G_i$ of $\pi G$. Conversely any set $X_i \subseteq V(G_i)$ (vertex $v_i \in V(G_i)$ respectively) will be denoted by $X$ (vertex $v$ respectively) in $G$.

2. Universal Doublers

The following lemma collects some useful facts about graphs with isolated vertices.

**Lemma 1.** (a) (Ore, see [6]) If $\gamma(G) > |V(G)|/2$, then $G$ has an isolated vertex.

(b) If $G$ has an isolated vertex $v$ and $\pi$ is a permutation of $V(G)$ such that $\pi(v) = v$, then $\gamma(\pi G) < 2\gamma(G)$.

**Proof.** (b) Suppose $G$ has an isolated vertex $v$. Let $G'$ be the (possibly empty) subgraph induced by the other vertices of $G$. Suppose $\pi$ is a permutation of $V(G')$ with $\pi(v) = v$, and let $\pi'$ be the permutation of $V(G')$ induced by $\pi$. Then $\gamma(\pi G) = \gamma(\pi' G') + 1 < 2\gamma(G') + 2 = 2\gamma(G)$. ■

**Proposition 2.** A graph $G$ is a universal doubler if and only if for each $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, $|V(G) - N[X]| \geq 2\gamma(G) - |X|$.

**Proof.** Consider any $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and let $Y = V(G) - N[X]$. Suppose $|Y| < 2\gamma(G) - |X|$. If $|Y| < \gamma(G)$, let $D$ be any $\gamma$-set of $G$, and if $|Y| \geq \gamma(G)$, let $D$ be any dominating set of $G$ with $|D| = |Y|$. Let $\pi$ be any permutation of $V(G)$ such that $\pi(Y) \subseteq D$. Then $W = X_1 \cup D_2 > \pi G$ (since $X_1 \succ V(G_1) - Y_1$ and $D_2 \succ V(G_2) \cup Y_1$), and $|W| = |X_1| + |D_2| < 2\gamma(G)$.

Conversely, let $\pi$ be a permutation of $V(G)$ such that $\gamma(\pi G) < 2\gamma(G)$ and let $W = X_1 \cup D_2$ be a $\gamma$-set of $\pi G$, where $X_1 = W \cap V(G_1)$ and $D_2 = W \cap V(G_2)$. Without loss of generality let $|X_1| < \gamma(G)$.

If $X_1 \neq \phi$, then $|D_2| = \gamma(\pi G) - |X| < 2\gamma(G) - |X|$. Since $D_2 \succ Y_1$ and each vertex of $D_2$ covers at most one vertex of $Y_1$, we have $|Y_1| \leq |D_2|$, so $|Y| < 2\gamma(G) - |X|$ as desired.

If $X_1 = \phi$, then $D_2 \succ V(\pi G)$, which means $D_2 = V(G_2)$, since each vertex of $G_2$ covers just one vertex of $G_1$. Then $|V(G)| = |D_2| < 2\gamma(G)$, so by Lemma 1(a), $G$ has an isolated vertex, say $v$. Let $u = \pi^{-1}(v)$. Then $N_{\pi G}[v_2] = \{v_2, u_1\}$, so if we define $X'_1 = \{u_1\}$ and $D'_2 = D_2 - \{v_2\}$, then
$W' = X'_1 \cup D'_2$ is a $\gamma$-set of $\pi G$ with $X'_1 \neq \emptyset$, reducing to the previous case.

The following corollary gives some degree properties of universal doublers. First, recall that $S \subseteq V(G)$ is a packing if the vertices in $S$ have mutually disjoint closed neighbourhoods, and that a packing which dominates $G$ is called an efficient dominating set \[4, \text{p. 108}\].

**Corollary 3.** Let $G$ be a universal doubler. Then $G$ has no isolated vertices and every vertex of $G$ that is contained in a minimum dominating set has degree at least $\gamma(G)$.

If $G$ has an efficient dominating set, then $\gamma(G) \leq \sqrt{|V(G)| + 0.25} - 0.5$. Otherwise, for any nonempty packing $X$ contained in a minimum dominating set of $G$ we have $\gamma(G) \leq |V(G)|/(|X| + 2)$.

**Proof.** Let $G$ satisfy the hypothesis. By Lemma 1(b), $G$ has no isolated vertex.

Let $w$ be a vertex of $G$ that is contained in some minimum dominating set $D$ of $G$. If $\gamma(G) = 1$, then since $w$ is not isolated, we have $\deg(w) \geq \gamma(G)$. If $\gamma(G) > 1$, let $X = D - \{w\}$. Then $0 < |X| < \gamma(G)$, so by Proposition 2, $|V(G) - N[X]| \geq 2\gamma(G) - |X| = \gamma(G) + 1$. Since $D$ is a dominating set, we see $\{w\} \succ V(G) - N[X]$, which implies $\deg(w) \geq \gamma(G)$.

