TOTAL DOMINATION EDGE CRITICAL GRAPHS WITH MAXIMUM DIAMETER

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Abstract

Denote the total domination number of a graph $G$ by $\gamma_t(G)$. A graph $G$ is said to be total domination edge critical, or simply $\gamma_t$-critical, if $\gamma_t(G + e) < \gamma_t(G)$ for each edge $e \in E(G)$. For $3_t$-critical graphs $G$, that is, $\gamma_t$-critical graphs with $\gamma_t(G) = 3$, the diameter of $G$ is either 2 or 3. We characterise the $3_t$-critical graphs $G$ with $\text{diam } G = 3$.

1. Introduction

Let $G = (V, E)$ be a graph with order $|V| = n$. The open neighbourhood of a vertex $v$ is the set of vertices adjacent to $v$, that is, $N(v) = \{w \mid vw \in E(G)\}$, and the closed neighbourhood of $v$ is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$ we define the open and closed neighbourhoods $N(S)$ and $N[S]$ of $S$ by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$, respectively. The private neighbourhood of $x \in S$, $S \subseteq V(G)$, consists of all vertices in the closed neighbourhood of $x$ but not in the closed neighbourhood of $S - \{x\}$, and is denoted by $\text{pn}(x, S)$, that is, $\text{pn}(x, S) = N[x] - N[S - \{x\}]$. If $v \in \text{pn}(x, S)$, then $v$ is called a private neighbour of $x$ relative to $S$, or simply a private neighbour of $x$, if confusion is unlikely. If $G$ is a graph with $\text{diam } G = k$ and $d(u, v) = k$, then...
we say that \( u \) and \( v \) are diametrical vertices. A shortest \( u-v \) path in \( G \) is a diametrical path. Two subsets \( X \) and \( Y \) of \( V \) are called diametrical sets if \( d(x, y) = \text{diam } G \) for each \( x \in X \) and \( y \in Y \). If \( X \) and \( Y \) are diametrical sets, then \( (X, Y) \) is a maximal diametrical pair if for each \( z \in V - (X \cup Y) \), \( d(x, z) < \text{diam } G \) for some \( x \in X \) and \( d(y, z) < \text{diam } G \) for some \( y \in Y \).

For sets \( S, X \subseteq V \), if \( N[S] = X \) (\( N(S) = X \), respectively), we say that \( S \) dominates \( X \), written \( S \succ X \) (\( S \succ_t X \), respectively, written \( S \succ_t X \)). If \( S = \{s\} \) or \( X = \{x\} \), we also write \( s \succ X \), \( S \succ_t x \), etc. If \( S \succ V \) (\( S \succ_t V \), respectively), we say that \( S \) is a dominating set (total dominating set) of \( G \), and we also write \( S \succ G \) (\( S \succ_t G \), respectively). The cardinality of a minimum dominating (minimum total dominating) set of \( G \) is called the domination number (total domination number) of \( G \) and is denoted by \( \gamma(G) \) (\( \gamma_t(G) \), respectively); if \( S \) is a minimum dominating (minimum total dominating) set, we also call \( S \) a \( \gamma \)-set (\( \gamma_t \)-set) of \( G \). We note that the parameter \( \gamma_t(G) \) is only defined for graphs \( G \) without isolated vertices. Domination-related concepts not defined here can be found in [2].

The addition of an edge to a graph can change the domination number by at most one. Sumner and Blitch [5, 6] studied domination edge critical graphs \( G \), that is, graphs \( G \) for which \( \gamma(G) = \gamma(G) - 1 \) for each \( e \in E(G) \). We consider the same concept for total domination. A graph \( G \) is total domination edge critical or just \( \gamma_t \)-critical if \( \gamma_t(G + e) < \gamma_t(G) \) for any edge \( e \in E(G) \neq \emptyset \). It is shown in [3] that the addition of an edge to a graph can change the total domination number by at most two.

