Abstract

Let $T$ be a hamiltonian bipartite tournament with $n$ vertices, $\gamma$ a hamiltonian directed cycle of $T$, and $k$ an even number. In this paper, the following question is studied: What is the maximum intersection with $\gamma$ of a directed cycle of length $k$? It is proved that for an even $k$ in the range $4 \leq k \leq \frac{n+4}{2}$, there exists a directed cycle $C_h(k)$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(C_h(k)) \cap A(\gamma)| \geq h(k)-3$ and the result is best possible.

In a forthcoming paper the case of directed cycles of length $k$, $k$ even and $k > \frac{n+4}{2}$ will be studied.

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1. Introduction

The subject of pancyclicity has been studied by several authors (e.g. [1, 2, 5, 10, 12, 14]). Three types of pancyclicity have been considered. A digraph $D$ is pancyclic if it has directed cycles of all the possible lengths; $D$ is vertex-pancyclic if given any vertex $v$ there are directed cycles of every length containing $v$; and $D$ is arc-pancyclic if given any arc $e$ there are directed cycles of every length containing $e$.

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily
arc-pancyclic (see e.g. [3, 11, 13]). Within the concept of cycle-pancyclicity the following question is studied: Given a directed cycle $\gamma$ of a digraph $D$, find the maximum number of arcs which a directed cycle of length $k$ (if such a directed cycle exists) contained in $D[V(\gamma)]$ (the subdigraph of $D$ induced by $V(\gamma)$) has in common with $\gamma$. Cycle-pancyclicity in tournaments has been studied in [6, 7, 8] and [9]. In this paper, the cycle-pancyclicity in bipartite tournaments is investigated. In order to do so, it is sufficient to consider a hamiltonian bipartite tournament $T$ where $\gamma$ is a hamiltonian directed cycle (because we are looking for directed cycles of length $k$ contained in $D[V(\gamma)]$ whose arcs intersect the arcs of $\gamma$ the most possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set $V = \{0, 1, \ldots, n-1\}$ and an arc set $A$. Also we assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian directed cycle of $T$; $k$ will be an even number; $C_{h(k)}$ will denote a directed cycle of length $h(k)$ with $h(k) \in \{k, k-2\}$ and $\mathcal{I}(C_{h(k)}) = |A(C_{h(k)}) \cap A(\gamma)|$. This paper is the first part of the study of the existence of a directed cycle $C_{h(k)}$ where $\mathcal{I}(C_{h(k)})$ is the maximum. For general concepts we refer the reader to [4].

2. Preliminaries

A chord of a cycle $\mathcal{C}$ is an arc not in $\mathcal{C}$ but with both terminal vertices in $\mathcal{C}$. The length of a chord $f = (u, v)$ of $\mathcal{C}$ denoted $\ell(f)$ is equal to the length of $\langle u, \mathcal{C}, v \rangle$, where $\langle u, \mathcal{C}, v \rangle$ denotes the $uv$-directed path contained in $\mathcal{C}$. We say that $f$ is a $c$-chord if $\ell(f) = c$ and $f = (u, v)$ is an $a-c$-chord if $\ell(\langle v, \mathcal{C}, u \rangle) = c$. Observe that if $f$ is a $c$-chord, then it is also an $a-(n-c)$-chord. All the chords considered in this paper are chords of $\gamma$. We will denote by $\mathcal{C}_k$ a directed cycle of length $k$. In what follows all notation is taken modulo $n$.

For any $a$, $2 \leq a \leq n-2$, denote by $t_a$ the largest integer such that $a + t_a(k-2) < n-1$. The important case of $t_{k-1}$ is denoted by $t$ in the rest of the paper. Let $r$ be defined as follows: $r = n - [k-1 + t(k-2)]$. Notice that: If $a \leq b$ then $t_a \geq t_b$; $t \geq 0$ and $3 \leq r \leq k-1$, $r$ is odd.

**Lemma 2.1.** If the $a$-chord with an initial vertex $0$ (0 being an arbitrary vertex of $T$) is in $A$, then at least one of the two following properties holds.

(i) There exists a directed cycle $\mathcal{C}_k$ with $\mathcal{I}(\mathcal{C}_k) = k - 2$. 
(ii) For every $0 \leq i \leq t_a$, the $a + i(k - 2)$-chord with an initial vertex 0 is in $A$.

**Proof.** Suppose that (ii) in the lemma is false, and let 

$$j = \min\{i \in \{1, 2, \ldots, t_a\} \mid (a + i(k - 2), 0) \in A\},$$

then 

$$C_k = (0, a + (j - 1)(k - 2)) \cup (a + (j - 1)(k - 2), \gamma, a + j(k - 2)) \cup (a + j(k - 2), 0)$$

is a directed cycle with $I(C_k) = k - 2$. 