Finally, suppose that $X$ satisfies the hypotheses. If $X$ dominates $G$, then since $X$ is a packing, $|V(G)| \geq \gamma(G)(\gamma(G) + 1)$, from which the desired bound follows. Otherwise Proposition 2 applies to $X$, giving $|V(G)| - |X|\gamma(G) + 1 \geq 2\gamma(G) - |X|$ as needed.

Any vertex of $P_6$ or $C_6$ dominates at most three vertices and thus leaves at least three vertices undominated; hence by Proposition 2, if $G \in \{P_6, C_6\}$, then $\gamma(\pi G) = 4$ for each permutation $\pi$ of $V(G)$. The next corollary allows us to produce more universal doublers.

**Corollary 4.** Suppose $G$ is an $r$-regular graph that has an efficient dominating set and $r \geq \gamma(G)$. Then $\gamma(\pi G) = 2\gamma(G)$ for each permutation $\pi$ of $V(G)$.

**Proof.** Let $X \subseteq V(G)$, with $0 < |X| < \gamma(G)$; then $|N[X]| \leq (r + 1)|X|$. Since $G$ has an efficient dominating set, $|V(G)| = (r + 1)\gamma(G)$. Thus $|V(G) - N[X]| \geq (r + 1)(\gamma(G) - |X|)$, which is at least $2\gamma(G) - |X|$ since $r \geq \gamma(G)$, and the conclusion follows from Proposition 2.
We use circulant graphs to obtain examples of universal doublers. Let $r$, $k$ be positive integers, and let $n = k(r + 1)$. Let $p$ be the largest odd divisor of $r + 1$, and let $m = (r + 1)/p$. Define a graph $G_{r,k}$ as follows. Let $V(G_{r,k}) = \{0, 1, \ldots, n-1\}$. Two distinct vertices $v, w$ are adjacent if there is some $i$ with $|i| \leq (p-1)/2$ and $v - w \equiv i \pmod{p}$. (Less formally, for each vertex $v$, the closed neighbourhood $N[v]$ consists of $m$ runs of $p$ vertices, with equal spacing between the runs and one run centered on $v$.) Then $G_{r,k}$ is $r$-regular; since $G_{r,k}$ has the efficient dominating set $\{ip : 0 \leq i \leq k - 1\}$, $\gamma(G_{r,k}) = k$. For $r \geq k$, Corollary 4 implies $\gamma(\pi G_{r,k}) = 2\gamma(G_{r,k})$ for each permutation $\pi$ of $V(G_{r,k})$.

The graph $G_{r,k}$ is connected except when $r = 2j - 1$ for some positive integer $j$, when $p = 1$ and $G_{r,k}$ consists of $k$ disjoint copies of the complete graph $K_{r+1}$. When $r > 1$, and $r, k$ are odd, we can change the edge set to obtain a connected graph: say distinct $v, w$ are adjacent if either $|v - w| = n/2$ or there is $i$ with $|i| \leq (r-1)/2$ and $v - w \equiv i \pmod{n}$. The resulting graph $G_{r,k}^*$ is connected and $r$-regular with an efficient dominating set $\{ir + 1 : 0 \leq i \leq k - 1\}$; thus $\gamma(G_{r,k}^*) = k$. For $r \geq k$, Corollary 4 implies $\gamma(\pi G_{r,k}^*) = 2\gamma(G_{r,k}^*)$ for each permutation $\pi$ of $V(G_{r,k})$.

Also, the 3-cube $G = (K_2)^3$ satisfies the hypotheses of Corollary 4 with $r = 3$ and $\gamma(G) = 2$, hence $\gamma(\pi G) = 4$ for each permutation $\pi$ of $V(G)$.

3. Prism Doublers

In this section we consider graphs $G$ where the domination number of the (usual) prism of $G$ is twice that of $G$. We begin with a lemma which will allow us to study dominating sets of $G$ instead of those of its prism.

**Lemma 5.** The set $X$ with $X \cap V(G_1) = A_1$, $X \cap V(G_2) = B_2$ dominates $K_2 \times G$ if and only if in $G$, every vertex not in $A \cup B$ is adjacent to a vertex in $A$ and to a vertex in $B$.

