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MINIMAL REGULAR GRAPHS WITH GIVEN GIRTHS AND CROSSING NUMBERS

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Abstract

This paper investigates on those smallest regular graphs with given girths and having small crossing numbers.

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1. Introduction

Let G be a graph. The *girth* of G is the length of a smallest cycle in G. The *crossing number* of G, denoted $cr(G)$, is the minimum number of pairwise intersections of its edges when G is drawn in the plane.

An (r, g)-*graph* is an r-regular graph with girth g. Let $f(r, g)$ denote the minimum number of vertices in an (r, g)-graph. An (r, g)-graph with the minimum number of vertices is known as an (r, g)-*cage*. The problem of determining $f(r, g)$ or finding an (r, g)-cage is an old problem in graph theory. (See [11].) Most of the cages have high crossing numbers as is shown by inequality (2) below.

Let G be a graph. The *removal number* of G, denoted $rem(G)$, is defined to be the minimum number of edges in G whose removal results in a planar graph. Obviously $cr(G) \geq rem(G)$.

Recall Euler's formula for plane graphs. This states that $n+f \geq m+2$ where n, m and f denote the number of vertices, edges and faces in the plane graph G. Equality holds if and only if G is a connected graph.

Let n and m denote the number of vertices and edges respectively in an (r,g)-graph G. Then $rn = 2m$. Let G^* denote the planar graph obtained from G by deleting $rem(G)$ edges. Let f denote the number of faces in G^*. Then from Euler's formula and from the Hand-Shaking Lemma for plane graphs, we have $n - (m - rem(G)) + f \geq 2$ and $2(m - rem(G)) \geq gf$ respectively. By getting rid of m and f, we have

$$rem(G) \geq \frac{4g + (r(g-2) - 2g)n}{2(g-2)} \tag{1}$$

Since $cr(G) \geq rem(G)$ and $n \geq f(r,g)$, we have

$$cr(G) \geq \frac{4g + (r(g-2) - 2g)f(r,g)}{2(g-2)}\,. \tag{2}$$

In this paper, we are interested in those (r,g)-graphs having small crossing numbers.

Call a graph an (r,g,c)-*graph* if it is an (r,g)-graph with crossing number c. Let $f(r,g,c)$ denote the minimum number of vertices in an (r,g,c)-graph. Our problem is to determine $f(r,g,c)$ and, if possible, a smallest (r,g,c)-graph. Note that for a given c, an (r,g,c)-graph may not exist.

Obviously, $f(2,g,0) = g$ for any $g \geq 3$ and the g-cycle is the only smallest $(2,g,0)$-graph. Note that $f(2,g,c)$ does not exist for any $c \geq 1$. In what follows, we shall assume that r, $g \geq 3$. There are three cases which we wish to consider. The case $c = 0$ is treated in Section 2 and the cases $c = 1$ and $c = 2$ in the subsequent sections.

Throughout this paper, we let K_s denote a complete graph on s vertices and $K_{s,t}$ the complete bipartite graphs whose two partite sets have s and t vertices.

2. The Case $c = 0$

In this case, all the graphs under consideration are planar graphs. Let G be an $(r,g,0)$-graph. Then by counting the number of edges and the number

of vertices in G, and by using Euler's formula for plane graphs, we have

$$\sum_{i \geq g} (2i - (i-2)r) f_i \quad = \quad 4r \tag{3}$$

where f_i is the number of i-sided faces in G. From this formula, it is easily deduced that $g \leq 5$ and that $r \leq 5$. A closer look at the above formula reveals that there are only five possible pairs of (r, g), namely $(3,3), (3,4), (3,5), (4,3)$ and $(5,3)$. For each pair of (r, g), we shall determine the value of $f(r, g, 0)$ and the corresponding possible smallest $(r, g, 0)$-graphs.