**Corollary 2.2.** At least one of the two following properties holds

(i) There exists a directed cycle $C_k$ with $I(C_k) \geq k - 2$.

(ii) For every $0 \leq i \leq t$, every $(k - 1 + i(k - 2))$-chord is in $A$.

**Proof.** Clearly, for any vertex 0, $(0, k - 1) \in A$ since otherwise $(k - 1, 0) \in A$ and $C_k = (0, \gamma, k - 1) \cup (k - 1, 0)$ is a directed cycle with $I(C_k) = k - 1$ and thus (i) holds. Now applying Lemma 2.1 with $a = k - 1$ we have that (i) or (ii) holds.

3. The Cases $k = 4, 6, 8$

**Theorem 3.1.** There exists a directed cycle $C_4$ with $I(C_4) \geq 2$.

**Proof.** It follows from Corollary 2.2 for $k = 4$ that we may assume that for every $0 \leq i \leq t_3$, every $(3 + 2i)$-chord is in $A$; now recall $3 \leq r \leq k - 1$, $r$ is odd and $r = n - [k - 1 + t(k - 2)]$. Hence $r = 3$ and we conclude that $C_4 = (0, 3 + 2t_3, 3 + 2t_3 + 1, 3 + 2t_3 + 2, 0)$ has $I(C_4) = 3$.

**Theorem 3.2.** There exists a directed cycle $C_{h(6)}$ with $I(C_{h(6)}) \geq h(6) - 2$.

**Proof.** It follows from Corollary 2.2 for $k = 6$ that we can assume that for every $i$, $0 \leq i \leq t_5$, every $(5 + 4i)$-chord is in $A$; recall $3 \leq r \leq 5$, and $r$ is odd; so $r \in \{3, 5\}$. When $r = 3$, $C_4 = (0, 5 + 4t_5) \cup (5 + 4t_5, \gamma, 0)$ satisfies $I(C_4) = 3$; and when $r = 5$, $C_6 = (0, 5 + 4t_5) \cup (5 + 4t_5, \gamma, 0)$ satisfies $I(C_6) = 5$. 

Theorem 3.3. There exists a directed cycle \( C_{8(h)} \) with \( \mathcal{I}(C_{8(h)}) \geq h(8) - 3 \).

Proof. It follows from Corollary 2.2 for \( k = 8 \) that we may assume that for every \( i, \, 0 \leq i \leq t_7 \) every \( (7 + 6i) \)-chord is in \( A \); recall \( 3 \leq r \leq k - 1 = 7 \) and \( r \) is odd, so \( r \in \{3, 5, 7\} \). When \( r = 7 \) we obtain \( C_8 = (0, 7 + 6t_7) \cup (7 + 6t_7, \gamma, 0) \) a directed cycle with \( \mathcal{I}(C_8) = 7 \). When \( r = 5 \) we have \( C_6 = (0, 7 + 6t_7) \cup (7 + 6t_7, \gamma, 0) \) is a directed cycle with \( \mathcal{I}(C_6) = 5 \). When \( r = 3 \), since \( n \geq 2k - 4 = 12 \) and \( r = 3 \) we have \( t_7 \geq 1 \) and every -9-chord is in \( A \) (notice that \( (0, n - 9) \in A \) because \( n - 9 = 7 + (t_7 - 1)6 \) and 0 is an arbitrary vertex) in particular \( (n - 2, n - 11) \in A \); now observe that we can assume \( (n - 7, 0) \in A \) (otherwise \( (0, n - 7) \in A \) and \( C_8 = (0, n - 7) \cup (n - 7, \gamma, 0) \) is a directed cycle with \( \mathcal{I}(C_8) = 7 \); also observe that \( r = 3 \) implies \( (0, n - 3) \in A \). We conclude that \( C_8 = (n - 2, n - 11) \cup (n - 11, \gamma, n - 7) \cup (n - 7, 0, n - 3, n - 2) \) is a directed cycle with \( \mathcal{I}(C_8) = 5 \).

4. The Case \( n = 2k - 4 \)

Theorem 4.1. If \( n = 2k - 4 \), then there exists a directed cycle \( C_{h(k)} \) with \( \mathcal{I}(C_{h(k)}) = h(k) - 1 \).

Proof. Consider the arc between 0 and \( k - 3 \); when \((0, k - 3) \in A \) we have \( C_k = (0, k - 3) \cup (k - 3, \gamma, 0) \) a directed cycle with \( \mathcal{I}(C_k) = k - 1 \) and when \((k - 3, 0) \in A \) we obtain \( C_{k-2} = (k - 3, 0) \cup (0, \gamma, k - 3) \) a directed cycle with \( \mathcal{I}(C_{k-2}) = k - 3 \).