**Proof.** Suppose $X \succ K_2 \times G$ and consider $v \in V(G) - (A \cup B)$. Now, $v_1 \notin B_1$, thus $B_2 \neq v_1$ and therefore $A_1 \succ v_1$, that is, $v$ is adjacent in $G$ to a vertex in $A$. Similarly, since $v_2 \notin A_2$, $v$ is adjacent to a vertex in $B$. Conversely, consider any $x_i \in V(G_i)$. If $x_i \notin A_1 \cup B_1$, then obviously $X \succ x_i$. If $x_i \notin A_1 \cup B_1$, then $x$ is adjacent to a vertex in $A$ and to a vertex in $B$ and hence $A_1 \succ x_1$, $B_2 \succ x_2$. \[\square\]
If $A$ and $B$ are sets such that $A \cup B$ satisfies the conditions of Lemma 5, we say that $(A, B)$ is a dominating pair of $G$. Clearly $\gamma(K_2 \times G) = \min\{|A| + |B| : (A, B)$ is a dominating pair of $G\}$. Say that a dominating pair $(A, B)$ is minimum if $|A| + |B| = \gamma(K_2 \times G)$. Given a set $A$, we say we extend $A$ to a dominating pair of $G$ if we find a set $B$ such that $(A, B)$ is such a pair. Note that $(A, \phi)$ is a dominating pair if and only if $A = V(G)$, and $(A, A)$ is a dominating pair if and only if $A \supseteq G$. More generally we have the following result which we formulate as a corollary for easy reference.

**Corollary 6.** The pair $(A, B)$ is a dominating pair of $G$ if and only if $V(G) - N[A] \subseteq B$ and $V(G) - N[B] \subseteq A$.

If $\gamma(G) = 1$ and $G$ has at least two vertices, then $\gamma(K_2 \times G) = 2$. Now consider the case $\gamma(G) = 2$; we assume that $G$ has no isolated vertices, for otherwise it is easy to see that $\gamma(K_2 \times G) < 2\gamma(G)$.

Construct the class $\mathcal{H}$ of graphs as follows. For any $H \in \mathcal{H}$, $V(H) = S \cup T \cup \{u, v, w\}$ and $E(H)$ consists of edges such that

- $N(u) = S$, $N(v) = T$, $N(w) = S \cup T$,
- $E(\langle S \rangle)$ and $E(\langle T \rangle)$ consist of any edges such that at least one of $\langle S \rangle$ and $\langle T \rangle$ has a universal vertex,
- $E(S, T)$ is arbitrary.

Note that if (without loss of generality) $x \in S$ is a universal vertex of $\langle S \rangle$, then $\{x, v\}$ is a $\gamma$-set of $H$ for any $H \in \mathcal{H}$, while $\langle \{u, v\}, \{w\} \rangle$ is a dominating pair of $H$. Hence $\gamma(H) = 2$ and $\gamma(K_2 \times H) \leq 3$.

**Proposition 7.** If $\gamma(G) = 2$, then $\gamma(K_2 \times G) = 4$ if and only if $G \notin \mathcal{H}$ and for each $\gamma$-set $X = \{x_1, x_2\}$ of $G$, $|\text{pn}(x_i, X)| \geq 2$ for each $i$, where $|\text{pn}(x_i, X)| = 2$ implies that $\text{pn}(x_i, X)$ does not dominate $G - x_j$, $i \neq j$.

**Proof.** Suppose $\gamma(K_2 \times G) = 4$; clearly $G \notin \mathcal{H}$. Let $X = \{x, x'\}$ be any $\gamma$-set of $G$. If $\text{pn}(x, X) = \{y\}$, then $\langle \{x', y\}, \{x'\} \rangle$ is a dominating pair of $G$ and $\gamma(K_2 \times G) \leq 3$, a contradiction. Hence each vertex in $X$ has at least two $X$-pns. If $Y = \text{pn}(x, X)$ dominates $G - x'$ and $|Y| = 2$, then $(Y, \{x'\})$ is a dominating pair, again a contradiction.

Conversely, suppose $\gamma(K_2 \times G) < 4$ and suppose that $(A, B)$ is a minimum dominating pair of $G$ such that $|A| \leq |B|$. If $A$ is empty then $B = V(G)$ so $|V(G)| \leq 3$. Then $\gamma(G) = 2$ and Lemma 1(a) imply $G$ has an isolated}
vertex, and the conclusion follows. So we may assume \(|A| = 1\) and \(|B| \leq 2\), say \(A = \{w\}, Y = V(G) - N[w]\). If \(Y = \{y\}\), then \(\{y, w\}\) is a \(\gamma\)-set of \(G\) in which \(y\) has itself as its only \(pn\) and we are done.

By Corollary 6, \(Y \subseteq B\) and so \(|Y| \leq |B| \leq 2\). Hence we may assume that \(|Y| = 2\); say \(Y = \{u, v\}\). Then \(B = \{u, v\}\) and since \((A, B)\) is a dominating pair, \(B \succ G - w\) and \(w \succ G - \{u, v\}\). Therefore, if there is a vertex \(y\) in \(G\) that dominates \(\{u, v\}\), then \(\{y, w\}\) is a \(\gamma\)-set of \(G\) such that \(pn(y, \{y, w\}) = B \succ G - w\). If there is no such vertex \(y\), then \(d(u, v) \geq 3\). If \(d(u, v) \geq 5\), then in any shortest \((u, v)\)-path there are at least two vertices that are not dominated by \(\{u, v\}\), contradicting the fact that \(B \succ G - w\).

Hence \(3 \leq d(u, v) \leq 4\).