Note that in an $(r, g, 0)$-graph G with n vertices, m edges and f faces, we have $rn = 2m \geq gf$. Substituting these into Euler's formula for plane graphs, we have $m \leq \frac{g}{g-2}(n-2)$ where equality holds if and only if every face of G is a g-sided face. Replacing m by $\frac{rn}{2}$, we have

$$n \quad \geq \quad \frac{4g}{2r + 2g - rg}. \tag{4}$$

Immediate from the above inequality, it is deduced that $f(3,3,0) \geq 4$, $f(3,4,0) \geq 8$, $f(3,5,0) \geq 20$, $f(4,3,0) \geq 6$ and $f(5,3,0) \geq 12$. It is easy to see that K_4 is the only smallest $(3,3,0)$-graph and so $f(3,3,0) = 4$. Also, it is easily shown that the cube is the only smallest $(3,4,0)$-graph and so $f(3,4,0) = 8$.

Likewise, it is readily verified that the octahedron is the only smallest $(4,3,0)$-graph and so $f(4,3,0) = 6$.

The icosahedron shows that $f(5,3,0) = 12$. To see that it is the only smallest $(5,3,0)$-graph, note that any $(5,3,0)$-graph on 12 vertices is maximal planar. This means that every face is a triangle and that $f_3 = 20$. The only graph that satisfies these conditions is the icosahedron. (See [1] pp. 159–161.)

The dodecahedron shows that $f(3,5,0) = 20$. To see that it is the only smallest $(3,5,0)$-graph, note that any $(3,5,0)$-graph G on 20 vertices has 30 edges. Since $m = \frac{g(n-2)}{g-2}$, every face in G is a 5-sided face. Hence $f_5 = 12$. The only graph that satisfies these conditions is the dodecahedron. (See [1] pp. 159–161.)

We shall summarize the above observations in the following theorem.

Theorem 1. *The five platonic solids are the only smallest $(r, g, 0)$-graphs.*

3. The Case $c = 1$

Let G be an $(r, g, 1)$-graph with n vertices and m edges. Then $cr(G) = 1 = rem(G)$ and so there exists an edge e in G such that $G - e$ is planar with degree sequence $(r-1, r-1, r, \ldots, r)$. Note that the girth of $G-e$ is at least g. Let f denote the number of faces in $G - e$ and let f_i denote the number of i-sided faces in $G - e$. Then $2(m-1) = \sum_{i \geq g} i f_i = 2(r-1) + r(n-2)$ and $f = \sum_{i \geq g} f_i$. Substituting these into Euler's formula for plane graphs, we have

$$\sum_{i \geq g} (2i - (i-2)r) f_i \quad \geq \quad 4r - 4. \tag{5}$$

Like the case of planar graphs, it is readily deduced that there are only five possible pairs of (r, g), namely $(3,3), (3,4), (3,5), (4,3)$ and $(5,3)$.

Notice that the number of vertices in G satisfies the following inequality

$$n \quad \geq \quad \frac{2g+4}{2r+2g-rg}. \tag{6}$$

It is easy to see that $f(4,3,1) = 5$ because K_5 is the smallest $(4,3,1)$-graph. Also, $f(3,4,1) = 6$ because $K_{3,3}$ is the smallest $(3,4,1)$-graph.

Now $f(3,3,1) \geq 8$. To see this, let H be a $(3,3,1)$-graph on no more than 6 vertices. Then H must be isomorphic to the complete bipartite graph $K_{3,3}$. But this is not possible because $K_{3,3}$ contains no triangle.

A smallest $(3,3,1)$-graph is obtained from $K_{3,3}$ by replacing a vertex by a triangle with edges joining the triangle in a corresponding way (see Figure 1). To see the uniqueness of this graph, let H be a smallest $(3,3,1)$-graph.

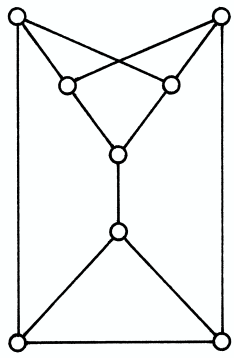


Figure 1. Smallest $(3,3,1)$-graph

Then H contains a triangle. Replace this triangle by a vertex of degree 3, the resulting graph is a non-planar cubic graph on 6 vertices which must be the complete bipartite graph $K_{3,3}$.