**Proposition 1** [3]. For any edge \( e \in E(G) \),

\[
\gamma_t(G) - 2 \leq \gamma_t(G + e) \leq \gamma_t(G).
\]

Graphs \( G \) with the property \( \gamma_t(G + e) = \gamma_t(G) - 2 \) for any \( e \in E(G) \) are called supercritical and are characterised in [4].

In this paper, we restrict our attention to \( 3_t \)-critical graphs \( G \), that is, \( \gamma_t(G) = 3 \). Note that since \( \gamma_t(G) \geq 2 \) for any graph \( G \), the addition of an edge to a \( 3_t \)-critical graph reduces the total domination number by exactly one. Also, observe that any graph \( G \) with \( \gamma_t(G) = 3 \) is connected. Sharp bounds on the diameter of a \( 3_t \)-critical graph are determined in [3].

**Proposition 2** [3]. If \( G \) is a \( 3_t \)-critical graph, then

\[
2 \leq \text{diam } G \leq 3.
\]
The graphs in Figures 1 and 2 illustrate sharpness of these bounds. Our goal is to investigate the $3_t$-critical graphs with diameter three.

![Figure 1. A $3_t$-critical graph $G$ with $diam \ G = 2$](image1)

2. $3_t$-Critical Graphs with Diameter Three

In [3] the authors showed that any $3_t$-critical graph $G$ with a cutvertex has exactly one cutvertex and it is adjacent to an endvertex. Moreover, they proved that such graphs $G$ have $diam \ G = 3$ and are the only $3_t$-critical graphs with an endvertex. Figure 2 illustrates a $3_t$-critical graph with an endvertex.

![Figure 2. A $3_t$-critical graph with an endvertex](image2)
Theorem 3 [3]. A graph $G$ with a cutvertex $v$ is $3_t$-critical if and only if $v$ is adjacent to an endvertex $x$, and for $W = N(v) - \{x\}$ and $Y = V - N[v]$, 

1. $\langle W \rangle$ is complete and $|W| \geq 2$,
2. $\langle Y \rangle$ is complete and $|Y| \geq 2$,

and

3. every vertex in $W$ is adjacent to $|Y| - 1$ vertices in $Y$ and every vertex in $Y$ is adjacent to at least one vertex in $W$.

We begin with a straightforward but useful observation.

Observation 4. For any $3_t$-critical graph $G$ and non-adjacent vertices $u$ and $v$, either

1. $\{u, v\}$ dominates $G$ or
2. (without loss of generality) $\{u, w\}$ dominates $G - v$, but not $v$, for some $w \in N(u)$. In this case, we write $uw \mapsto v$.

Next we develop some structural properties of $3_t$-critical graphs $G$ with $\text{diam } G = 3$. Although it is possible in a $3_t$-critical graph $G$ of diameter two for every pair of nonadjacent vertices to dominate $G$ (see Figure 1, for example), we now show this is not possible if $\text{diam } G = 3$.

Proposition 5. If $G$ is a $3_t$-critical graph with $\text{diam } G = 3$, then $G$ has a pair of nonadjacent vertices that does not dominate $G$.

Proof. Let $G$ be a $3_t$-critical graph with $\text{diam } G = 3$ and suppose that every pair of nonadjacent vertices of $G$ dominates $G$. Let $x$ and $y$ be diametrical vertices of $G$ where $x, a, b, y$ is a shortest $x$-$y$ path. Since $\{x, b\} \triangleright G$, every neighbour of $y$ is also dominated by $b$. Similarly, every neighbour of $x$ is dominated by $a$. Hence $\{a, b\}$ is a total dominating set of $G$, contradicting the fact that $\gamma_t(G) = 3$.