5. The Cases \( r = k - 1 \) and \( r = k - 3 \)

Theorem 5.1. If \( r = k - 1 \) or \( r = k - 3 \), then there exists a directed cycle \( C_{h(k)} \) with \( \mathcal{I}(C_{h(k)}) = h(k) - 1 \).

Proof. If \( r = k - 1 \), then \( (0, n - (k - 1)) \in A \) and \( C_k = (0, n - (k - 1)) \cup (n - (k - 1), \gamma, 0) \) is a directed cycle with \( \mathcal{I}(C_k) = k - 1 \). If \( r = k - 3 \), then \( (0, n - (k - 3)) \in A \) and \( C_{k-2} = (0, n - (k - 3)) \cup (n - (k - 3), \gamma, 0) \) is a directed cycle with \( \mathcal{I}(C_{k-2}) = k - 3 \).

Corollary 5.2. If \( t = 0 \), then there exists a directed cycle \( C_{h(k)} \) with \( \mathcal{I}(C_{h(k)}) = h(k) - 1 \).
Proof. If \( t = 0 \), then \( n = k - 1 + r \), where \( k - 3 \leq r \leq k - 1 \), \( r \) is odd (because \( n \geq 2k - 4 \)), so \( r \in \{ k - 3, k - 1 \} \). If \( r = k - 3 \), then the assertion follows by Theorem 4.1. If \( r = k - 1 \) and thus \( n = 2k - 2 \), then we can distinguish the cases \((0, k - 1) \in A \) and \((0, k - 1) \notin A \). If \((0, k - 1) \in A \), then \( C_k = (0, k - 1) \cup (k - 1, \gamma, n) \) is a cycle with \( I(C_k) = k - 1 \). The other case follows analogously.

6. The General Case

In this section, we assume \( r \leq k - 5 \), \( t \geq 1 \), and \( k \geq 10 \) (so \( n \geq 16 \)) in view of the results in previous sections.

Lemma 6.1. If the \((k - 1 + \alpha)\)-chord, \( \alpha \leq r + 1 \), with an initial vertex \( 0 \) is in \( A \), then at least one of the two following properties holds.

(i) There exists a directed cycle \( C_k \) with \( I(C_k) = k - 2 \).

(ii) For every \( 0 \leq i \leq t - 1 \), the \( k - 1 + \alpha + i(k - 2)\)-chord with an initial vertex \( 0 \) is in \( A \).

It follows directly from Lemma 2.1; only observe that since \( 3 \leq r \leq k - 5 \) we have \( k - 1 + r + 1 + (t - 1)(k - 2) \leq k + k - 5 + (t - 1)(k - 2) = k - 1 + t(k - 2) - 2 \leq k - 1 + t(k - 2) + r - 5 = n - 5 \).

Lemma 6.2. At least one of the two following properties holds

(i) There exists a directed cycle \( C_{h(k)} \) with \( I(C_{h(k)}) \geq h(k) - 3 \).

(ii) All the following chords are in \( A \): (a) Every \((k - 1)\)-chord. (b) Every \((-r)\)-chord. (c) Every \((k - 3)\)-chord and (d) Every \(-(r + 2)\)-chord.

Proof. Assume that (i) is false. Let us prove that (ii) holds. The proof of (a) follows directly from Corollary 2.2. The proof of (b) follows from Corollary 2.2, observing that \( n - r = k - 1 + t(k - 2) \). To prove (c) assume that there is a \(-(k - 3)\)-chord, say \( f = (y, x) \). It follows from (a) that \((x - 2, y) \in A \), and it follows from (b) that \((x - 2 + r, x - 2) \in A \). Hence, there exists a vertex \( z \) in \((x - 2 + r, \gamma, y - 1) \) such that \((z, x - 2) \in A \) and \((x - 2, z + 2) \in A \). We conclude

\[ C_{k-2} = (y, x) \cup (x, y, z) \cup (z, x - 2) \cup (x - 2, z + 2) \cup (z + 2, y) \]

is a directed cycle with \( I(C_{k-2}) = k - 5 = (k - 2) - 3 \).
Finally to prove (d), let \((y, x)\) be a \((r + 2)\)-chord. If follows from (c) and
Lemma 2.1 that every \(k - 3 + i(k - 2)\)-chord is in \(A\) for \(0 \leq i \leq t\) in
particular \(x + k - 2, x + k - 2 + k - 3 + (t - 1)(k - 2)\) ∈ \(A\), now observe that
\(x + k - 2 + k - 3 + (t - 1)(k - 2) = y + r + 2 + k - 2 + k - 3 + (t - 1)(k - 2) =
y + r + k - 1 + t(k - 2) = y + n = y, \) so \((x + k - 2, y) \in A\): we conclude that
\(\mathcal{E}_k = (y, x) \cup (x, \gamma, x + k - 2) \cup (x + k - 2, y)\) is a directed cycle with
\(\mathcal{J}(\mathcal{E}_k) = k - 2\).

\[\boxdot\]

**Lemma 6.3.** Let \(0 \leq i \leq r + 1, i\) being even. If all the \(−r\)-chords, \(−(r + 2)\)-chords, \((k - 3 + i)\)-chords and \((k - 1 + i)\)-chords are in \(T\), then at least one of the following properties holds.