It follows from the inequality (6) that $f(3,5,1) \geq 14$. The graph depicted in Figure 2 is a $(3,5,1)$-graph and so $f(3,5,1) \leq 18$. In [10], Royle has listed all the $(3,5)$-graphs on 14 and 16 vertices. We check that all these graphs have crossing number at least 2. There are eight $(3,5)$-graphs with 14 vertices. Six of them contains a subdivision of the Petersen graph as a subgraph. One of the two remaining graphs in the list is the graph G_7 of Figure 6 (which has crossing number 2 as will be explained later in the last section). The last graph in the list has removal number at least 2 because the resulting graph obtained by deleting any edge from this graph contains a subdivision of $K_{3,3}$ as subgraph. There are 48 $(3,5)$-graphs on 16 vertices. Except for the last graph, all of them has crossing number at least 2 because they all contain a subdivision of the graph G_3 (which has crossing number 2 as will be explained later in the section) of Figure 6 as subgraph. The last graph in the list contains a subdivision of the Petersen graph as a subgraph. Hence all of these graphs have crossing number at least 2. Thus we may conclude that $f(3,5,1) = 18$. However we do not know whether or not the graph in Figure 2 is the only smallest $(3,5,1)$-graph.

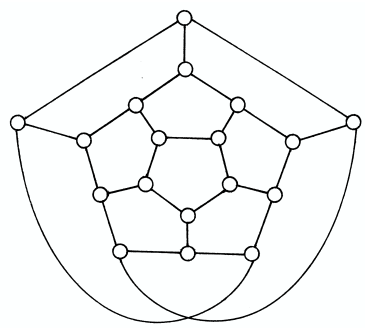


Figure 2. Smallest $(3,5,1)$-graph

The graph in Figure 3 is a $(5,3,1)$-graph and so $f(5,3,1) \leq 14$. Since $10 \leq f(5,3,1)$, there are only three possible values of $f(5,3,1)$. In the next section, we shall prove that $f(5,3,1) \geq 12$. We do not know whether or not a smallest $(5,3,1)$-graph is unique.

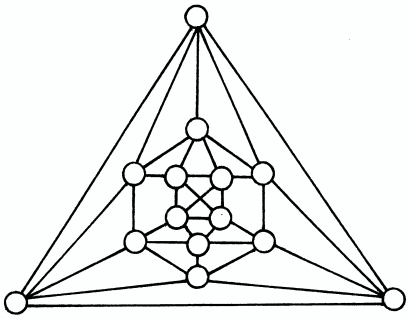


Figure 3. A $(5,3,1)$-graph

We shall summarize the above observations in the following theorem.

Theorem 2. *$f(3,3,1) = 8$, $f(3,4,1) = 6$, $f(3,5,1) = 18$, $f(4,3,1) = 5$ and $12 \leq f(5,3,1) \leq 14$.*

4. $(5,3,1)$-Graphs

In this section, we prove that $(5,3,1)$-graphs do not exist if the number of vertices is no more than 10.

Proposition 1. *Let G be a 5-regular graph on 10 vertices. Then $cr(G) \geq 2$.*

The proof of this proposition is by contradiction. It consists of a series of lemmas that we shall now prove. It is easy to see that if G is 5-regular and has 10 vertices, then it is non-planar and so $cr(G) \geq 1$. In this section, unless otherwsie stated, we shall assume that G is a 5-regular graph on 10 vertices and that $cr(G) = 1$ and obtain a contradiction.

Let H be a regular graph and let v be a vertex of H. Let A_v denote the subgraph of H induced by the set of vertices adjacent to v. Also, let B_v denote the subgraph obtained by deleting $A_v \cup \{v\}$ from H.

Lemma 1. *Let H be an r-regular graph on $2r$ vertices. Suppose $v \in V(H)$. Then A_v and B_v have the same number of edges.*

Proof. Note that A_v and B_v have r and $r-1$ vertices respectively. Let $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_{r-1})$ denote the degree sequences of A_v and B_v respectively.

Note that the number of edges in the subgraph $H - v$ is $r(r-1)$ and that it is also equal to $|E(A_v)| + |E(B_v)|+$ number of edges from A_v to B_v.

