MINIMAL REGULAR GRAPHS WITH GIVEN GIRTHS AND CROSSING NUMBERS

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Abstract

This paper investigates on those smallest regular graphs with given girths and having small crossing numbers.

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1. Introduction

Let $G$ be a graph. The girth of $G$ is the length of a smallest cycle in $G$. The crossing number of $G$, denoted $cr(G)$, is the minimum number of pairwise intersections of its edges when $G$ is drawn in the plane.

An $(r, g)$-graph is an $r$-regular graph with girth $g$. Let $f(r, g)$ denote the minimum number of vertices in an $(r, g)$-graph. An $(r, g)$-graph with the minimum number of vertices is known as an $(r, g)$-cage. The problem of determining $f(r, g)$ or finding an $(r, g)$-cage is an old problem in graph theory. (See [11].) Most of the cages have high crossing numbers as is shown by inequality (2) below.
Let $G$ be a graph. The *removal number* of $G$, denoted $\text{rem}(G)$, is defined to be the minimum number of edges in $G$ whose removal results in a planar graph. Obviously $\text{cr}(G) \geq \text{rem}(G)$.

Recall Euler’s formula for plane graphs. This states that $n + f \geq m + 2$ where $n$, $m$ and $f$ denote the number of vertices, edges and faces in the plane graph $G$. Equality holds if and only if $G$ is a connected graph.

Let $n$ and $m$ denote the number of vertices and edges respectively in an $(r, g)$-graph $G$. Then $rn = 2m$. Let $G^*$ denote the planar graph obtained from $G$ by deleting $\text{rem}(G)$ edges. Let $f$ denote the number of faces in $G^*$. Then from Euler’s formula and from the Hand-Shaking Lemma for plane graphs, we have

$$n - (m - \text{rem}(G)) + f \geq 2$$

respectively. By getting rid of $m$ and $f$, we have

$$\text{rem}(G) \geq \frac{4g + (r(g - 2) - 2g)n}{2(g - 2)}$$

Since $\text{cr}(G) \geq \text{rem}(G)$ and $n \geq f(r, g)$, we have

$$\text{cr}(G) \geq \frac{4g + (r(g - 2) - 2g)f(r, g)}{2(g - 2)}$$

In this paper, we are interested in those $(r, g)$-graphs having small crossing numbers.

Call a graph an $(r, g, c)$-graph if it is an $(r, g)$-graph with crossing number $c$. Let $f(r, g, c)$ denote the minimum number of vertices in an $(r, g, c)$-graph. Our problem is to determine $f(r, g, c)$ and, if possible, a smallest $(r, g, c)$-graph. Note that for a given $c$, an $(r, g, c)$-graph may not exist.

Obviously, $f(2, g, 0) = g$ for any $g \geq 3$ and the $g$-cycle is the only smallest $(2, g, 0)$-graph. Note that $f(2, g, c)$ does not exist for any $c \geq 1$. In what follows, we shall assume that $r, g \geq 3$. There are three cases which we wish to consider. The case $c = 0$ is treated in Section 2 and the cases $c = 1$ and $c = 2$ in the subsequent sections.

Throughout this paper, we let $K_s$ denote a complete graph on $s$ vertices and $K_{s,t}$ the complete bipartite graphs whose two partite sets have $s$ and $t$ vertices.

2. The Case $c = 0$

In this case, all the graphs under consideration are planar graphs. Let $G$ be an $(r, g, 0)$-graph. Then by counting the number of edges and the number
of vertices in $G$, and by using Euler’s formula for plane graphs, we have

\[ \sum_{i \geq g} (2i - (i - 2)r) f_i = 4r \]

where $f_i$ is the number of $i$-sided faces in $G$. From this formula, it is easily deduced that $g \leq 5$ and that $r \leq 5$. A closer look at the above formula reveals that there are only five possible pairs of $(r, g)$, namely $(3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$. For each pair of $(r, g)$, we shall determine the value of $f(r, g, 0)$ and the corresponding possible smallest $(r, g, 0)$-graphs.