(i) There exists a directed cycle \(\mathcal{E}_k\) with \(\mathcal{J}(\mathcal{E}_k) \geq k - 3\).

(ii) All the \(−(2r - i + 1)\)-chords, \(−(2r - i + 3)\)-chords and \(−(2r - i + 5)\)-chords are in \(T\).

**Proof.** Assume that the hypothesis of the Lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the \([(k - 3) + i]\)-chords and all the \([(k - 1) + i]\)-chords are in \(T\), it follows from Lemma 6.1 (taking \(\alpha = i - 2\) that every \([k - 3 + i + (t - 1)(k - 2)]\)-chord is in \(T\), and that (taking \(\alpha = i\)) every \([k - 1 + i + (t - 1)(k - 2)]\)-chord is in \(T\). Thus the following arcs are in \(T\): \((r, 0), (r + 2, 0), (0, k - 1 + (t - 1)
\(\mathcal{J}(\mathcal{E}_k) = k - 2\).

\[\boxdot\]

First, we prove that every \(−(2r - i + 1)\)-chord is in \(A\). Suppose that
there exists a \(2r - i + 1\)-chord. We can assume without loss of generality
that \((x_7, x_1)\) is that chord. Hence \(\mathcal{E}_k = (x_7, x_1, x_1 + 1 = r, 0, x_4) \cup (x_4, \gamma, x_7)\)
is a directed cycle with \(\mathcal{J}(\mathcal{E}_k) = k - 3\).

Now we prove that every \(−(2r - i + 3)\)-chord is in \(A\). Assume the contrary; we may assume that \((x_7, x_2)\) is a \(2r - i + 3\)-chord. Then \(\mathcal{E}_k =
(x_7, x_2 = r + 1, r + 2, 0, x_4) \cup (x_4, \gamma, x_7)\) is a directed cycle with \(\mathcal{J}(\mathcal{E}_k) = k - 3\).
Finally, we prove that every \(-(2r - i + 5)\)-chord is in \(T\). Assuming the opposite, we may consider that \((x_8, x_2)\) is a \((2r - i + 5)\)-chord. Then \(C_k = (x_8, x_2 = r + 1, r + 2, 0, x_3) \cup (x_3, \gamma, x_8)\) is a directed cycle with \(I(C_k) = k - 3\).

**Lemma 6.4.** At least one of the following properties holds.

(i) There exists a directed cycle \(C_k\) with \(I(C_k) \geq k - 3\).

(ii) For any even vertex \(x\) (resp. odd) there exist at most \(\frac{k-4}{2}\) consecutive odd (resp. even) vertices in \(\gamma\) which are in-neighbors of \(x\).

**Proof.** Assume that (i) does not hold. Assume without loss of generality that \(x = 0\). It follows from Corollary 2.2 that the vertices \(k - 1 + i(k - 2)\), for \(0 \leq i \leq t\) are not in-neighbors of 0.

So, there are at most \(\frac{k-4}{2}\) odd vertices consecutive in \(\langle k-1, \gamma, 0 \rangle\) which are in-neighbors of 0. Since \((0, 1) \in A\), also in \(\langle 0, \gamma, k-1 \rangle\) there are at most \(\frac{k-4}{2}\) odd vertices consecutive in-neighbors of 0.

The following corollary is a directed consequence of this Lemma (only observe that the hypothesis \(n \geq 2k - 4\) is not needed in the Lemma).

**Corollary 6.5.** Let \(T\) be a bipartite tournament with \(n\) vertices and \(\gamma\) a hamiltonian cycle of \(T\). For each even (resp. odd) vertex \(x\) of \(T\) such that the number of consecutive odd (resp. even) in-neighbors of \(x\) in \(\gamma\) is at least \(\frac{k-2}{2}\), \(3 \leq k \leq n\), \(k\) even, there exists a directed cycle \(C_k\) containing \(x\) with \(I(C_k) \geq k - 2\).

**Lemma 6.6.** If every \((k + 1)\)-chord is in \(A\) then at least one of the two following properties holds.

(i) There exists a directed cycle \(C_{h(k)}\) with \(I(C_{h(k)}) \geq h(k) - 3\).