Therefore,

$$\frac{\sum_{i=1}^{r} a_i}{2} + \frac{\sum_{i=1}^{r-1} b_i}{2} + \sum_{i=1}^{r}(r-1-a_i) = r(r-1)$$

which, on simplification, leads to $\frac{\sum_{i=1}^{r} a_i}{2} = \frac{\sum_{i=1}^{r-1} b_i}{2}$. Thus A_v and B_v have the same number of edges. ■

Lemma 2. *If $rem(G) = 1$, then there is an edge e in G such that $G - e$ is a planar triangulation.*

Proof. Since $rem(G) = 1$, there is an edge e in G such that $G - e$ is planar. Since $G - e$ has 10 vertices and 24 edges, $G - e$ is a triangulation. ■

Lemma 3. *If G contains a 4-cycle $a_0a_1a_2a_3$, then either $a_ia_{i+2} \in E(G)$ for some i or else there is a vertex $u \in V(G)$ such that u is adjacent to a_i for all i. Here i is considered modulo 4.*

Proof. This is a consequence of Lemma 2 because otherwise $G - e$ is not a triangulation for any edge e in G. ■

Lemma 4. *A_v is connected for any vertex v in G.*

Proof. Suppose A_v is disconnected for some vertex v in G. Let w_1 and w_2 be two vertices in two different components of A_v, say G_1 and G_2 respectively.

Since $|V(G_1)| + |V(G_2)| \leq 5$, we may assume that w_1 is of degree d where $d \leq 1$ and w_2 is of degree $3 - d$ in A_v. Therefore w_1 and w_2 must be adjacent to a common vertex z in B_v. But then w_1zw_2v is a 4-cycle in G not satisfying Lemma 3. This contradiction proves that A_v is connected. ■

Lemma 4 implies that A_v has at least 4 edges because it has 5 vertices. By Lemma 1, because A_v and B_v have the same number of edges, it follows that A_v has at most 6 edges because B_v has 4 vertices. Next, we shall dispose of some degree sequences of A_v.

Lemma 5. *For any $v \in V(G)$, A_v contains no vertices of degree 4.*

Proof. Suppose on the contrary that A_v contains a vertex u of degree 4. Evidently there exist two non-adjacent vertices x and y in $A_v - u$.

If x and y are adjacent to a common vertex z in B_v (call this condition (*)), then $xzyu$ is a 4-cycle in G not satisfying Lemma 3.

Notice that the condition (*) is always satisfied if $|E(A_v)| \leq 5$.

So assume that $|E(A_v)| = 6$ and that the condition (*) is not satisfied. Then A_v is the graph of Figure 4(a). In this case, the only graph in which condition (*) is not satisfied is the graph of Figure 4(b). However, the removal number of this graph is at least 2 because it contains a subdivision of $K_{3,4}$ (the subgraph induced by the thick edges) as a subgraph. This contradiction proves the lemma. ■

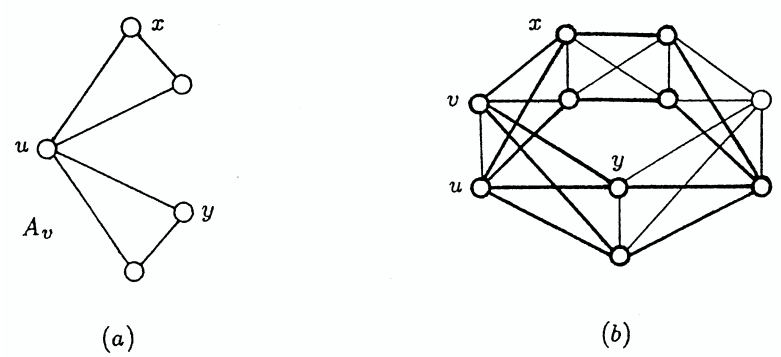


Figure 4. Graph not satisfying condition (*)

Lemma 6. *It is not possible that A_v contains two non-adjacent vertices v_1 and v_2 such that $d_{A_v}(v_1) = 1$ and $d_{A_v}(v_2) \leq 2$.*

Proof. If these conditions are satisfied, then v_1 must be adjacent to three vertices of B_v and v_2 must be adjacent to at least two vertices of B_v. This means that they must be adjacent to a common vertex z of B_v. But then G contains a 4-cycle vv_1zv_2 not satisfying Lemma 3. ■

Lemmas 5 and 6 imply that the only possible degree sequences for A_v left to be considered are $(2,2,2,2,2)$ and $(2,2,2,3,3)$. There are only three graphs associated with these degree sequences. However, these three cases are disposed of in the next three lemmas.