Note that in an $(r, g, 0)$-graph $G$ with $n$ vertices, $m$ edges and $f$ faces, we have $rn = 2m \geq gf$. Substituting these into Euler’s formula for plane graphs, we have

\[ m \leq \frac{g}{g-2} (n-2) \]

where equality holds if and only if every face of $G$ is a $g$-sided face. Replacing $m$ by $\frac{r}{2}$, we have

\[ n \geq \frac{4g}{2r + 2g - rg} \]  

Immediate from the above inequality, it is deduced that $f(3, 3, 0) \geq 4$, $f(3, 4, 0) \geq 8$, $f(3, 5, 0) \geq 20$, $f(4, 3, 0) \geq 6$ and $f(5, 3, 0) \geq 12$. It is easy to see that $K_4$ is the only smallest $(3, 3, 0)$-graph and so $f(3, 3, 0) = 4$.

Also, it is easily shown that the cube is the only smallest $(3, 4, 0)$-graph and so $f(3, 4, 0) = 8$.

Likewise, it is readily verified that the octahedron is the only smallest $(4, 3, 0)$-graph and so $f(4, 3, 0) = 6$.

The icosahedron shows that $f(5, 3, 0) = 12$. To see that it is the only smallest $(5, 3, 0)$-graph, note that any $(5, 3, 0)$-graph on 12 vertices is maximal planar. This means that every face is a triangle and that $f_3 = 20$. The only graph that satisfies these conditions is the icosahedron. (See [1] pp. 159–161.)

The dodecahedron shows that $f(3, 5, 0) = 20$. To see that it is the only smallest $(3, 5, 0)$-graph, note that any $(3, 5, 0)$-graph $G$ on 20 vertices has 30 edges. Since $m = \frac{g(n-2)}{g-2}$, every face in $G$ is a 5-sided face. Hence $f_5 = 12$. The only graph that satisfies these conditions is the dodecahedron. (See [1] pp. 159–161.)

We shall summarize the above observations in the following theorem.

**Theorem 1.** The five platonic solids are the only smallest $(r, g, 0)$-graphs.
3. The Case $c = 1$

Let $G$ be an $(r, g, 1)$-graph with $n$ vertices and $m$ edges. Then $cr(G) = 1 = \text{rem}(G)$ and so there exists an edge $e$ in $G$ such that $G - e$ is planar with degree sequence $(r - 1, r - 1, r, \ldots, r)$. Note that the girth of $G - e$ is at least $g$. Let $f$ denote the number of faces in $G - e$ and let $f_i$ denote the number of $i$-sided faces in $G - e$. Then $2(m - 1) = \sum_{i \geq g} i f_i = 2(r - 1) + r(n - 2)$ and $f = \sum_{i \geq g} f_i$. Substituting these into Euler’s formula for plane graphs, we have

$$
\sum_{i \geq g} (2i - (i - 2)r) f_i \geq 4r - 4. 
$$

Like the case of planar graphs, it is readily deduced that there are only five possible pairs of $(r, g)$, namely $(3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$.

Notice that the number of vertices in $G$ satisfies the following inequality

$$
n \geq \frac{2g + 4}{2r + 2g - rg}.
$$

It is easy to see that $f(4, 3, 1) = 5$ because $K_5$ is the smallest $(4, 3, 1)$-graph. Also, $f(3, 4, 1) = 6$ because $K_{3,3}$ is the smallest $(3, 4, 1)$-graph.

Now $f(3, 3, 1) \geq 8$. To see this, let $H$ be a $(3, 3, 1)$-graph on no more than 6 vertices. Then $H$ must be isomorphic to the complete bipartite graph $K_{3,3}$. But this is not possible because $K_{3,3}$ contains no triangle.

A smallest $(3, 3, 1)$-graph is obtained from $K_{3,3}$ by replacing a vertex by a triangle with edges joining the triangle in a corresponding way (see Figure 1). To see the uniqueness of this graph, let $H$ be a smallest $(3, 3, 1)$-graph.