If \(d(u, v) = 4\), then \(w\) is the unique vertex of \(G\) not dominated by \(\{u, v\}\), and \(w\) lies on each \((u, v)\)-path. Since \(\gamma(G) = 2\), it follows that to dominate \(\{u, v, w\}\), there is at least one vertex that dominates (without loss of generality) both \(u\) and \(w\), as well as each central vertex on any \((u, w)\)-path of length two. Thus \(G \in \mathcal{H}\) with \(E(S, T) = \phi\). Similarly, if \(d(u, v) = 3\), then \(G \in \mathcal{H}\) with \(E(S, T) \neq \phi\).

This result can be generalized as follows. (Recall that \(N\{X\} = N[X] - X\).)

**Theorem 8.** A graph \(G\) is a prism doubler if and only if for each pair of sets \(X, Y \subseteq V(G)\) with \(0 < |X| < \gamma(G)\) and \(Y = V(G) - N[X]\), either

(a) \(|Y| \geq 2\gamma(G) - |X|\), or

(b) \(|Y| = 2\gamma(G) - |X| - d\) for some \(d\), \(1 \leq d \leq |X|\), and at least \(d\) vertices (necessarily in \(N[X]\)) are required to dominate \(N\{X\} - N[Y]\).

**Proof.** Suppose \(\gamma(K_2 \times G) = 2\gamma(G)\) and consider any \(X \subseteq V(G)\) with \(0 < |X| < \gamma(G)\). Note that \((X, X \cup Y)\) is a dominating pair of \(G\). If \(|Y| \geq 2\gamma(G) - |X|\), we are done. If \(|Y| < 2\gamma(G) - 2|X|\), then \(|X| + |X \cup Y| < 2|X| + 2\gamma(G) - 2|X| = 2\gamma(G)\). Hence we assume that \(2\gamma(G) - 2|X| \leq |Y| \leq 2\gamma(G) - |X| - 1\); say \(|Y| = 2\gamma(G) - |X| - d\), where \(1 \leq d \leq |X|\). Suppose there is \(Z \subseteq N[X]\) such that \(|Z| \leq d - 1\) and \(Z \succ N\{X\} - N[Y]\). Then \(N\{X\} \subseteq N[Y] \cup N[Z]\), hence \(V(G) - N[Y \cup Z] = V(G) - (N[Y] \cup N[Z]) \subseteq X\) and so by Corollary 6, \((X, Y \cup Z)\) is a dominating pair of \(G\). But \(|X| + |Y \cup Z| \leq |X| + (2\gamma(G) - |X| - d) + (d - 1) = 2\gamma(G) - 1\), a contradiction. Hence (b) holds.

Conversely, suppose \(\gamma(K_2 \times G) < 2\gamma(G)\) and consider any minimum dominating pair \((X, D)\) of \(G\). Since \(|X| + |D| \leq 2\gamma(G)\), we may assume
Further, |X| < \gamma(G) and |D| < 2\gamma(G) - |X|. If X is empty then D = V(G) so 
\gamma(K_2 \times G) = |V(G)|. Thus |V(G)| < 2\gamma(G), so by Lemma 1(a), G has an
isolated vertex, say v. Then \{v\}, D - \{v\} is a minimum dominating pair, so
we may assume X is nonempty. Let Y = V(G) - N[X]. By Corollary 6, Y \subseteq D
and so |Y| < 2\gamma(G) - |X|; hence (a) does not hold. If |Y| < 2\gamma(G) - 2|X|,
then (b) does not hold either (because d is not in the stated range) and we
are done. Hence suppose |Y| = 2\gamma(G) - |X| - d for some d with 1 \leq d \leq |X|.
Let Z = D - Y; then Y \cup Z = D > V(G) - X and so Z > N(X) - N[Y].
Further, |Z| = |D| - |Y| < (2\gamma(G) - |X|) - (2\gamma(G) - |X| - d) = d. Hence
(b) does not hold.

We now apply Proposition 7 and Theorem 8 to paths and cycles to show
that with the exception of P_3, C_3 (which have \gamma = 1), P_6 and C_6 (see
Section 2), no path or cycle is a prism doubler. Let P_n, C_n have vertex
sequence 1, 2, \ldots, n.

- Note that P_4, C_4 have \gamma-sets in which a vertex has only one pn; hence
  by Proposition 7, \gamma(K_2 \times P_4), \gamma(K_2 \times C_4) < 4. Also, P_5 \in \mathcal{H}, hence
  \gamma(K_2 \times P_5) < 4. For any two non-adjacent vertices x and y of C_5,
  X = \{x, y\} is a \gamma-set of C_5, where x has two pns (one of which is x) and
  pn(x, X) dominates V(G) - \{y\}. Thus \gamma(K_2 \times C_5) < 4.