Lemma 7. *A_v is not the graph $K_{2,3}$.*

Proof. In this case, B_v is the complete graph on 4 vertices. Let x_1, x_2 and x_3 denote the three vertices of degree 2 in $K_{2,3}$. Then each x_i must be adjacent to two vertices in B_v. But this means that there are x_i and x_j $(i \neq j)$ which are adjacent to a common vertex z in B_v giving rise to a 4-cycle vx_izx_j in G not satisfying Lemma 3. ■

Lemma 8. *A_v is not a cycle on 5 vertices.*

Proof. Suppose A_v is a cycle on 5 vertices whose vertices are labelled as $v_1, v_2, \ldots, v_5$ in cyclic order.

Note that B_v is the graph obtained from the complete graph on 4 vertices by deleting an edge. So B_v contains two vertices x_1 and x_2 of degree 2, and two vertices y_1 and y_2 of degree 3. Clearly each x_i is adjacent to three vertices of A_v.

Now, x_1 and x_2 are adjacent to a common vertex w in A_v. But then $x_1wx_2y_1$ or $x_1wx_2y_2$ is a 4-cycle in G not satisfying Lemma 3. ■

Lemma 9. *A_v is not the graph of Figure* 5(a).

Proof. Suppose A_v is the graph of Figure 5(a). In this case, B_v is the complete graph on 4 vertices. Let $V(B_v) = \{u_1, \ldots, u_4\}$.

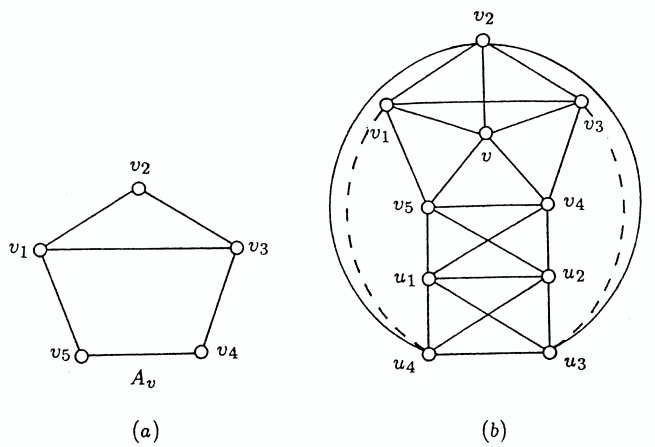


Figure 5

Note that any two non-adjacent vertices x and y in A_v must not be adjacent to a common vertex z in B_v. This is because otherwise we have a 4-cycle $vxzy$ in G not satisfying Lemma 3.

Hence we may assume without loss of generality that v_2 is adjacent to u_3 and u_4 and that v_4 is adjacent to u_1 and u_2. Since v_5 and v_2 are non-adjacent, v_5 is adjacent to u_1 and u_2 (see Figure 5(b)).

Now v_1 is adjacent either to u_3 or u_4. However in either case, we have a 4-cycle $v_1v_5u_1w$ not satisfying Lemma 3, where $w \in \{u_3, u_4\}$. This contradiction proves the lemma. ■

5. The Case $c = 2$

Let G be an $(r, g, 2)$-graph with n vertices. Then $1 \leq rem(G) \leq 2$. Let $f_i(r, g, c)$ denote the minimum number of vertices in an (r, g, c)-graph with removal number i, $i = 1, 2$.

The rest of this paper is to prove the following theorem.