![Figure 1. Smallest (3, 3, 1)-graph](image-url)
Then $H$ contains a triangle. Replace this triangle by a vertex of degree 3, the resulting graph is a non-planar cubic graph on 6 vertices which must be the complete bipartite graph $K_{3,3}$.

It follows from the inequality (6) that $f(3,5,1) \geq 14$. The graph depicted in Figure 2 is a $(3,5,1)$-graph and so $f(3,5,1) \leq 18$. In [10], Royle has listed all the $(3,5)$-graphs on 14 and 16 vertices. We check that all these graphs have crossing number at least 2. There are eight $(3,5)$-graphs with 14 vertices. Six of them contains a subdivision of the Petersen graph as a subgraph. One of the two remaining graphs in the list is the graph $G_7$ of Figure 6 (which has crossing number 2 as will be explained later in the last section). The last graph in the list has removal number at least 2 because the resulting graph obtained by deleting any edge from this graph contains a subdivision of $K_{3,3}$ as subgraph. There are 48 $(3,5)$-graphs on 16 vertices. Except for the last graph, all of them has crossing number at least 2 because they all contain a subdivision of the graph $G_3$ (which has crossing number 2 as will be explained later in the section) of Figure 6 as subgraph. The last graph in the list contains a subdivision of the Petersen graph as a subgraph. Hence all of these graphs have crossing number at least 2. Thus we may conclude that $f(3,5,1) = 18$. However we do not know whether or not the graph in Figure 2 is the only smallest $(3,5,1)$-graph.

![Figure 2. Smallest $(3,5,1)$-graph](image)

The graph in Figure 3 is a $(5,3,1)$-graph and so $f(5,3,1) \leq 14$. Since $10 \leq f(5,3,1)$, there are only three possible values of $f(5,3,1)$. In the next section, we shall prove that $f(5,3,1) \geq 12$. We do not know whether or not a smallest $(5,3,1)$-graph is unique.
We shall summarize the above observations in the following theorem.

**Theorem 2.** \( f(3, 3, 1) = 8, f(3, 4, 1) = 6, f(3, 5, 1) = 18, f(4, 3, 1) = 5 \) and \( 12 \leq f(5, 3, 1) \leq 14 \).

## 4. \((5, 3, 1)\)-Graphs

In this section, we prove that \((5, 3, 1)\)-graphs do not exist if the number of vertices is no more than 10.

**Proposition 1.** Let \( G \) be a 5-regular graph on 10 vertices. Then \( cr(G) \geq 2 \).

The proof of this proposition is by contradiction. It consists of a series of lemmas that we shall now prove. It is easy to see that if \( G \) is 5-regular and has 10 vertices, then it is non-planar and so \( cr(G) \geq 1 \). In this section, unless otherwise stated, we shall assume that \( G \) is a 5-regular graph on 10 vertices and that \( cr(G) = 1 \) and obtain a contradiction.

Let \( H \) be a regular graph and let \( v \) be a vertex of \( H \). Let \( A_v \) denote the subgraph of \( H \) induced by the set of vertices adjacent to \( v \). Also, let \( B_v \) denote the subgraph obtained by deleting \( A_v \cup \{v\} \) from \( H \).

**Lemma 1.** Let \( H \) be an \( r \)-regular graph on \( 2r \) vertices. Suppose \( v \in V(H) \). Then \( A_v \) and \( B_v \) have the same number of edges.

**Proof.** Note that \( A_v \) and \( B_v \) have \( r \) and \( r - 1 \) vertices respectively. Let \((a_1, \ldots, a_r)\) and \((b_1, \ldots, b_{r-1})\) denote the degree sequences of \( A_v \) and \( B_v \) respectively.
Note that the number of edges in the subgraph $H - v$ is $r(r - 1)$ and that it is also equal to $|E(A_v)| + |E(B_v)| + \text{number of edges from } A_v \text{ to } B_v$.

