GENERALIZED CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \nsubseteq \mathcal{Q}^{n-1}$. We investigate the generalized chromatic numbers of the well-known properties of graphs $I_k$, $O_k$, $W_k$, $S_k$ and $D_k$.

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1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$. A property of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$.

Throughout the text we will call a component of a graph that is a spanning supergraph of a path $P_k$ of order $k$ a $k$-component. Let $G$ be a graph and $V_1 \subseteq V(G)$. We say that a vertex $v \in V(G) - V_1$ is adjacent to a $k$-component of $G[V_1]$ if $v$ is adjacent to a vertex of some $k$-component of $G[V_1]$. 

Example. For a positive integer $k$ we define the following well-known properties:

\[ \mathcal{O} = \{ G \in \mathcal{I} : E(G) = \emptyset \}, \]
\[ \mathcal{I}_k = \{ G \in \mathcal{I} : G \text{ does not contain } K_{k+2} \}, \]
\[ \mathcal{O}_k = \{ G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \}, \]
\[ \mathcal{W}_k = \{ G \in \mathcal{I} : \text{each path in } G \text{ has at most } k + 1 \text{ vertices} \}, \]
\[ \mathcal{S}_k = \{ G \in \mathcal{I} : \text{the maximum degree of } G \text{ is at most } k \}, \]
\[ \mathcal{T}_k = \{ G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or } K_{\lceil \frac{k+3}{2} \rceil, \lfloor \frac{k+3}{2} \rfloor} \}, \]
\[ \mathcal{D}_k = \{ G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., every subgraph of } G \text{ has a vertex of degree at most } k \}. \]

For every additive hereditary property $\mathcal{P} \neq \mathcal{I}$ there is a smallest integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \not\in \mathcal{P}$, called the completion of $\mathcal{P}$. Note that all the properties in the above example, except $\mathcal{O}$, are of completeness $k$. The set $\mathbf{F}(\mathcal{P})$ of minimal forbidden subgraphs is defined by

\[ \{ G \in \mathcal{I} : G \in \mathcal{P} \text{ and } H \in \mathcal{P} \text{ for all } H \subset G \}. \]

Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be arbitrary hereditary properties of graphs. A vertex $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-partition of a graph $G$ is a partition $\{ V_1, V_2, \ldots, V_n \}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{Q}_i$. The property $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is defined as the set of all graphs having a vertex $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-partition. It is easy to see that if $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is additive and hereditary too. If $\mathcal{Q}_1 = \mathcal{Q}_2 = \cdots = \mathcal{Q}_n = \mathcal{Q}$, then we write $\mathcal{Q}^n = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$.

The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \not\subseteq \mathcal{Q}^{n-1}$.

As an example of the non-existence of $\chi_{\mathcal{Q}}(\mathcal{P})$ we have $\chi_{\mathcal{O}}(\mathcal{I}_1)$ since it is well known that there exist triangle-free graphs of arbitrary chromatic number. The following theorem, due to J. Nešetřil and V. Rödl (see [12]), implies that for some additive hereditary properties $\mathcal{P}$ we have that $\chi_{\mathcal{Q}}(\mathcal{P})$ exists if and only if $\chi_{\mathcal{Q}}(\mathcal{P}) = 1$. In particular, $\chi_{\mathcal{O}}(\mathcal{I}_k)$ exists if and only $\chi_{\mathcal{O}}(\mathcal{I}_k) = 1$.

Theorem 1.1 [12]. Let $\mathbf{F}(\mathcal{P})$ be a finite set of 2-connected graphs. Then for every graph $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{P}$ such that for any partition $\{ V_1, V_2 \}$ of $V(H)$ there is an $i$, $i = 1$ or $i = 2$, for which $G \leq H[V_i]$. $\blacksquare$
Corollary 1.2. If $F(P)$ is a finite set of 2-connected graphs, then for any additive hereditary property $Q$ it follows that $\chi_Q(P)$ exists if and only if $P \subseteq Q$. ■

The value of $\chi_Q(P)$ is known for various choices of $P$ and $Q$. In the remainder of this section we mention some simple results, most of which are known or follow immediately from well-known results. See for example [2] and [5].