Theorem 3. *$f_1(3, 3, 2) = 12$, $f_1(3, 4, 2) = 10$, $f_1(3, 5, 2) = 14$, $f_1(4, 3, 2) = 8$, $12 \leq f_1(5, 3, 2) \leq 16$, $f_2(3, 3, 2) = 12$, $f_2(3, 4, 2) = 12$, $f_2(3, 5, 2) = 10$, $f_2(4, 3, 2) = 7$ and $f_2(5, 3, 2) = 8$.*

Suppose $rem(G) = 1$. Then inequalities (5) and (6) also hold for G and so there are only five possible pairs of (r, g), namely $(3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$.

Suppose $rem(G) = 2$. Let x_1 and x_2 be two edges of G such that $G - \{x_1, x_2\}$ is planar. Then the degree sequence of $G - \{x_1, x_2\}$ is either $(r-2, r-1, r-1, r, \cdots, r)$ or $(r-1, r-1, r-1, r-1, r, \cdots, r)$. Following similar argument as was done for the case $c = 1$, we have

$$\sum_{i \geq g} (2i - (i-2)r) f_i \quad \geq \quad 4r - 8 \tag{7}$$

and the number of vertices in G satisfies the following inequality

$$n \quad \geq \quad \frac{8}{2r + 2g - rg}. \tag{8}$$

Again, only the same five pairs of (r, g) satisfy inequality (7).

Let G be a non-planar graph and let e be an edge in G. Then e is called a *p-critical edge* if $G - e$ is a planar graph.

Lemma 10. *Let G be a non-planar graph. If G contains a unique p-critical edge, then $cr(G) \geq 2$.*

Proof. If $cr(G) = 1$, then there exist two edges e_1 and e_2 in G which intersect each other and such that $G - \{e_i\}$ is planar for $i = 1, 2$. But this contradicts the uniqueness of the removal edge of G. ■

5..1 $(3, g, 2)$-graphs

In [9] (p. 647–648), Royle has listed all connected cubic graphs of order up to and including 10. It is easily seen that there are only two cubic graphs in the list with $c > 1$, the Petersen graph G_8 and the graph G_3 in Figure 6. Thus $f_i(3, g, 2) \geq 10$. Further, Royle's list also indicates that $f_2(3, 4, 2) \geq 12$ and that $f_i(3, 3, 2) \geq 12$.

Clearly, the Petersen graph is a $(3, 5, 2)$-graph. It has removal number 2 because removing any one of its edges yields a non-planar graph. Therefore $f_2(3, 5, 2) = 10$ and the Petersen graph is the only smallest $(3, 5, 2)$-graph with removal number 2.

Note that the graph G_3 in Figure 6 has crossing number 2. To see this, we note that G_3 contains two vertex-disjoint graphs $K_{2,3}$ which are not outerplanar. If $cr(G_3) = 1$, then at least one of these subgraphs has a planar drawing. This subgraph does not separate the vertices of the other $K_{2,3}$ subgraph and, because of outerplanarity, its edges are crossed by at least one of the edges e_1, e_2 and e_3. A similar argument can be used for the other $K_{2,3}$ subgraph. Hence G_3 is a $(3, 4, 2)$-graph. Moreover it has removal number 1 because $G_3 - e_i$ is planar for any $1 \leq i \leq 3$. Therefore $f_1(3, 4, 2) = 10$ and G_3 is the only smallest $(3, 4, 2)$-graph with removal number 1.

$f_1(3, 5, 2) \geq 14$ follows from inequality (1). In Royle's list of cubic graphs on 14 vertices [10], there are eight graphs with girth equal to 5. All but the graph G_7 contain a subdivision of the Petersen graph as a subgraph. This means that except for the graph G_7, they all have removal number at least 2. It is a routine exercise to verify that the edge e is the only p-critical edge in G_7. By Lemma 10, $cr(G_7) \geq 2$. Therefore $f_1(3, 5, 2) = 14$ and G_7 is the only smallest $(3, 5, 2)$-graph with removal number 1.

Let H be a smallest $(3, g, 2)$-graph with removal number i and $g \neq 3$. We may obtain a $(3, 3, 2)$-graph with removal number i by replacing a vertex of degree 3 from H by a triangle with edges joining the triangle in a corresponding way. The graph G_1 (respectively G_2) is obtained from G_3 (respectively the Petersen graph) in this way. Combining this with the previous

observations, we have $f_1(3,3,2) = 12$ and $f_2(3,3,2) = 12$. The uniqueness of these graphs follows from that of the graph G_3 and the Petersen graph.