Therefore,

$$\sum_{i=1}^{r} a_i \frac{r}{2} + \sum_{i=1}^{r-1} b_i \frac{r-1}{2} + \sum_{i=1}^{r} (r - 1 - a_i) = r(r - 1)$$

which, on simplification, leads to $\sum_{i=1}^{r} a_i \frac{r}{2} = \sum_{i=1}^{r-1} b_i \frac{r-1}{2}$. Thus $A_v$ and $B_v$ have the same number of edges.

Lemma 2. If $\text{rem}(G) = 1$, then there is an edge $e$ in $G$ such that $G - e$ is a planar triangulation.

Proof. Since $\text{rem}(G) = 1$, there is an edge $e$ in $G$ such that $G - e$ is planar. Since $G - e$ has 10 vertices and 24 edges, $G - e$ is a triangulation.

Lemma 3. If $G$ contains a 4-cycle $a_0a_1a_2a_3$, then either $a_ia_{i+2} \in E(G)$ for some $i$ or else there is a vertex $u \in V(G)$ such that $u$ is adjacent to $a_i$ for all $i$. Here $i$ is considered modulo 4.

Proof. This is a consequence of Lemma 2 because otherwise $G - e$ is not a triangulation for any edge $e$ in $G$.

Lemma 4. $A_v$ is connected for any vertex $v$ in $G$.

Proof. Suppose $A_v$ is disconnected for some vertex $v$ in $G$. Let $w_1$ and $w_2$ be two vertices in two different components of $A_v$, say $G_1$ and $G_2$ respectively.

Since $|V(G_1)| + |V(G_2)| \leq 5$, we may assume that $w_1$ is of degree $d$ where $d \leq 1$ and $w_2$ is of degree $3 - d$ in $A_v$. Therefore $w_1$ and $w_2$ must be adjacent to a common vertex $z$ in $B_v$. But then $w_1 zw_2 v$ is a 4-cycle in $G$ not satisfying Lemma 3. This contradiction proves that $A_v$ is connected.

Lemma 4 implies that $A_v$ has at least 4 edges because it has 5 vertices. By Lemma 1, because $A_v$ and $B_v$ have the same number of edges, it follows that $A_v$ has at most 6 edges because $B_v$ has 4 vertices. Next, we shall dispose of some degree sequences of $A_v$.

Lemma 5. For any $v \in V(G)$, $A_v$ contains no vertices of degree 4.
Proof. Suppose on the contrary that $A_v$ contains a vertex $u$ of degree 4. Evidently there exist two non-adjacent vertices $x$ and $y$ in $A_v - u$.

If $x$ and $y$ are adjacent to a common vertex $z$ in $B_v$ (call this condition (*)), then $xyz$ is a 4-cycle in $G$ not satisfying Lemma 3.

Notice that the condition (*) is always satisfied if $|E(A_v)| \leq 5$.

So assume that $|E(A_v)| = 6$ and that the condition (*) is not satisfied. Then $A_v$ is the graph of Figure 4(a). In this case, the only graph in which condition (*) is not satisfied is the graph of Figure 4(b). However, the removal number of this graph is at least 2 because it contains a subdivision of $K_{3,4}$ (the subgraph induced by the thick edges) as a subgraph. This contradiction proves the lemma.

![Diagram](image)

Figure 4. Graph not satisfying condition (*)

Lemma 6. It is not possible that $A_v$ contains two non-adjacent vertices $v_1$ and $v_2$ such that $d_{A_v}(v_1) = 1$ and $d_{A_v}(v_2) \leq 2$.

Proof. If these conditions are satisfied, then $v_1$ must be adjacent to three vertices of $B_v$ and $v_2$ must be adjacent to at least two vertices of $B_v$. This means that they must be adjacent to a common vertex $z$ of $B_v$. But then $G$ contains a 4-cycle $v_1zv_2$ not satisfying Lemma 3.

Lemmas 5 and 6 imply that the only possible degree sequences for $A_v$ left to be considered are $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 3, 3)$. There are only three graphs associated with these degree sequences. However, these three cases are disposed of in the next three lemmas.