Also, it is possible for a $3_t$-critical graph $G$ with $\text{diam } G = 2$ to have the property that for every pair of nonadjacent vertices $u$ and $v$, there is a vertex $x$ such that $ux \mapsto v$, and there is a vertex $y$ such that $vy \mapsto u$. See Figure 3 for an example. We now show that a $3_t$-critical graph with diameter three cannot have this property.
Proposition 6. If $G$ is a $3_t$-critical graph with $\text{diam } G = 3$, then $G$ has a pair of nonadjacent vertices $u$ and $v$ such that $ux \leftrightarrow v$, for some $x \in V$, but there is no vertex $y$ such that $vy \leftrightarrow u$.

Proof. Let $G$ be a $3_t$-critical graph with diameter three. Let $x$ and $y$ be diametrical vertices of $G$ where $x, a, b, y$ is a shortest $x$-$y$ path. By the proof of Proposition 5, at least one of $\{x, b\}$ and $\{a, y\}$ does not dominate $G$. Assume then, without loss of generality, that $\{x, b\}$ does not dominate $G$. If $xw \leftrightarrow b$, then $w \in N(x)$ by Observation 4 and $w \in N(y)$ to dominate $y$, thus $d(x, y) \leq 2$, a contradiction. Hence the only possibility is that $bw \leftrightarrow x$.

It is useful to know more about the diametrical sets of vertices of a $3_t$-critical graph with diameter three.

Theorem 7. If $G$ is a $3_t$-critical graph with $\text{diam } G = 3$, then $G$ has a unique maximal diametrical pair $(X, Y)$. Moreover, $X$ (say) has cardinality one and $\langle Y \rangle$ is complete.

Proof. Let $G$ be a $3_t$-critical graph with $\text{diam } G = 3$. The proof of the theorem is a direct consequence of the following three lemmas.

Lemma 8. For any maximal diametrical pair $(Y_1, Y_2)$ of $G$, $\langle Y_i \rangle$ is complete for each $i$ and $|Y_i| = 1$ for at least one $i$.

Proof. Let $(Y_1, Y_2)$ be a maximal diametrical pair of $G$. First we show that if $|Y_i| \geq 2$, then $\langle Y_i \rangle$ is complete. Let $x \in Y_1$ and $\{y, z\} \subseteq Y_2$ and
suppose that $yz \notin E(G)$. Since $\{y, z\} \not\subseteq G$, we may assume without loss of generality that $yw \rightarrow z$ for some vertex $w$, contradicting the fact that $d(x, y) = 3$. Hence $\langle Y_2 \rangle$ is complete. A similar argument shows that $\langle Y_1 \rangle$ is complete.

Next we show that $|Y_i| = 1$ for at least one $i$. Suppose to the contrary that both $Y_1$ and $Y_2$ have cardinality at least two. Let $x \in Y_1$ and $y \in Y_2$ and consider $\{x, y\}$. Since $|Y_i| \geq 2$ for $i \in \{1, 2\}$, there is no vertex $w$ such that $xw \rightarrow y$ or $yw \rightarrow x$. It follows that $\{x, y\} > G$. This is the case for every $x \in Y_1$ and every $y \in Y_2$. Let $A$ ($B$, respectively) be the set of vertices that are distance one from every vertex of $Y_1$ ($Y_2$, respectively). If both $\langle A \rangle$ and $\langle B \rangle$ are complete, then $\gamma_t(G) = 2$, a contradiction. Thus let $a, b \in A$ where $ab \notin E(G)$. Consider $\{a, y\}$. Since neither $a$ nor $y$ is adjacent to $b$, $\{a, y\} \not\subseteq G$. Hence, $yc \rightarrow a$ or $ac \rightarrow y$. Since no vertex in $N[y]$ dominates $Y_1$, $ac \rightarrow y$. Therefore, $c$ dominates $Y_2 - \{y\}$. Furthermore, since $\{x, y\} > G$, $c$ is adjacent to $x$, implying that $y$ is the only vertex at distance three from $x$, contradicting the fact that $|Y_i| > 1$ for $i \in \{1, 2\}$. \[\blacksquare\]

Consider the maximal diametrical pair $\langle \{x\}, Y \rangle$ of $G$. Note that by Lemma 8 and the definition of maximal diametrical pair, $Y = \{y \in V \mid d(x, y) = 3\}$.