We now look at Royle's list of cubic graphs on 12 vertices with girth 4 (see [10]). There are twenty such cubic graphs. However there are only three graphs G_4, G_5 and G_6 from this list with removal number at least 2. Since these three graphs can all be drawn on the plane with only two crossings, they are the smallest $(3,4,2)$-graphs with removal number 2. Thus $f_2(3,4,2) = 12$.

5..2 $(4,3,2)$-graphs

Let G be a $(4,3,2)$-graph on n vertices. Then clearly $n \geq 6$.

If $n = 6$, then G is the complementary graph of $3K_2$ and is planar. Hence $n \geq 7$.

If $n = 7$, then $\overline{G}$ is a 2-regular graph which is either $C_3 \cup C_4$ or C_7. If $\overline{G}$ is $C_3 \cup C_4$, then G is the graph G_{10} and $rem(G_{10}) = 2$ because G_{10} contains $K_{3,4}$ as a subgraph. If $\overline{G}$ is C_7, then $cr(G) = 1$ (see [2]). Therefore $f_2(4,3,2) = 7$ and G_{10} is the only smallest $(4,3,2)$-graph with removal number 2.

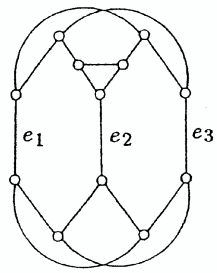
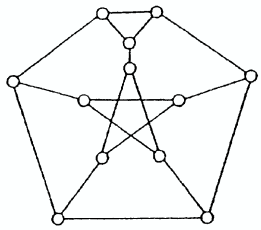
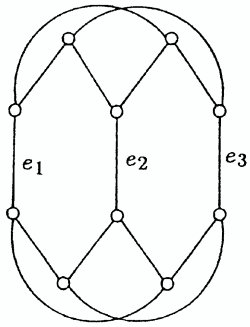
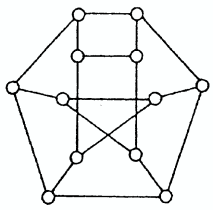
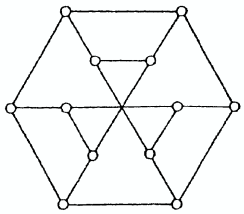
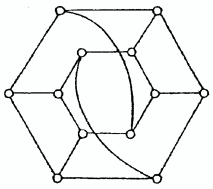
If $n = 8$, then $\overline{G}$ is a cubic graph on 8 vertices. There are precisely five cubic graphs on 8 vertices. If $\overline{G}$ is the cube, then G is the graph G_9 which is the cartesian product $K_4 \times K_2$ and has crossing number 2 (see [8]). Now, $rem(G_9) = 1$ because $G_9 - e_i$ is planar for each $i = 1,2,3$. If $\overline{G}$ is not the cube, we have checked, by direct verification that either $rem(G) \geq 2$ or else $cr(G) \leq 1$. Therefore $f_1(4,3,2) = 8$ and G_9 is the only smallest $(4,3,2)$-graph with removal number 1.

5..3 $(5,3,2)$-graphs

It follows from inequality (1) that $f_1(5,3,2) \geq 10$. However the proof of Proposition 1 implies that $f_1(5,3,2) \geq 12$. Clearly, the graph G_{11} in Figure 6 has removal number 1. We have checked that the edge e is the only removal edge. By Lemma 10, $cr(G_{11}) = 2$. Hence G_{11} is a $(5,3,2)$-graph with $rem(G_{11}) = 1$ and so $12 \leq f_1(5,3,2) \leq 16$.

Let G be a 5-regular graph on 8 vertices. Then $\overline{G}$ is a 2-regular graph. Hence there are only three 5-regular graphs on 8 vertices namely, $\overline{C_8}$, $\overline{C_3 \cup C_5}$ and $\overline{C_4 \cup C_4}$. Since $\overline{C_r \cup C_s}$ contains $K_{r,s}$ as a subgraph, it follows that $cr(\overline{C_3 \cup C_5}) \geq 4$ and $cr(\overline{C_4 \cup C_4}) \geq 4$.