It is easy to see that $O_{a+b+1} \subseteq O_a \circ O_b$ and $D_{a+b+1} \subseteq D_a \circ D_b$ (see for example [9]), which implies that $\chi_Q(O_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $Q$ of completeness $n$, and $\chi_D_n(P) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $P$ such that $O_k \subseteq P \subseteq D_k$. Note that Corollary 1.2 implies that the latter equality does not extend to $c(P) = n$.

The well-known theorem of Lovász states:

Theorem 1.3 [10]. $S_{a+b+1} \subseteq S_a \circ S_b$ for all $a, b \geq 0$. ■

This implies that $\chi_{S_n}(S_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$. (See [5].)

It is also easy to see that if $O_k \subseteq P \subseteq O^{k+1}$, then $\chi_{I_n}(P) = \left\lceil \frac{k+1}{n+1} \right\rceil$.

The next result is interesting since it shows that the value of $\chi_{S_n}(D_k)$ is independent of $n$.

Theorem 1.4. For all $k$ and $n$ we have $\chi_{S_n}(D_k) = k + 1$.

Proof. Since $D_k \subseteq O^{k+1} \subseteq S_n^{k+1}$ we have the upper bound. We prove the lower bound by induction on $k$. The result is true for $k = 1$ since $D_1 \not\subseteq S_n$.

Assume, therefore, that $D_k \not\subseteq S_n$ and let $H \in D_k$ such that $H \not\subseteq S_n^k$. Let $G = (n+1)H + K_1$. Since every subgraph of $(n+1)H$ has a vertex of degree at most $k$, every subgraph of $G$ has a vertex of degree at most $k+1$. Thus $G \in D_{k+1}$.

Also, $G \not\subseteq S_n^{k+1}$: Suppose, to the contrary, that $\{V_1, V_2, \ldots, V_{k+1}\}$ is an $S_n^{k+1}$-partition of $V(G)$. Let $v$ be the universal vertex of $G$ and suppose, without loss of generality, that $v \in V_1$. Since $G[V_1] \in S_n$ it follows that $|V_1| \leq n + 1$. Since there are $n + 1$ copies of $H$ in $G$ we have that for some copy $F$ of $H$, $F \cap V_1 = \emptyset$. This contradicts the fact that $H \not\subseteq S_n^k$. ■

The lattice of (additive) hereditary properties is discussed in [1] — we use the supremum and infimum of properties in our next result without further discussion.
Theorem 1.5. Let $P_1, P_2$ and $Q$ be additive hereditary properties such that \(\chi_Q(P_1)\) and \(\chi_Q(P_2)\) are finite. The following hold:

(i) \(\chi_Q(P_1 \cup P_2) = \chi_Q(P_1 \lor P_2) = \max\{\chi_Q(P_1), \chi_Q(P_2)\}\).

(ii) \(\chi_Q(P_1 \cap P_2) \leq \min\{\chi_Q(P_1), \chi_Q(P_2)\}\).

(iii) \(\max\{\chi_Q(P_1), \chi_Q(P_2)\} \leq \chi_Q(P_1 \circ P_2) \leq \chi_Q(P_1) + \chi_Q(P_2)\).

We remark that the inequality in Theorem 1.5(ii) may be strict. For example \(\chi_O(T_3) = 4\) and \(\chi_O(I_1)\) is infinite but \(\chi_O(T_3 \cap I_1) = 3\). (See [2].)

2. Results on \(W_k\)

In this section we investigate the value of \(\chi_{W_n}(W_k)\). The problem of determining it has been discussed in (or is related to problems in) several papers (see for example [3], [4], [6], [7], [8] and [11]) and the following conjecture has been made in at least three of them:

Conjecture 2.1 [3], [6], [7]. \(W_{a+b+1} \subseteq W_a \circ W_b\) for all positive integers \(a\) and \(b\).