(ii) For every odd \(\alpha\), \(0 < \alpha r < k\); every \(-(\alpha + 1)r + 1\)-chord is in \(A\). And for every even \(\alpha\), \(0 \leq \alpha r < k\); every \(-(\alpha + 1)r\)-chord is in \(A\).

**Proof.** For \(\alpha = 0\), we can assume that every \(-r\)-chord is in \(A\) (otherwise it follows from Lemma 6.2 that (i) holds and we are done). For \(\alpha = 1\), suppose that \((x_1, x_0)\) is a \((2r + 1)\)-chord, let \((x_0, x_2)\) the \(-r\)-chord with an initial vertex \(x_0\) and \((x_2, x_3)\) the \([(k - 1) + (t - 1)(k - 2)]\)-chord with an initial vertex \(x_2\) (It follows from Corollary 2.2 that we can assume such a chord exists); clearly, \(\ell(x_1, \gamma, x_2) = r + 1\) and \(\ell(x_3, \gamma, x_1) = n - (r + 1) - [k - 1 + (t - 1)(k - 2)] = k - 3\). Now notice that \(x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}\)
Proof. Suppose (i) does not hold, we shall prove that property (ii) holds by induction on \( \alpha \). Since \( \alpha \geq 2 \), at least one of the following properties holds.

Lemma 6.7. Let \( (x_1, x_0) \) be an \((\alpha + 1)\)-chord in \( A \).

We have proved the assertion of Lemma 6.6 for \( \alpha = 0 \) and \( \alpha = 1 \). To complete the proof, assume (i) does not hold for some \( \alpha ' \geq 2 \) and we show that (i) holds. Let \( \alpha \) be the least integer \( \alpha \geq 2 \) for which (ii) does not hold. We analyze two possible cases.

**Case 1.** \( \alpha \) is odd.

We have \( \alpha \geq 3 \), \( 0 < \alpha r < k \) and there exists an \((\alpha + 1)\)-chord in \( A \). Since \( \alpha - 1 \) is even, the choice of \( \alpha \) implies that every \(-((\alpha - 1) + 1)\)-chord is in \( A \).

Let \((x_1, x_0)\) be an \((\alpha + 1)\)-chord, \((x_0, x_2)\) the \(-\alpha r\)-chord with an initial vertex \( x_0 \) and \((x_2, x_3)\) the \((k + 1) + (t - 1)(k - 2)\)-chord with an initial vertex \( x_2 \) (it follows from the hypothesis and Lemma 2.1 that these chords are in \( A \)). Clearly, \( \ell(x_1, \gamma, x_2) = (\alpha + 1)r - \alpha \) and \( \ell(x_3, \gamma, x_1) = n - (r + 1) - [(k + 1) + (t - 1)(k - 2)] = k - 3 \). Notice \( x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\} \) because \( \alpha r < k \), and \( \ell(x_2, \gamma, x_3) \geq k + 1 \). We conclude that \( \mathcal{C}_{k-2} = \langle x_3, \gamma, x_1 \rangle \) is a directed cycle with \( \mathcal{J}(\mathcal{C}_{k-2}) \geq k - 5 \).

**Case 2.** \( \alpha \) is even.

We have \( \alpha \geq 2 \), \( 0 < \alpha r < k \), and there exists an \((\alpha + 1)\)-chord in \( A \). Since \( \alpha - 1 \) is odd, the choice of \( \alpha \) implies that every \(-[(\alpha - 1) + 1]r\)-chord is in \( A \). Let \((x_1, x_0)\) be an \((\alpha + 1)\)-chord, \((x_0, x_2)\) the \(-\alpha r\)-chord with an initial vertex \( x_0 \), and \((x_2, x_3)\) the \((k + 1) + (t - 1)(k - 2)\)-chord with an initial vertex \( x_2 \).

Clearly, \( \ell(x_1, \gamma, x_2) = (\alpha + 1)r - \alpha = r - 1 \) and \( \ell(x_3, \gamma, x_1) = n - (r + 1) - [k + 1 + (t - 1)(k - 2)] = k - 3 \). Moreover, \( x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\} \) because \( \ell(x_2, \gamma, x_0) = \alpha r + 1 \), \( 0 < \alpha r + 1 \leq k \) and \( \ell(x_2, \gamma, x_3) \geq k + 1 \). We obtain \( \mathcal{C}_k = \langle x_3, \gamma, x_1 \rangle \) is a directed cycle with \( \mathcal{J}(\mathcal{C}_k) = k - 3 \).

**Lemma 6.7.** At least one of the following properties holds.

(i) There exists a directed cycle \( \mathcal{C}_{h(k)} \) with \( \mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3 \).

(ii) For \( i \) even \( -2 \leq i \leq r + 1 \); every \(-(2r - 1 + i)\)-chord and every \((k - 1 + i)\)-chord is in \( A \).