Lemma 7. $A_v$ is not the graph $K_{2,3}$. 
**Proof.** In this case, $B_v$ is the complete graph on 4 vertices. Let $x_1, x_2$ and $x_3$ denote the three vertices of degree 2 in $K_{2,3}$. Then each $x_i$ must be adjacent to two vertices in $B_v$. But this means that there are $x_i$ and $x_j$ ($i \neq j$) which are adjacent to a common vertex $z$ in $B_v$ giving rise to a 4-cycle $vx_izx_j$ in $G$ not satisfying Lemma 3.

**Lemma 8.** $A_v$ is not a cycle on 5 vertices.

**Proof.** Suppose $A_v$ is a cycle on 5 vertices whose vertices are labelled as $v_1, v_2, \ldots, v_5$ in cyclic order.

Note that $B_v$ is the graph obtained from the complete graph on 4 vertices by deleting an edge. So $B_v$ contains two vertices $x_1$ and $x_2$ of degree 2, and two vertices $y_1$ and $y_2$ of degree 3. Clearly each $x_i$ is adjacent to three vertices of $A_v$.

Now, $x_1$ and $x_2$ are adjacent to a common vertex $w$ in $A_v$. But then $x_1wx_2y_1$ or $x_1wx_2y_2$ is a 4-cycle in $G$ not satisfying Lemma 3.

**Lemma 9.** $A_v$ is not the graph of Figure 5(a).

**Proof.** Suppose $A_v$ is the graph of Figure 5(a). In this case, $B_v$ is the complete graph on 4 vertices. Let $V(B_v) = \{u_1, \ldots, u_4\}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

Note that any two non-adjacent vertices $x$ and $y$ in $A_v$ must not be adjacent to a common vertex $z$ in $B_v$. This is because otherwise we have a 4-cycle $vxzy$ in $G$ not satisfying Lemma 3.
Hence we may assume without loss of generality that \( v_2 \) is adjacent to \( u_3 \) and \( u_4 \) and that \( v_4 \) is adjacent to \( u_1 \) and \( u_2 \). Since \( v_5 \) and \( v_2 \) are non-adjacent, \( v_5 \) is adjacent to \( u_1 \) and \( u_2 \) (see Figure 5(b)).

Now \( v_1 \) is adjacent either to \( u_3 \) or \( u_4 \). However in either case, we have a 4-cycle \( v_1v_5u_1w \) not satisfying Lemma 3, where \( w \in \{u_3, u_4\} \). This contradiction proves the lemma.

5. The Case \( c = 2 \)

Let \( G \) be an \((r, g, 2)\)-graph with \( n \) vertices. Then \( 1 \leq \text{rem}(G) \leq 2 \). Let \( f_i(r, g, c) \) denote the minimum number of vertices in an \((r, g, c)\)-graph with removal number \( i \), \( i = 1, 2 \).

The rest of this paper is to prove the following theorem.

**Theorem 3.** \( f_1(3, 3, 2) = 12, f_1(3, 4, 2) = 10, f_1(3, 5, 2) = 14, f_1(4, 3, 2) = 8, 12 \leq f_1(5, 3, 2) \leq 16, f_2(3, 3, 2) = 12, f_2(3, 4, 2) = 12, f_2(3, 5, 2) = 10, f_2(4, 3, 2) = 7 \) and \( f_2(5, 3, 2) = 8 \).

Suppose \( \text{rem}(G) = 1 \). Then inequalities (5) and (6) also hold for \( G \) and so there are only five possible pairs of \((r, g)\), namely \((3, 3),(3, 4),(3, 5),(4, 3)\) and \((5, 3)\).

Suppose \( \text{rem}(G) = 2 \). Let \( x_1 \) and \( x_2 \) be two edges of \( G \) such that \( G - \{x_1, x_2\} \) is planar. Then the degree sequence of \( G - \{x_1, x_2\} \) is either \((r-2, r-1, r-1, r, \cdots, r)\) or \((r-1, r-1, r-1, r-1, r-1, r, \cdots, r)\). Following similar argument as was done for the case \( c = 1 \), we have

\[
\sum_{i \geq g} (2i - (i - 2)r) f_i \geq 4r - 8
\]

and the number of vertices in \( G \) satisfies the following inequality

\[
n \geq \frac{8}{2r + 2g - rg}.
\]

Again, only the same five pairs of \((r, g)\) satisfy inequality (7).