This conjecture implies the following for \(\chi_{W_n}(W_k)\):

Conjecture 2.2. For every \(n, k \geq 1\), the following holds:

\[\chi_{W_n}(W_k) = \left\lceil \frac{k + 1}{n + 1} \right\rceil.\]

In [6] the bound \(\chi_{W_n}(W_k) \leq \left\lfloor \frac{k-n+1}{2} \right\rfloor + 2\) is proved. The following theorem will enable us to improve on this bound.

Theorem 2.3. \(W_{\left\lfloor \frac{a+b}{2} \right\rfloor + b+1} \subseteq W_a \circ W_b\) for all \(a \geq 15\) and \(b \geq 1\).

Proof. Consider any graph \(G\) in \(W_{\left\lfloor \frac{a+b}{2} \right\rfloor + b+1}\). Take \(V_1\) to be a maximal subset of \(V(G)\) such that \(G[V_1]\) is in \(W_a\). Let \(V_2 = V(G) - V_1\). Suppose that there is a path \(P\) in \(G[V_2]\) of length \(b+1\) and let \(v_1\) and \(v_2\) denote the end-vertices of \(P\). Since \(V_1\) is maximal in \(W_a\) it follows that there is a path \(P_1\) of length \(a+1\) in \(G[V_1 \cup \{v_1\}]\) and a path \(P_2\) of length \(a+1\) in \(G[V_1 \cup \{v_2\}]\). Note that if either \(v_1\) or \(v_2\) is an end-vertex of \(P_1\) or \(P_2\) respectively, then in both cases we get a path of length at least \(a+b+3\).
in $G$, a contradiction. Therefore the vertices $v_1$ and $v_2$ are not end-vertices of their respective paths. Let $P_{11}$ and $P_{12}$ denote the paths on either side of $v_1$ such that $P_{11} \cup \{v_1\} \cup P_{12} = P_1$. Similarly, let $P_{21} \cup \{v_2\} \cup P_{22} = P_2$. Now suppose, without loss of generality, that $x = |E(P_{11})| + 1 \leq y = |E(P_{12})| + 1$, so that $x + y = a + 1$.

It is easily seen that if $y \geq \lceil \frac{2a+2}{3} \rceil + 1$, then by simply taking the path $P_{12} \cup P$, we get a path of length at least $\lceil \frac{2a+2}{3} \rceil + 1 + b + 1 \geq \frac{2a+2}{3} + b + 2 \geq \left\lfloor \frac{3a}{2} \right\rfloor + b + 1$ in $G$, a contradiction. Therefore $\lceil \frac{a+1}{2} \rceil \leq y \leq \lceil \frac{2a+2}{3} \rceil$. Moreover, each $P_{ij}$, $i, j \in \{1, 2\}$ has length at least $\left\lceil \frac{a-5}{3} \right\rceil$, since $x = a + 1 - y \geq a - \left\lceil \frac{2a+2}{3} \right\rceil + 1 \geq a - \frac{2a+5}{3} = \frac{a-8}{3} \geq \left\lceil \frac{a-5}{3} \right\rceil$.

Note that $P_{11}$ and $P_{12}$ are necessarily disjoint as are $P_{21}$ and $P_{22}$, and that $v_1$ and $v_2$ are not on any of these paths.

$P_{12}$ must intersect both $P_{21}$ and $P_{22}$: Firstly, $P_{12}$ must intersect the longer of $P_{21}$ and $P_{22}$ since otherwise we get a too long path in $G$; containing the two longer paths and $P$. Furthermore, if $P_{12}$ does not intersect the shorter of $P_{21}$ and $P_{22}$, then we get a path of length at least $\lceil \frac{a+1}{2} \rceil + b + 1 + \left\lceil \frac{a-5}{3} \right\rceil \geq \frac{a+1}{2} + \frac{a-7}{3} + b + 1 = \frac{5}{6}(a - 1) > \left\lceil \frac{2a}{3} \right\rceil + 1 + b$ (since $a \geq 15$) in $G$; containing $P_{12}$, $P$, and the shorter of $P_{21}$ and $P_{22}$, a contradiction. Similarly, the longer of $P_{21}$ and $P_{22}$ must intersect both $P_{11}$ and $P_{12}$.