**Proof.** Suppose (i) does not hold, we shall prove that property (ii) holds by induction on \( i \). We start with \( i = -2 \) and \( i = 0 \); namely, we prove that the
following chords are in $A$: (a) every $(k - 3)$-chord, (b) every $(k - 1)$-chord, 
(c) every $-(2r + 3)$-chord and (d) every $-(2r + 1)$-chord.

The proof of (a) and (b) follows directly from Lemma 6.2. Let 0 be any vertex of $T$. It follows from Lemma 6.1 (with $\alpha = 0$) and from Lemma 6.2 (part (b) and (d)) that the following chords are in $A$: $(0, k-1+(t-1)(k-2))$, $(r+2, 0)$ and $(r, 0)$. Part (c): every $-(2r+3)$-chord is in $A$. If $(n-r-1, r+2) \in A$, then $\mathcal{E}_k = (n-r-1, r+2, 0, k-1+(t-1)(k-2))$ is a directed cycle of $\mathcal{I}(\mathcal{E}_k) = k - 3$, a contradiction. (Notice that $k-1+(t-1)(k-2) \in (r+2, \gamma, n-r-1)$. Hence letting $k-1+(t-1)(k-2) = k-3$;

Part (d): every $-(2r+1)$-chord is in $A$. If $(n-r-2, r-1) \in A$, then $\mathcal{E}_k = (n-r-2, r-1, r, 0, k-1+(t-1)(k-2))$ is a directed cycle of $\mathcal{I}(\mathcal{E}_k) = k - 3$, a contradiction. (Notice that since $r \leq k-5 < k-1$ and $k \geq 10$ we have $r < k-1+(t-1)(k-2) < n-(r+2)$, and $\ell(k-1+(t-1)(k-2), \gamma, n-r-2) = n-(r+2)-[k-1+(t-1)(k-2)] = k-4$.

Assume that (ii) in Lemma 6.7 holds for each $i'$ even, $0 \leq i' \leq i$ and let us prove it for $i + 2$; namely, we prove

(a) Every $(k + 1 + i)$-chord is in $A$, $0 \leq i \leq r - 1$.
(b) Every $-(2r - 1 - i)$-chord is in $A$, $0 \leq i \leq r - 1$.

Proof of (a). It follows from the inductive hypothesis that for each $j$ even, $0 \leq j \leq i$, every $[(k - 1) + j]$-chord and every $[(k - 3) + j]$-chord is in $A$ (because for $j = 0$ we have proved that every $(k - 3)$-chord is in $A$). It follows from Lemma 6.2 that every $-(r)$-chord and every $-(r+2)$-chord is in $A$. Therefore it follows from Lemma 6.3 that for even $j$, $0 \leq j \leq i$ every $-(2r - j + 1)$-chord, $-(2r - j + 3)$-chord and $-(2r - j + 5)$-chord is in $A$. That means that for each even $j$, $0 \leq j \leq i \leq r - 1$ every $-(2r - j + 1)$-chord is in $A$. These are $\frac{1}{2} + 3$ chords with initial odd (resp. even) vertices consecutive in $\gamma$.

Assume by contradiction that $(x_3, 0)$ is a $-(k + 1 + i)$-chord, $i$ being even $0 \leq i \leq r - 1$. Let $x_0 = n - (2r - i - 1)$. Hence letting $x_2 = 2$ we have that $(x_2, x_0)$ is a $-(2r - i + 1)$-chord (we have observed that every $-(2r - i + 1)$-chord is in $A$).

First, we prove that $x_0 \in \langle x_3+1, \gamma, n-1 \rangle$: \ell(x_0, \gamma, 0) = 2r-i-1 \geq r \geq 3,
\ell(x_3, \gamma, x_0) = n - (k+1+i+2r-i-1) = k-1+t(k-2)+r-k-2r \geq k-1+k-2-r-k = k-3-r \geq 2$ (remember $3 \leq r \leq k-5$).
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Now, there exists an out-neighbor of $x_0$, say $x$, such that $x$ is in $\langle x_2, \gamma, x_3 - 1 \rangle$ this is a direct consequence of Lemma 6.4 and the fact that the number even vertices in $\langle x_2, \gamma, x_3 - 1 \rangle$ is at least $\frac{k+2}{2}$ (Notice that $x_0$ is odd, $\ell(x_2, \gamma, x_3 - 1) = k + 1 + i - 3 = k + i - 2 \geq k - 2$). Let $x_4$ be the smallest (the nearest to 0 in $\gamma$) such vertex.

Let $x_1 = 0$, we will prove that $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$. Since for each $j, \ j \ even\ -4 \leq j \leq i \leq r - 1,\ every -(2r - j + 1)-chord is in $A$, it follows that

$\{(2, x_0), (4, x_0), (6, x_0), \ldots, (i + 4, x_0), (i + 6, x_0)\} \subseteq A$.