- For n \geq 7, write n = 4i + r with 0 \leq r \leq 3. For G \in \{P_n, C_n\}, let
  X = \{4j - 1 : j = 1, \ldots, i\}, except for P_{4i} let X = \{4j - 1 : j = 1, \ldots, i - 1\} \cup \{4i\}. Then |X| = i in all cases. Let Y = V(G) - N[X].
  Except for G = P_{4i}, Y = \{4j - 3 : j = 1, \ldots, i\} \cup \{4i + j : 1 \leq j \leq r\}.
  For P_{4i}, Y = \{4j - 3 : j = 1, \ldots, i\} \cup \{4i - 2\}. Thus |Y| = i + r except
  for P_{4i}, when |Y| = i + 1. Then using \gamma(G) = \lceil n/3 \rceil, it is straightforward
to verify |X| < \gamma(G) and |Y| < 2\gamma(G) - |X|.

In all cases, X contains neither vertices at distance less than four nor,
when G = P_n, any vertex adjacent to an end vertex. Therefore Y
\supset V(G) - X. In particular, N(X) - N[Y] = \emptyset, so X does not satisfy the
conditions of Theorem 8, and thus \gamma(K_2 \times G) < 2\gamma(G).

As an example of a prism doubler that is not a universal doubler, con-
sider the double star S(k,l) with k, l \geq 2. Proposition 7 shows that
\gamma(K_2 \times S(k,l)) = 4. However, if k = 2, then Figure 1 illustrates a per-
mutation \pi such that \gamma(\pi S(k,l)) = 3. A similar result holds for the graph
K_n(k_1, \ldots, k_n), k_i \geq n, obtained by joining k_i new vertices to vertex v_i of
K_n with V(K_n) = \{v_1, \ldots, v_n\}.
We now consider prism doublers with the additional properties that they are regular and have efficient dominating sets; say that such a graph is a perfect doubler.

Above we have shown that $C_{3i}$ is a perfect doubler only when $i \leq 2$. Here is an infinite family of examples: for each positive integer $m$, set $G_m = K_2 \times C_{4m}$. Then $G_m$ is a 3-regular graph with $8m$ vertices, and has an efficient dominating set $\{(0, 4j), (1, 4j + 2) : 0 \leq j \leq m - 1\}$ of size $2m$. It can be proved by induction that $G_m$ is a perfect doubler for each $m$. Also, as shown in the previous section, the graphs $G_{r,k}$ and $G_{r,k}^*$ are perfect doublers when $r \geq k$.

A more interesting family of examples is provided by the hypercubes. For a positive integer $p$, consider the $(2^p - 1)$-dimensional binary hypercube $G_p = (K_2)^{2^p-1}$. This graph is $(2^p - 1)$-regular and has efficient dominating sets (also known as perfect single-error-correcting codes), for example the Hamming code $H_p$ (see for Example [7, p. 423]). Thus $\gamma(G_p) = 2^{2^p-1-p}$.

It has been shown (in [5] and independently in [8]) that $\gamma(K_2 \times G_p) = \gamma((K_2)^{2^p}) = 2^{2^p-p} = 2\gamma(G_p)$.

The ternary hypercubes with efficient dominating sets are also perfect doublers [1]. That is, for $p \geq 1$, let $T_p = (K_3)^{(3^p-1)/2}$. Then $\gamma(T_p) = 3^{((3^p-1)/2)-p}$ and $\gamma(K_2 \times T_p) = 2\gamma(T_p)$.

Here is a version of Theorem 8 useful for regular graphs with efficient dominating sets.
Corollary 9. Let $G$ be an $r$-regular graph with an efficient dominating set. Suppose that for each subset $X$ of $V(G)$ with $\frac{r-1}{r}\gamma(G) < |X| < \gamma(G)$ and $(r-1)\gamma(G) < |N[X]|$, there does not exist any set $W$ satisfying $V(G) - N[X] \subseteq W \subseteq V(G)$ and $|W| < 2\gamma(G) - |X|$ and $W > V(G) - X$. Then $G$ is a perfect doubler.

Proof. Suppose that $G$ is an $r$-regular graph with an efficient dominating set, but not a perfect doubler. Then there is some $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$ for which conditions (a) and (b) of Theorem 8 both fail. Set $Y = V(G) - N[X]$. Since (a) does not hold, $|Y| < 2\gamma(G) - |X|$. As $|V(G)| = (r + 1)\gamma(G)$, this implies $(r - 1)\gamma(G) < |N[X]| - |X| = |N\{X\}|$. The $r$-regularity of $G$ gives $|N\{X\}| \leq r|X|$, so $\frac{r-1}{r}\gamma(G) < |X|$.

Since (b) does not hold, $|Y| = 2\gamma(G) - |X| - d$ for some $d$, $1 \leq d \leq |X|$, and some vertex set $S$ satisfies $|S| < d$ and $S \supseteq N\{X\} - N[Y]$. Set $W = Y \cup S$. Then $|W| < 2\gamma(G) - |X|$ and $W > V(G) - X$.