Let \( G \) be a non-planar graph and let \( e \) be an edge in \( G \). Then \( e \) is called a \( p \)-critical edge if \( G - e \) is a planar graph.

**Lemma 10.** Let \( G \) be a non-planar graph. If \( G \) contains a unique \( p \)-critical edge, then \( \text{cr}(G) \geq 2 \).
Proof. If \( cr(G) = 1 \), then there exist two edges \( e_1 \) and \( e_2 \) in \( G \) which intersect each other and such that \( G - \{e_i\} \) is planar for \( i = 1, 2 \). But this contradicts the uniqueness of the removal edge of \( G \).

5.1 \((3, g, 2)\)-graphs

In [9] (p. 647–648), Royle has listed all connected cubic graphs of order up to and including 10. It is easily seen that there are only two cubic graphs in the list with \( c > 1 \), the Petersen graph \( G_8 \) and the graph \( G_3 \) in Figure 6. Thus \( f_i(3, g, 2) \geq 10 \). Further, Royle’s list also indicates that \( f_2(3, 4, 2) \geq 12 \) and that \( f_1(3, 3, 2) \geq 12 \).

Clearly, the Petersen graph is a \((3, 5, 2)\)-graph. It has removal number 2 because removing any one of its edges yields a non-planar graph. Therefore \( f_2(3, 5, 2) = 10 \) and the Petersen graph is the only smallest \((3, 5, 2)\)-graph with removal number 2.

Note that the graph \( G_3 \) in Figure 6 has crossing number 2. To see this, we note that \( G_3 \) contains two vertex-disjoint graphs \( K_{2,3} \) which are not outerplanar. If \( cr(G_3) = 1 \), then at least one of these subgraphs has a planar drawing. This subgraph does not separate the vertices of the other \( K_{2,3} \) subgraph and, because of outerplanarity, its edges are crossed by at least one of the edges \( e_1, e_2 \) and \( e_3 \). A similar argument can be used for the other \( K_{2,3} \) subgraph. Hence \( G_3 \) is a \((3, 4, 2)\)-graph. Moreover it has removal number 1 because \( G_3 - e_i \) is planar for any \( 1 \leq i \leq 3 \). Therefore \( f_1(3, 4, 2) = 10 \) and \( G_3 \) is the only smallest \((3, 4, 2)\)-graph with removal number 1.

\( f_1(3, 5, 2) \geq 14 \) follows from inequality (1). In Royle’s list of cubic graphs on 14 vertices [10], there are eight graphs with girth equal to 5. All but the graph \( G_7 \) contain a subdivision of the Petersen graph as a subgraph. This means that except for the graph \( G_7 \), they all have removal number at least 2. It is a routine exercise to verify that the edge \( e \) is the only \( p \)-critical edge in \( G_7 \). By Lemma 10, \( cr(G_7) \geq 2 \). Therefore \( f_1(3, 5, 2) = 14 \) and \( G_7 \) is the only smallest \((3, 5, 2)\)-graph with removal number 1.

Let \( H \) be a smallest \((3, g, 2)\)-graph with removal number \( i \) and \( g \neq 3 \). We may obtain a \((3, 3, 2)\)-graph with removal number \( i \) by replacing a vertex of degree 3 from \( H \) by a triangle with edges joining the triangle in a corresponding way. The graph \( G_1 \) (respectively \( G_2 \)) is obtained from \( G_3 \) (respectively the Petersen graph) in this way. Combining this with the previous
observations, we have \( f_1(3, 3, 2) = 12 \) and \( f_2(3, 3, 2) = 12 \). The uniqueness of these graphs follows from that of the graph \( G_3 \) and the Petersen graph.

We now look at Royle’s list of cubic graphs on 12 vertices with girth 4 (see [10]). There are twenty such cubic graphs. However there are only three graphs \( G_4, G_5 \) and \( G_6 \) from this list with removal number at least 2. Since these three graphs can all be drawn on the plane with only two crossings, they are the smallest \((3, 4, 2)\)-graphs with removal number 2. Thus \( f_2(3, 4, 2) = 12 \).