Now it follows from inequality (1) that $rem(\overline{C_8}) \geq 2$. Figure 6 depicts a drawing of $\overline{C_8}$ ($\cong G_{12}$) with two crossings and we have $rem(G_{12}) = 2 = cr(G_{12})$. Thus $f_2(5,3,2) = 8$ and G_{12} is the only smallest $(5,3,2)$-graph with removal number 2.

$(r,g,2)$	$rem(G) = 1$	$rem(G) = 2$
$(3,3,2)$	G_1  $f_1(3,3,2) = 12$	G_2  $f_2(3,3,2) = 12$
$(3,4,2)$	G_3  $f_1(3,4,2) = 10$	G_4  G_5  G_6  $f_2(3,4,2) = 12$

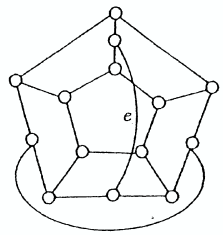
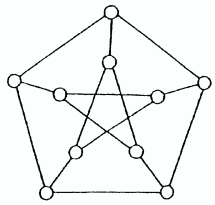
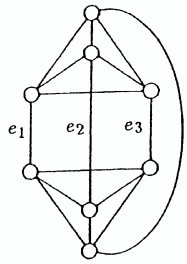
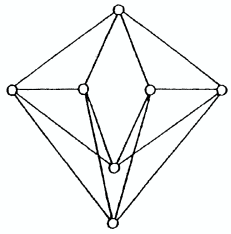
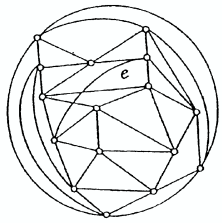
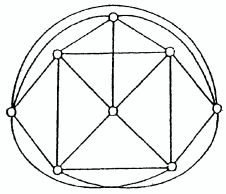
$(3,5,2)$	G_7  $f_1(3,5,2) = 14$	G_8  $f_2(3,5,2) = 10$
$(4,3,2)$	G_9  $f_1(4,3,2) = 8$	G_{10}  $f_2(4,3,2) = 7$
$(5,3,2)$	G_{11}  $12 \leq f_1(5,3,2) \leq 16$	G_{12}  $f_2(5,3,2) = 8$

Figure 6

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References

[1] G. Chartrand and L. Lesniak, Graphs & Digraphs (Third edition), (Chapman & Hall, New York 1996).

[2] R.K. Guy and A. Hill, *The crossing number of the complement of a circuit*, Discrete Math. **5** (1973) 335–344.

[3] D.J. Kleitman, *The crossing number of* $K_{5,n}$, J. Combin. Theory B **9** (1970) 315–323.

[4] M. Koman, *On nonplanar graphs with minimum number of vertices and a given girth*, Commentationes Math. Univ. Carolinae (Prague) **11** (1970) 9–17.

[5] D. McQuillan and R.B. Richter, *On* 3*-regular graphs having crossing number at least* 2, J. Graph Theory, **18** (1994) 831–839.

[6] M. Nihei, *On the girths of regular planar graphs*, Pi Mu Epsilon J. **10** (1995) 186–190.

[7] B. Richter, *Cubic graphs with crossing number two*, J. Graph Theory **12** (1988) 363–374.

[8] R.D. Ringeisen and L.W. Beineke, *On the crossing numbers of products of cycles and graphs of order four*, J. Graph Theory **4** (1980) 145–155.

[9] G.F. Royle, *Graphs and multigraphs*, in: C.J. Colbourn and J.H. Dinitz ed., The CRC Handbook of Combinatorial Designs, (CRC Press, New York, 1995) 644–653.

[10] G.F. Royle, Cubic cages, `http://www.cs.uwa.edu.au/~gordon/cages/index.hmtl`.

[11] P.K. Wong, *Cages — a survey*, J. Graph Theory **6** (1982) 1–22.

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