We need the following definition from [3] to give another example of a perfect doubler. Let $k$ be an integer, $k > 1$, and let $S = \{1, 2, \ldots, 2k - 1\}$. The vertices of the odd graph $O_k$ are the subsets of $S$ of cardinality $k - 1$, and two vertices of $O_k$ are adjacent if they are disjoint sets.

Thus $O_k$ is a $k$-regular graph with $C(2k - 1, k - 1)$ vertices. For example, $O_2 \cong K_3$ and $O_3$ is isomorphic to the Petersen graph. Biggs has shown [2, Sections 3j, 21b, 21j] that if $O_k$ has an efficient dominating set, then $k$ is even. Such sets are known to exist for $k = 2, 4, 6$ and it is conjectured [3] that there are no more.

We next show that $O_2$ and $O_4$ are perfect doublers. We will need the following result, equivalent to [3, Section 2.3]: the distance between vertices $v, w$ of $O_k$ is

$$d(v, w) = \begin{cases} 2|v \cap w| + 1 & \text{if } |v \cap w| < (k - 1)/2, \\ 2(k - 1 - |v \cap w|) & \text{otherwise}. \end{cases}$$

This implies that $O_k$ has diameter $k - 1$.

Proposition 10. For $G \in \{O_2, O_4\}$, $\gamma(K_2 \times G) = 2\gamma(G)$.

Proof. Since $O_2 \cong K_3$, it suffices to consider $O_4$, which is a 4-regular graph with 35 vertices, diameter 3, and domination number 7; an efficient
dominating set of $O_4$ is $\{123, 145, 167, 246, 257, 347, 356\}$. Here and later, we write vertices of $O_4$ as strings of length 3.

By Corollary 9, we need only examine vertex sets $X$ with $\frac{3}{4}\gamma(O_4) < |X| < \gamma(O_4)$, which means $|X| = 6$. We also may assume $3\gamma(O_4) < |N\{X\}|$, and then $|X| = 6$ implies $|N[X]| \geq 28$. Since $O_4$ is 4-regular and $|X| = 6$, $|N[X]| \leq 30$.

For a vertex set $X$, choose a maximum size packing $X'$ inside $X$. For distinct vertices $v, w$ of $X'$, it follows from (1) that $|v \cap w| = 1$.

With $X \subseteq V(O_4)$ of size 6, consider the possibility $|X'| \geq 5$. At least one of the indices 1, . . . , 7 must then occur at least 15/7 times among the vertices of $X'$; that is, some index occurs at least three times. Without loss of generality, we may assume that 123, 145, 167 are in $X'$. No other vertex of $X'$ can then contain 1 (otherwise such a vertex will have at least two indices in common with one of 123, 145 or 167). Since vertices of $X'$ are not adjacent, some index occurs at least twice in the remaining members of $X'$, so we may assume 246 and 257 are in $X'$. If $|X'| = 6$, it is not difficult to see that the last member of $X'$ is either 347 or 356.

If $|N[X]| = 30$ then $X = X'$, and by the previous paragraph, $X$ consists of all but one vertex, say $z$, of an efficient dominating set of $O_4$. Then $V(O_4) - N[X] = N[z]$ has 5 elements. Note that $N[z]$ dominates a vertex $w \in V(O_4) - N[z]$ if and only if $d(z, w) = 2$, that is, if and only if (by (1)) $|z \cap w| = 2$. Thus $N[z]$ dominates 12 vertices in $V(O_4) - N[z]$ and 17 vertices in total. Two more vertices will cover at most 10 more vertices, so it is not possible to find a set $W$ of 7 vertices including those of $N[z]$ that covers the 29 vertices of $V(O_4) - X$.

For the remainder of the proof, it is helpful to consider the situation where there is a vertex not in $X$ that is covered by more than one vertex in $X$. Without loss of generality, we may assume that 135 and 357 are in $X$; these both cover 246. The vertices that are at distance 3 from both 135 and 357 may be divided into three families: $\mathcal{F}_1 = \{234, 236, 346\}$, $\mathcal{F}_2 = \{245, 256, 456\}$, and $\mathcal{F}_3 = \{127, 147, 167\}$.

If $|N[X]| = 29$ then $V(O_4)$ contains one vertex doubly covered by $X$ and 28 that are singly covered. Thus one internal distance of the set $X$ is 2, and the other distances are 3. We may assume 135, 357 are the members of $X$ at distance 3; then the remaining four vertices of $X$ are in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Since the internal distances of each $\mathcal{F}_i$ are all 2, this is not possible.