Note that since $P_{11}$ and $P_{12}$ are disjoint and $P_{21}$ and $P_{22}$ are disjoint, $P_{2i}$, $i \in \{1, 2\}$ can only intersect one of $P_{11}$ and $P_{12}$ first and vice-versa.

Suppose that both $P_{21}$ and $P_{22}$ intersect $P_{12}$ first. Then we obtain a path of length at least $x + b + 1 + 1 + \left\lfloor \frac{5}{2} \right\rfloor \geq a + 1 - y + \left\lfloor \frac{y-1}{2} \right\rfloor + b + 2 \geq a - \frac{1}{2} \left\lceil \frac{2a+2}{3} \right\rceil + \frac{5}{2} + b \geq a - \frac{1}{2} \left( \frac{2a+2}{3} \right) + \frac{5}{2} + b = \frac{2a}{3} + \frac{13}{6} + b > \left\lceil \frac{2a}{3} \right\rceil + 1 + b$ in $G$; containing $P_{11}$, $P$, at least one edge of either $P_{21}$ or $P_{22}$ and at least a half of $P_{12}$, a contradiction.

Now, suppose that $P_{21}$ or $P_{22}$ intersects $P_{11}$ first, say $P_{21}$. Then we obtain a path of length at least $y + \left\lfloor \frac{5}{2} \right\rfloor + b + 1 + 1 = y + \left\lfloor \frac{1}{2} (a + 1 - y) \right\rfloor + b + 2 \geq y + \frac{a+1-y-1}{2} + b + 2 = \frac{a}{2} + \frac{9}{2} + b + 2 \geq \frac{1}{2} \left\lceil \frac{a+1}{2} \right\rceil + \frac{9}{2} + b + 2 \geq \frac{2a+9}{4} + b > \left\lceil \frac{2a}{3} \right\rceil + 1 + b$ in $G$; containing $P_{12}$, $P$, at least one edge of $P_{21}$ and at least a half of $P_{11}$, a contradiction.

**Theorem 2.4.** $\chi_{W_n}(W_k) \leq \left\lceil \frac{3k}{2n+3} \right\rceil$ for all $n \geq 15$ and $k \geq 1$.

**Proof.** $W_{\left\lceil \frac{2n+3}{4} \right\rceil} \subseteq W_n^c$ for all positive integers $c$ and $n$: the proof is by induction on $c$. The result holds for $c = 1$. Suppose now that the result holds for $c$. Note that $W_{(c+1)\left\lceil \frac{2n+3}{4} \right\rceil} = W_{\left\lceil \frac{2n}{4} \right\rceil + 1 + c \left\lceil \frac{2n+3}{4} \right\rceil}$ which by Theorem 2.3 is
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contained in \( W_n \circ W^c_{c[2n+3]} \) which by the induction hypothesis is contained in \( W_n \circ W^c_n = W^c_{n+1} \).

Now, with \( c = \left\lceil \frac{3k}{2n+3} \right\rceil \), since \( k \leq \left\lceil \frac{3k}{2n+3} \right\rceil \left\lceil \frac{2n+3}{3} \right\rceil \) we have that \( W_k \subseteq W^c_{c[2n+3]} \) \( \subseteq W^c_n \).

This result is close to the bound \( \chi_{W_n}(W_k) \leq \left\lceil \frac{3(k-n)}{2n+2} \right\rceil + 1 \) presented in [7] but our method of proof is completely different.

3. Results Relating \( S_k \) and \( W_n \)

**Theorem 3.1.** For positive integers \( n \) and \( k \) we have that

\[
\left\lceil \frac{k+1}{n+1} \right\rceil \leq \chi_{W_n}(S_k) \leq \left\lceil \frac{k+1}{2} \right\rceil.
\]

**Proof.** The left inequality holds since \( K_{k+1} \in S_k \). The right inequality follows as a corollary to Theorem 1.3.