Hence the selection of $x_4$ implies $x_1 \geq i + 8$, so $x_4 - i - 4 > 3$.

Finally, since $\ell(x_4, \gamma, x_3) + \ell(x_1, \gamma, x_4 - i - 4) = k + 1 + i - (i + 4) = k - 3$ it follows that $C_k = (x_4 - i - 4, x_0, x_4) \cup (x_4, \gamma, x_3) \cup (x_3, x_1) \cup (x_1, \gamma, x_4 - i - 4)$ is a directed cycle with $\ell(C_k) = k - 3$ (Notice $x_4 - i - 4, x_0 \in A$ by the choice of $x_4$ and the fact $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$).

**Proof of (β).** Part (β) follows from Lemma 6.3 (taking $i + 2$ instead of $i$) and the following facts:

Every $(k - 1 + i)$-chord is in $A$ for even $i, -2 \leq i \leq r + 1$ (it follows from part (α)).

Every $(k - 3 + i)$-chord is in $A$ for even $i, 0 \leq i \leq r + 1$ (it follows from the assertion of above).

Every $(-r)$-chord and every $-(r + 2)$-chord is in $A$ (it follows from Lemma 6.2).

**Theorem 6.8.** If $n \geq 2k - 4$, then there exists a directed cycle $C_{h(k)}$ with $\ell(C_{h(k)}) \geq h(k) - 3$.

**Proof.** The case $n = 2k - 4$ is considered in Section 4. Assume $n > 2k - 4$ and suppose by contradiction that there is no directed cycle $C_{h(k)}$ with $\ell(C_{h(k)}) \geq h(k) - 3$.

It follows from Lemma 6.7 that for each even $i, -2 \leq i \leq r + 1$ every $(k - 1 + i)$-chord is in $A$, in particular

(1) $\{(0, k - 3), (0, k - 1), (0, k + 1), (0, k + 3), \ldots, (0, k + r - 2), (0, k + r)\} \subseteq A$.

(Notice that $k + r < n - 1$ because $t \geq 1$ and $k \geq 10$).
It follows from Lemma 6.2 that every \((-r)\)-chord is in \(A\), and by Lemma 6.7 that every \((k+1)\)-chord is in \(A\). Therefore, by Lemma 6.6 we have: For every odd \(\alpha\), \(0 < \alpha r < k\), every \(-[(\alpha+1)r+1]\)-chord is in \(A\). And for every even \(\alpha\), \(0 \leq \alpha r < k\), every \(-\alpha r\)-chord is in \(A\). Let \(\alpha_0 = \max\{\alpha \in \mathbb{N} \mid \alpha r < k\}\). Clearly, \(\alpha_0 r < k\). We will analyze the two possible cases:

Case 1. \(\alpha_0\) is even.
It follows from Lemma 6.6 that every \(-\alpha_0 (\alpha_0 + 1)\)-chord is in \(A\), in particular \((\alpha_0 + 1)r, 0) \in A\). On the other hand, \(\alpha_0 r < k\) and the selection of \(\alpha_0\) implies \(k < (\alpha_0 + 1)r < k + r\). Thus \(y = (\alpha_0 + 1)r \in \{k + 1, k + 3, k + 5, \ldots, k + r\}\); thus we have \((y, 0) \in A\) and (1) implies \((0, y) \in A\). A contradiction.

Case 2. \(\alpha_0\) is odd.
It follows from Lemma 6.6 that every \(-[(\alpha_0 + 1)r + 1]\)-chord is in \(A\), in particular \(((\alpha_0 + 1)r + 1), 0) \in A\). On the other hand, \(\alpha_0 r < k\) and the choice of \(\alpha_0\) implies \(k + 1 < (\alpha_0 + 1)r + 1 \leq k + r\), \(y = (\alpha_0 + 1)r + 1\) is odd and \(y \in \{k + 1, k + 3, k + 5, \ldots, k + r\}\). So it follows from (1) that \(0, y) \in A\) and we have proved \((y, 0) \in A\). A contradiction.

7. Remarks
In this section, it is proved that the hypothesis of Theorem 6.8 is tight.

**Definition 7.1.** A digraph \(D\) with vertex set \(V\) is called *cyclically \(p\)-partite complete* \((p \geq 3)\) provided one can partition \(V = V_0 + V_1 + \cdots + V_{p-1}\) so that \((u, v)\) is an arc of \(D\) if and only if \(u \in V_i, v \in V_{i+1}\) (notation modulo \(p\)).