If $|N[X]| = 28$, then from $V(O_4) - N[X] \subseteq W$ and $|W| < 2\gamma(O_4) - |X|$ we see that $W = V(O_4) - N[X]$.
If some vertex not in $X$ is multiply covered by $X$, we may again assume that $135, 357 \in X$ and $246 \not\in X$. It then follows from $|N[X]| \geq 28$ that three vertices of $X$, say $x_1, x_2, x_3$, are at distance 3 from each other and from each of $135, 357$. Since the internal distances of each $F_i$ are all 2, each $F_i$ contains one $x_j$. However, the four neighbours of 246 are 135 and 357 (which are in $X$), 157 (which is adjacent to every member of $F_1$), and 137 (adjacent to every member of $F_2$). Thus $N(246) \subseteq N[X]$, so $W = V(O_4) - N[X]$ does not cover 246.

The only other conceivable way to have $|N[X]| = 28$ is for $X$ to have one internal distance of 1, with the other distances being 3. Without loss of generality, we may assume 123 and 567 are the adjacent members of $X$. Then the other vertices $v_1, v_2, v_3, v_4$ of $X$ have the form $e_i4e_j$ with $e_i \in \{1, 2, 3\}$ and $e_j \in \{5, 6, 7\}$. But this implies that for some $h, k$ we have $|v_h \cap v_k| = 2$, and then $d(v_h, v_k) = 2$. So $|N[X]| = 28$ cannot be achieved in this way. □

Our work leads us to believe that something along the following lines is true.

**Conjecture 11.** Let $G$ be a regular graph with an efficient dominating set. If every sufficiently large packing in $G$ extends to an efficient dominating set, then $G$ is a perfect doubler.

### 4. Partial Doublers

Recall that if $\gamma(\pi G) = 2\gamma(G)$ for some but not all permutations $\pi$ of $V(G)$, then $G$ is called a partial doubler. As noted above, $\gamma(K_2 \times P_3) < 4$. However, if $\pi$ is the transposition which maps the central vertex of $P_3$ to one of its neighbours and vice versa, then $\gamma(\pi P_3) = 4$, so $P_3$ is a partial doubler. A natural question which now arises is whether, for each integer $k \geq 2$, there exists a graph $G$ and permutations $\pi_1, \pi_2$ such that $\gamma(\pi_1 G) = \gamma(G) = k$ and $\gamma(\pi_2 G) = 2k$, or more generally, permutations $\pi_i$, $0 \leq i \leq \gamma(G)$, such that $\gamma(\pi_i G) = \gamma(G) + i$. We answer this question by constructing a large class $G$ of such graphs.

**Construction.** Let $G$ be any isolate-free bipartite graph of order $n$ with bipartition $(X, Y)$, $s = |X|$, $t = |Y|$, $s \leq t$, and note that $X$ is a (not necessarily minimum) dominating set of $G$ such that each edge is incident with exactly one vertex in $X$. Construct $G^*$ with $V(G^*) = X \cup Y \cup A \cup B$ as follows. (The construction is illustrated in Figure 2 for $G = C_4$, where the black vertices are in $X$ and the grey vertices are in $Y$.) First replace
each edge $xy \in E(G)$ with $k \geq 2$ (not necessarily fixed) multiple edges, then subdivide each of these new edges once (the white vertices of degree two in Figure 2, where $k = 2$ in each case). Denote the set of new degree two vertices incident with $x$ and $y$ by $A^{xy}$. Join each vertex $x \in X$ to a set $B^x$ of new vertices (the white vertices of degree one in Figure 2), where $|B^x| \geq 2$ and $\Sigma_{x \in X} (|B^x| - 2) \geq t$. Let $A = \bigcup_{xy \in E(G)} A^{xy}$ and $B = \bigcup_{x \in X} B^x$. Any graph $G^*$ obtained in this way is in $\mathcal{G}$.

![Figure 2. An example of a graph $C^*_4 \in \mathcal{G}$](image)

Note that $X \cup Y$ dominates $G^*$. Moreover, if $D$ is any $\gamma$-set of $G^*$, then $X \subseteq D$ to dominate the set $B$ of leaves. Since $V(G^*) - N[X] = Y$ and $N(y) \cap N(y') = \emptyset$ for distinct $y, y' \in Y$, at least $t$ vertices are required to dominate $Y$. Thus $\gamma(G^*) = s + t = n$. For each $k$ with $0 \leq k \leq n$ we now define a permutation $\pi_k$ of $V(G^*)$; we shall prove that $\gamma(\pi_k G^*) = \gamma(G^*) + k$.