The first inequality in Theorem 3.1 may be strict, for example \( \chi_{W_2}(S_2) = 2 > \left\lceil \frac{2+1}{2+1} \right\rceil \) (since \( S_2 \not\subseteq W_2 \)). Equality in both the inequalities may be achieved, for example, by Theorem 1.3 we have that \( \chi_{S_n}(S_k) = \left\lceil \frac{k+1}{n+1} \right\rceil \) and therefore \( \chi_{W_1}(S_k) = \left\lceil \frac{k+1}{2} \right\rceil \).

Note that whether or not the second inequality proved in Theorem 3.1 may be strict still remains an open problem.

We now start working towards bounds on \( \chi_{S_n}(W_k) \).

**Theorem 3.2.** \( W_4 \subseteq S_2 \circ S_1 \).

**Proof.** Consider any graph \( G \) in \( W_4 \). Take \( V_1 \) to be a subset of \( V(G) \) such that, in order of priority:

(i) \( G[V_1] \) is in \( S_2 \),

(ii) \( G[V_1] \) contains a maximum number of 4-components,

(iii) \( G[V_1] \) contains a maximum number of components isomorphic to \( K_3 \),

(iv) \( G[V_1] \) contains a maximum number of 2-components and

(v) \( G[V_1] \) contains a maximum number of isolated vertices.

(In other words, we consider all subsets \( V \) of \( V(G) \) such that \( G[V] \in S_2 \). Amongst these we consider all subsets \( V \) for which \( G[V] \) has a maximum
number of 4-components. Amongst these we consider all subsets inducing a maximum number of components isomorphic to $K_3$ etc.)

Let $V_2 = V(G) - V_1$. We will show that $G[V_2] \in S_1$. Suppose, to the contrary, that $G[V_2] \notin S_1$ and let $v$ be a vertex in $G[V_2]$ of degree at least two with $u$ and $w$ two of its neighbours in $G[V_2]$. Note that by choice of $V_1$ every component in $G[V_1]$ is a 4-component, $K_3$, $K_2$, or $K_1$.

Moreover, by (v) it follows that $u$, $v$ and $w$ each have at least one neighbour in $V_1$. Furthermore, $v$ cannot be adjacent to a 4-component in $G[V_1]$. This case is analogous to the above case since a 4-component will also contribute three vertices to give a $P_6$ in $G$. Moreover, neither $u$ nor $w$ are adjacent to 4-components or triangles in $G[V_1]$, since otherwise we obtain at least a $P_6$ in $G$.

Therefore $v$ must be adjacent to a $K_2$ in $G[V_1]$. Note that $u$ and $w$ must each have at least one neighbour on the $K_2$ adjacent to $v$ in $G[V_1]$, otherwise we obtain a $P_6$ in $G$. If $v$ is adjacent to both vertices on the $K_2$ in $G[V_1]$, then we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii).

Thus $v$ has only one neighbour on any $K_2$ in $G[V_1]$. If $u$ or $w$ is adjacent to the same vertex as $v$ on the $K_2$ adjacent to $v$ in $G[V_1]$, then once again we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii). Therefore, both $u$ and $w$ are adjacent to the vertex on the $K_2$ in $G[V_1]$ that is not adjacent to $v$. However, then we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a 4-component; containing the $K_2$ in $G[V_1]$ and the vertices $v$ and either $u$ or $w$; still satisfying (i), but contradicting (ii). Therefore $G[V_2] \in S_1$.

\textbf{Corollary 3.3.} For all $n \geq 2$ and $k$, we have that $\chi_{S_n}(W_k) \leq 2 \left\lceil \frac{k+1}{5} \right\rceil$. 
Proof. It is known that $W_{4+k+1} \subseteq W_4 \circ W_k$ (see [3]). Similar to the proof of Theorem 1.3 it follows that $W_k \subseteq W_{\lceil \frac{k+1}{2} \rceil}$. The result now follows from Theorem 3.2.