**Remark 7.2.** The cyclically 4-partite complete digraph \(T_4\) is a bipartite tournament and clearly every directed cycle of \(T_4\) has length \(\equiv 0 (\text{mod} 4)\). So for \(k = 4m + 2\), \(T_4\) has no directed cycles of length \(k\) and for \(k = 4m\), \(T_4\) has no directed cycles of length \(k - 2\).

Now we consider the following simple lemma.

**Lemma 7.3.** Let \(C_{h(k)}\) be a directed cycle with \(I(C_{h(k)}) = h(k) - 2\). If \(f_1 = (0, x_1), f_2 = (y_1, y_2)\) are the arcs of \(C_{h(k)}\) not in \(\gamma\), then \(y_2 = y_1 + n - (h(k) - 2 + x_1)\). Namely, \(f_2\) is a \(-((x_1 + (h(k) - 2))\)-chord of \(\gamma\).
Remark 7.4. For $n \geq 5$, $k \geq 5$, such that $n \neq k + s(k-2) + m(k-4)$ and $n \neq s(k-2) + m(k-4)$ with $s, m \in \mathbb{N}$, there exists a bipartite hamiltonian tournament $T_n$ with no directed cycles $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 2$.

Proof. Define $T_n$ as follows:

Let

$$C = \{(i, i+k-1+s(k-2)+m(k-4)) \mid i \in \{0,1,\ldots,n-1\}, \ s, m \in \mathbb{N} \text{ with } (k-1)+s(k-2)+m(k-4) < n-1\}$$

and

$$F = \{(i, i+k-3+s(k-2)+m(k-4)) \mid i \in \{0,1,\ldots,n-1\}, \ s, m \in \mathbb{N} \text{ with } (k-3)+s(k-2)+m(k-4) < n-1\},$$

$$A(T_n) = C \cup F \cup \left(\left\{(i+j, i) \mid j \in \left\{2, 3, \ldots, \left\lfloor\frac{n-1}{2}\right\rfloor\right\}\right\} - (C \cup F)\right)$$

$$\cup \{(i, i+1) \mid i \in \{0,1,\ldots,n-1\}\} \cup \left\{(i + \frac{n}{2}, i) \mid i \in \left\{0,1,\ldots,\frac{n}{2} - 1\right\}\right\}.$$}

Clearly, there is no directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 1$ (Notice that $T_n$ has every $(k-1)$-chord and every $(k-3)$-chord). Now assume for contradiction that $\mathcal{C}_{h(k)}$ is a directed cycle of $T_n$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = k - 2$, and let $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ the only arcs of $\mathcal{C}_k$ not in $\gamma$. Without loss of generality we can assume $\ell(f_1) < \frac{n}{2}$. The definition of $T_n$ implies that $x_1 = k-1+s(k-2)+m(k-4)$ or $x_1 = k-3+s(k-2)+m(k-4)$. It follows from Lemma 7.3 that $y_2$ has one of the following forms:

(a) $y_2 = y_1 + n - [k-1+(s+1)(k-2)+m(k-4)].$

When $k-1+(s+1)(k-2)+m(k-4) < n-1$ we obtain that $f_2$ is a $-(k-1+(s+1)(k-2)+m(k-4))$-chord, contradicting the definition of $T_n$.

When $k-1+(s+1)(k-2)+m(k-4) \geq n-1$ we have that $\ell(x_1, y_0) \leq k-1$ and the fact $\mathcal{C}_k - \{(0, x_1), (y_1, y_2)\} \subseteq (x_1, y_0)$ implies $\ell(x_1, y_0) \geq k-3$; and since $\ell(x_1, y_0)$ is odd we have $\ell(x_1, y_0) \in \{k-1, k-3\}$. Now if $\ell(x_1, y_0) = k-1$, then $n = x_1 + k-1 = k-1+s(k-2)+m(k-4)+k-1 =$
$k + (s + 1)(k - 2) + m(k - 4)$, a contradiction. If $\ell(x_1, y, 0) = k - 3$, then 
$n = x_1 + k - 3 = k - 1 + s(k - 2) + m(k - 1) + k = s + (m + 1)(k - 4)$, a contradiction.

(b) $y_2 = y_1 + n - [(k - 1) + s(k - 2) + (m + 1)(k - 4)]$.

(c) $y_2 = y_1 + n - [(k - 3) + (s + 1)(k - 2) + m(k - 4)]$.

(d) $y_2 = y_1 + n - [(k - 3) + s(k - 2) + (m + 1)(k - 4)]$.

Cases (b), (c) and (d) can be analyzed in a completely analogous form as the case (a) to get a contradiction.

It is easy to verify that if $n = k + s(k - 2) + m(k - 4)$ or $n = s(k - 2) + m(k - 4)$ with $s, m \in \mathbb{N}$, then $T_n$ (any bipartite hamiltonian tournament with $n$ vertices) has a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 2$.

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References


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