5..2 \((4, 3, 2)\)-graphs

Let \( G \) be a \((4, 3, 2)\)-graph on \( n \) vertices. Then clearly \( n \geq 6 \).

If \( n = 6 \), then \( G \) is the complementary graph of \( 3K_2 \) and is planar. Hence \( n \geq 7 \).

If \( n = 7 \), then \( \overline{G} \) is a 2-regular graph which is either \( C_3 \cup C_4 \) or \( C_7 \). If \( G \) is \( C_3 \cup C_4 \), then \( G \) is the graph \( G_{10} \) and \( rem(G_{10}) = 2 \) because \( G_{10} \) contains \( K_{3,4} \) as a subgraph. If \( \overline{G} \) is \( C_7 \), then \( cr(G) = 1 \) (see [2]). Therefore \( f_2(4, 3, 2) = 7 \) and \( G_{10} \) is the only smallest \((4, 3, 2)\)-graph with removal number 2.

If \( n = 8 \), then \( \overline{G} \) is a cubic graph on 8 vertices. There are precisely five cubic graphs on 8 vertices. If \( \overline{G} \) is the cube, then \( G \) is the graph \( G_9 \) which is the cartesian product \( K_4 \times K_2 \) and has crossing number 2 (see [8]). Now, \( rem(G_9) = 1 \) because \( G_9 - e_i \) is planar for each \( i = 1, 2, 3 \). If \( \overline{G} \) is not the cube, we have checked, by direct verification that either \( rem(G) \geq 2 \) or else \( cr(G) \leq 1 \). Therefore \( f_1(4, 3, 2) = 8 \) and \( G_9 \) is the only smallest \((4, 3, 2)\)-graph with removal number 1.

5..3 \((5, 3, 2)\)-graphs

It follows from inequality (1) that \( f_1(5, 3, 2) \geq 10 \). However the proof of Proposition 1 implies that \( f_1(5, 3, 2) \geq 12 \). Clearly, the graph \( G_{11} \) in Figure 6 has removal number 1. We have checked that the edge \( e \) is the only removal edge. By Lemma 10, \( cr(G_{11}) = 2 \). Hence \( G_{11} \) is a \((5, 3, 2)\)-graph with \( rem(G_{11}) = 1 \) and so \( 12 \leq f_1(5, 3, 2) \leq 16 \).

Let \( G \) be a 5-regular graph on 8 vertices. Then \( \overline{G} \) is a 2-regular graph. Hence there are only three 5-regular graphs on 8 vertices namely, \( \overline{C_8}, C_3 \cup C_5 \) and \( C_4 \cup \overline{C_4} \). Since \( C_r \cup C_s \) contains \( K_{r,s} \) as a subgraph, it follows that \( cr(C_3 \cup C_5) \geq 4 \) and \( cr(C_4 \cup \overline{C_4}) \geq 4 \).
Now it follows from inequality (1) that \( rem(C_8) \geq 2 \). Figure 6 depicts a drawing of \( C_8 \) (\( \cong G_{12} \)) with two crossings and we have \( rem(G_{12}) = 2 = cr(G_{12}) \). Thus \( f_2(5, 3, 2) = 8 \) and \( G_{12} \) is the only smallest \((5, 3, 2)\)-graph with removal number 2.

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<tr>
<td></td>
<td>(f_1(3, 3, 2) = 12)</td>
<td>(f_2(3, 3, 2) = 12)</td>
</tr>
<tr>
<td>((3, 4, 2))</td>
<td>(G_3)</td>
<td>(G_4)</td>
</tr>
<tr>
<td></td>
<td>(f_1(3, 4, 2) = 10)</td>
<td>(f_2(3, 4, 2) = 12)</td>
</tr>
</tbody>
</table>

\(e_1\), \(e_2\), and \(e_3\) are the crossing edges in the drawing of \( C_8 \).
Figure 6

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References


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