The inequality in Corollary 3.3 may be strict, for example $\chi_{S_2}(W_1) = 1 < 2 = 2\lceil \frac{2}{3} \rceil$. Equality may also be obtained, for example $\chi_{S_2}(W_2) = 2 = 2\lceil \frac{3}{5} \rceil$.

Having proved Corollary 3.3 we naturally ask: Can this bound be improved and if so under what conditions? Corollary 3.5 gives us an answer for $n \geq 5$ and Theorem 3.7 for $n \geq 9$.

Theorem 3.4. For an additive hereditary property $Q$ with $c(Q) \geq 5$, the following holds: $W_k \subseteq Q^{\lceil \frac{k}{3} \rceil} \circ O$.

Proof. Let $c = \lceil \frac{k}{3} \rceil$. Consider any graph $G$ in $W_k$. Take $V_1$ to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $Q$,
(ii) $G[V_1]$ contains a maximum number of 6-components,
(iii) $G[V_1]$ contains a maximum number of 4-components,
(iv) $G[V_1]$ contains a maximum number of 2-components and
(v) $G[V_1]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take $V_i$ to be a subset of $V(G) - \bigcup_{j=1}^{i-1} V_j$ such that for each $i$, $G[V_i]$ satisfies the above list. Let $S = V(G) - \bigcup_{j=1}^{c-1} V_j$. We will show that $G[S] \in O$. Suppose, to the contrary, that $G[S] \notin O$ and let $v$ be a vertex in $G[S]$ of degree at least one and $u$ be a neighbour of $v$ in $G[S]$. Suppose that $v$ is not an end-vertex of a $P_4$ in $G[V_c \cup \{v\}]$ and that $u$ is not an end-vertex of a $P_5$ in $G[V_c \cup \{u\}]$. Note that for every $i$, the choice of $V_i$ gives that every component in $G[V_i]$ is a 6-component, a 4-component, $K_2$ or $K_1$. Moreover, by (iv) it follows that $u$ and $v$ have at least one neighbour in $V_c$ each and by (iv) both $u$ and $v$ are adjacent to nontrivial components in $G[V_c]$. Since $v$ is not an end-vertex of a $P_4$ in $G[V_c \cup \{v\}]$ it follows that $v$ is not adjacent to a 6-component or a 4-component in $G[V_c]$. Similarly, $u$ is not adjacent to a 6-component in $G[V_c]$.

Suppose that $u$ is adjacent to a 4-component in $G[V_c]$. Then, since $v$ is not adjacent to a 4-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u$ and $v$ with a 6-component; still satisfying (i) since $K_6 \in Q$, but contradicting (ii), since neither $u$ nor $v$ is adjacent to a 6-component in $G[V_c]$.\[\]
Therefore \( u \) is adjacent to a 2-component in \( G[V_c] \). However, then we can replace the components in \( G[V_c] \) that are adjacent to \( u \) and \( v \) with a 4-component; satisfying (i) and (ii) but contradicting (iii).

Therefore, \( u \) is an end-vertex of a \( P_5 \) in \( G[V_c \cup \{u\}] \) or \( v \) is an end-vertex of a \( P_4 \) in \( G[V_c \cup \{v\}] \). In both cases it follows that there is a path \( P \) of length four in \( G[S \cup V_c] \). Let \( x \) be the end-vertex of \( P \) in \( V_c \) and \( y \) the neighbour of \( x \) on \( P \). By repeating this argument it follows that \( x \) is an end-vertex of a \( P_4 \) in \( G[V_c \setminus \{x\}] \) or \( y \) is an end-vertex of a \( P_5 \) in \( G[V_{c-1} \cup \{y\}] \). Continuing in this way we obtain a path of length at least \( 3c+1 \geq k+1 \) in \( G \), a contradiction. Therefore, \( G[S] \notin \mathcal{O} \).

**Corollary 3.5.** For an additive hereditary property \( Q \) with \( c(Q) \geq 5 \), the following holds: \( \chi_Q(W_k) \leq \left\lceil \frac{k}{3} \right\rceil + 1 \).