Let $\pi_s = 1$, the identity. For each $i$ with $1 \leq i \leq s$ we define $\pi_{s-i}$ recursively by means of transpositions. Choose $x \in X$, $y \in Y$ such that $\pi_{s-i+1}(x) = x$, $\pi_{s-i+1}(y) = y$, let $\rho_i$ be the transposition $(x, y)$ and define $\pi_{s-i} = \rho_i \circ \pi_{s-i+1}$. (Since $|X| = s$ and $\pi_s = 1$, this choice of $x$ and $y$ is always possible.) Similarly, for each $j$ with $1 \leq j \leq t$ we define $\pi_{s+j}$ as follows. Choose any $y \in Y$ such that $\pi_{s+j-1}(y) = y$. Further, choose $x \in X$ such that $\pi_{s+j-1}$ fixes three distinct vertices $u, v, w \in B^x$. Since $\Sigma_{x \in X} (|B^x| - 2) \geq t$, this choice of $x$ is always possible. Let $\sigma_j$ be the
transposition \((y, u)\) and define \(\pi_{s+j} = \sigma_j \circ \pi_{s+j-1}\). The graphs \(\pi_s C_4^*, \pi_{s-1} C_4^*\) and \(\pi_{s+1} C_4^*\) corresponding to \(C_4^*\) of Figure 2 are illustrated in Figure 3.

\[\gamma(\pi_k G^*) = \gamma(G^*) + k.\]

**Proof.** We begin by noting that the result holds for \(\pi_s = 1\), for if \(D\) is any \(\gamma\)-set of \(K_2 \times G^*\), then \(X_1 \cup X_2 \subseteq D\) to dominate the vertices in \(B_1 \cup B_2\). This leaves the vertices in \(Y_1 \cup Y_2\) undominated, and it is easy to see that at
least \( t \) vertices are required to dominate \( Y_1 \cup Y_2 \). But \( Y_2 \) dominates \( Y_1 \cup Y_2 \); hence \( \gamma(K_2 \times G^*) = 2s + t = \gamma(G^*) + s \).

For arbitrary \( i, 1 \leq i \leq s \), consider any \( \gamma \)-set \( D \) of \( \pi_{s-i}G^* \) and note that as above, \( X_1 \cup X_2 \subseteq D \) to dominate \( B_1 \cup B_2 \). Let \( X' = \{ x \in X : \pi_{s-i}(x) \in Y \} \) and \( Y' = \pi_{s-i}(X') \); observe that \( |X'| = |Y'| = i \). In \( \pi_{s-i}G^* \), \( X_1 \cup X_2 \) dominates all vertices except \( (Y_1 - Y_1') \cup (Y_2 - Y_2') \); again it is easy to see that at least \( |Y - Y'| = t - i \) vertices are required to dominate these. Using (for example) \( Y_2 - Y_2' \), we obtain \( \gamma(\pi_{s-i}G^*) = 2s + t - i = \gamma(G^*) + s - i \).

Finally, for arbitrary \( j, 1 \leq j \leq t \), let \( D \) be a \( \gamma \)-set of \( \pi_{s+j}G^* \) with \( |D \cap (X_1 \cup X_2)| \) maximum, consider any \( x \in X \) and suppose \( |\{x_1, x_2\} \cap D| < 2 \). By definition of \( \pi_{s+j} \) there are at least two vertices \( v, w \in B^x \) that are fixed by \( \pi_{s+j} \). Hence if (say) \( x_1 \notin D \), then to dominate \( v_1 \) and \( w_1, \{v_1, v_2\} \cap D \neq \phi \) and \( \{w_1, w_2\} \cap D \neq \phi \). But then \( (D - \{v_1, v_2, w_1, w_2\}) \cup \{x_1, x_2\} \) is a \( \gamma \)-set which contradicts the choice of \( D \). Therefore \( X_1 \cup X_2 \subseteq D \). As in the case of \( \pi_sG^* \), this leaves the vertices in \( Y_1 \cup Y_2 \) undominated. Let \( Y' = \{ y \in Y : \pi_{s+j}(y) \in B \} \) and note that \( |Y'| = j \). Then for any \( y', y'' \in Y' \), \( N[y'] \cap N[y''] = \phi \) and in \( \pi_{s+j}G^* \), \( N[Y'] \cap N[Y''] = \phi \). Therefore a set \( Z \) of at least \( j \) vertices are needed to dominate \( Y_1' \) in \( \pi_{s+j}G^* \), and \( j \) vertices distinct from those in \( Z \) to dominate \( Y_2' \). Further, at least \( |Y - Y'| = t - j \) vertices are required to dominate \( (Y_1 - Y_1') \cup (Y_2 - Y_2') \). Hence

\[
\gamma(\pi_{s+j}G^*) \geq |X_1| + |X_2| + 2j + t - j
= 2s + j + t
= \gamma(G^*) + s + j.
\]

Obviously \( X_1 \cup X_2 \cup Y_1' \cup Y_2 \) dominates \( \pi_{s+j}G^* \) and so \( \gamma(\pi_{s+j}G^*) = \gamma(G^*) + s + j \), as required.

Theorem 12 also holds if we require \( \Sigma_{x \in X}(|B^x| - 1) \geq t \) in the construction of \( G^* \), but the proof is technically more difficult. The simplest example is obtained by taking \( G = K_2 \) with \( V(G) = \{x, y\} \), replacing \( xy \) with \( K_{2,2} \) and joining \( x \) to two new vertices \( u \) and \( v \). Then \( \pi_0 = (x, y), \pi_1 = 1 \) and \( \pi_2 = (y, u) \).

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