The inequality in Corollary 3.5 may be strict, for example we have that \( \chi_{\mathcal{T}_6}(W_k) = \left\lceil \frac{k+1}{6} \right\rceil < \frac{k+3}{3} \leq \left\lceil \frac{k}{3} \right\rceil + 1 \) with \( K_6 \in \mathcal{T}_5 \) and \( K_7 \notin \mathcal{T}_5 \). Equality can also be obtained: In Theorem 3.8 (still to follow) we prove that \( \chi_{S_n}(W_6) \geq 3 \) and by Corollary 3.5 we have \( \chi_{S_n}(W_6) \leq \left\lceil \frac{6}{3} \right\rceil + 1 = 3 \).

**Theorem 3.6.** For an additive hereditary property \( Q \) with \( c(Q) \geq 9 \), the following holds: \( W_k \subseteq Q^{\left\lceil \frac{k-1}{4} \right\rceil} \circ S_1 \).

**Proof.** Let \( c = \left\lceil \frac{k-1}{4} \right\rceil \). Consider any graph \( G \) in \( W_k \). Take \( V_1 \) to be a subset of \( V(G) \) such that, in order of priority:

(i) \( G[V_1] \) is in \( Q \),
(ii) \( G[V_1] \) contains a maximum number of 10-components,
(iii) \( G[V_1] \) contains a maximum number of 8-components,
(iv) \( G[V_1] \) contains a maximum number of 6-components,
(v) \( G[V_1] \) contains a maximum number of 4-components,
(vi) \( G[V_1] \) contains a maximum number of 2-components and
(vii) \( G[V_1] \) contains a maximum number of isolated vertices.

Now, for \( 2 \leq i \leq c \) take \( V_i \) to be a subset of \( V(G) - \bigcup_{j=1}^{i-1} V_j \) such that for each \( i \), \( G[V_i] \) satisfies the above list. Let \( S = V(G) - \bigcup_{j=1}^{c-1} V_j \). We will show that \( G[S] \in S_1 \). Suppose, to the contrary, that \( G[S] \notin S_1 \) and let \( v \) be a vertex in \( G[S] \) of degree at least two with \( u \) and \( w \) neighbours of \( v \) in \( G[S] \).
Suppose that $u$ is not an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ and that $v$ is not an end-vertex of a $P_6$ in $G[V_c \cup \{v\}]$ and that $w$ is not an end-vertex of a $P_5$ in $G[V_c \cup \{w\}]$. Note that for every $i$, the choice of $V_i$ gives that every component in $G[V_i]$ is a 10-component, an 8-component, a 6-component, a 4-component, $K_2$ or $K_1$. Moreover, by (vii) it follows that $u$, $v$ and $w$ have at least one neighbour in $V_c$ each and by (vi) each of $u$, $v$ and $w$ is adjacent to a nontrivial component in $G[V_c]$. Since $u$ is not an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ it follows that $w$ is not adjacent to a 10-component in $G[V_c]$. Similarly, $v$ is not adjacent to a 10-component or an 8-component in $G[V_c]$ and $w$ is not adjacent to a 10-component, an 8-component or a 6-component in $G[V_c]$.

Suppose that $u$ is adjacent to an 8-component in $G[V_c]$. Then, since neither $v$ nor $w$ are adjacent to an 8-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u$, $v$ and $w$ with a 10-component; still satisfying (i) but contradicting (ii).

Suppose that $v$ is adjacent to a 6-component in $G[V_c]$. Then, since $w$ is not adjacent to a 6-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u$, $v$ and $w$ with an 8-component; still satisfying (i) and (ii) but contradicting (iii), since none of $u$, $v$ and $w$ is adjacent to a 10-component or an 8-component in $G[V_c]$. Similarly, $u$ is not adjacent to a 6-component in $G[V_c]$.

Suppose that $v$ is adjacent to a 4-component in $G[V_c]$. Note that since $u$, $v$ and $w$ are not adjacent to 6-components in $G[V_c]$ it follows that neither $u$ nor $w$ is adjacent to a 4-component in $G[V_c]$ — otherwise we can replace the components in $G[V_c]$ that are adjacent to $u$, $v$ and $w$ with a 6-component; satisfying (i) through (iii) but contradicting (iv). Therefore, since $u$ and $w$ are not adjacent to 4-components in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u$, $v$ and $w$ with a 6-component; containing three vertices of the 4-component, $u$ and its neighbour in $V_c$.

Therefore $v$ is adjacent to a 2-component in $G[V_c]$. However, then we can replace the components in $G[V_c]$ that are adjacent to $u$, $v$ and $w$ with a 4-component; satisfying (i) through (iv) but contradicting (v).

Therefore, $u$ is an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ or $v$ is an end-vertex of a $P_6$ in $G[V_c \cup \{v\}]$ or $w$ is an end-vertex of a $P_5$ in $G[V_c \cup \{w\}]$. In each case it follows that there is a path $P$ of length 6 in $G[S \cup V_c]$. Let $z$ be the end-vertex of $P$ in $V_c$, $y$ the neighbour of $z$ on $P$ and $x$ the other neighbour of $y$ on $P$. By repeating the above argument it follows that $z$ is an end-vertex of a $P_5$ in $G[V_{c-1} \cup \{z\}]$ or $y$ is an end-vertex of a $P_6$ in $G[V_{c-1} \cup \{y\}]$ or $x$
is an end-vertex of a $P_7$ in $G[V_{c-1} \cup \{x\}]$. Continuing in this way we obtain a path of length at least $4c + 2 \geq k + 1$ in $G$, a contradiction. Therefore, $G[S] \in S_1$.

**Theorem 3.7.** For $n \geq 9$, the following holds:

$$\left\lfloor \frac{k + 1}{n + 1} \right\rfloor \leq \chi_{S_n}(W_k) \leq \left\lceil \frac{k - 1}{4} \right\rceil + 1.$$  

**Proof.** The left inequality holds since $K_{k+1} \in W_k$. The right inequality follows as a corollary of Theorem 3.6 since $K_10 \in S_n$ for each $n \geq 9$.

Our next result improves on the lower bound in Theorem 3.7 for large values of $n$.

**Theorem 3.8** For all positive integers $k$ and $n$, $\chi_{S_n}(W_k) \geq \lfloor \log_2(k + 2) \rfloor$.

**Proof.** We first prove, by induction on $m$, that for all positive integers $m$ and $n$, $W_{2m-1-2} \not\subseteq S_n^m$. For the case where $m = 1$ the result holds since $W_1 \not\subseteq S_n$. Assume therefore that the result holds for $m - 1$, thus there exists a graph $H$ such that $H \in W_{2m-2}$ and $H \not\subseteq S_n^{m-1}$. Now let $G = (n + 1)H + K_1$. Clearly $G \in W_{2(2m-1-2)+2} = W_{2m+1-2}$. As in the proof of Theorem 1.4 $G \not\subseteq S_n^m$.

Now, let $k$ and $n$ be any positive integers. We have that $W_k \supseteq W_{2\lfloor \log_2(k+2) \rfloor - 2} \not\subseteq S_n^{\lfloor \log_2(k+2) \rfloor - 1}$ and the result follows.

Corollary 3.5 and Theorem 3.7 seem to suggest that for every $k$ and $m$ we can get $W_k \subseteq S_n^{\left\lceil \frac{k}{m} \right\rceil + 1}$ for all $n$ sufficiently large. However, Theorem 3.8 implies that $W_6 \not\subseteq S_n^{\left\lceil \frac{6}{2} \right\rceil + 1}$ for all $n$ since $\chi_{S_n}(W_6) \geq \lfloor \log_2(8) \rfloor = 3$. The method of proof in Theorem 3.6 does not extend. If we try to maximize with respect to 12-components, 10-components etc. the argument fails, and assuming that $k$ is large makes no difference.

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