**Lemma 9.** For every vertex $u \in V - \{x\}$, $d(u, y) \leq 2$ for every $y \in Y$.

**Proof.** If $|Y| = 1$, then $x$ is the only vertex at distance three from $Y$. Assume then that $|Y| \geq 2$. Let $y, z \in Y$ and suppose there is a vertex $u$ such that $d(u, y) = 3$ and $d(u, z) = 2$; note that $u \neq x$. Let $uaby$ be a $u$-$y$ path and let $ucz$ be a $u$-$z$ path ($c$ may equal $a$). Note that $cy \notin E(G)$. Since neither $x$ nor $y$ is adjacent to $c$, $xw \rightarrow y$ or $yw \rightarrow x$. If $xw \rightarrow y$, then $d(x, z) = 2$, contradicting that $z \in Y$ and that $\{x\}$ and $Y$ are diametrical sets. If $yw \rightarrow x$, then $d(u, y) = 2$, again a contradiction. \[\blacksquare\]

**Lemma 10.** $\langle \{x\}, Y \rangle$ is the unique maximal diametrical pair of $G$.

**Proof.** Consider any maximal diametrical pair $\langle \{u\}, W \rangle$ of $G$. If $u = x$, then $W = \{w \in V \mid d(u, w) = 3\} = \{w \in V \mid d(x, w) = 3\} = Y$ and we are done. If $u \in Y$, then $d(x, u) = 3$, i.e., $x \in W$ and by Lemma 9, $d(u, z) \leq 2$ for each $z \in V - \{x\}$. Hence $W = \{x\}$ and since $\langle \{u\}, \{x\} \rangle$ is a maximal diametrical pair, it follows that $Y = \{u\}$ and the result follows. Hence we may assume that $u \notin Y \cup \{x\}$. It follows from Lemma 9 that $W \cap (Y \cup \{x\}) = \emptyset$. 

Consider any \( w \in W \) and suppose firstly that \( \{u, w\} \succ G \). Note that no vertex is adjacent to \( x \) as well as to a vertex in \( Y \). Hence either \( ux \in E(G) \) and \( wy \in E(G) \) for each \( y \in Y \), or \( wx \in E(G) \) and \( uy \in E(G) \) for each \( y \in Y \). Suppose the former case holds and consider an arbitrary vertex \( y \in Y \). By Lemma 9, \( d(u, y) = 2 \) and \( d(w, x) = 2 \). Let \( uay \) and \( wbx \) be a \( u-y \) path and a \( w-x \) path, respectively and note that \( \{ub, yb\} \subseteq E(G) = \emptyset \). Thus \( \{u, y\} \not\succ G \) and so \( uc \mapsto y \) or \( yc \mapsto u \) for some vertex \( c \). If \( uc \mapsto y \), then \( cw \in E(G) \) and so \( d(u, w) = 2 \), a contradiction since \( u \) and \( w \) are diametrical vertices. If \( yc \mapsto u \), then \( d(x, y) = 2 \), also a contradiction. Similarly, the case \( wx \in E(G) \) and \( uy \in E(G) \) for each \( y \in Y \) is impossible. We conclude that \( \{u, w\} \not\succ G \).

Thus there is some vertex \( d \) such that \( \{u, w, d\} \) is independent. Since neither \( d \) nor \( u \) is adjacent to \( w \), \( uc \mapsto d \) or \( dc \mapsto u \). If \( uc \mapsto d \), then \( d(u, w) = 2 \), a contradiction. Thus we may assume that \( dc \mapsto u \). Then without loss of generality, \( d \in N(Y) \) and \( c \in N(x) \). Now we consider \( \{x, d\} \). Since \( d \) is not adjacent to \( u \) or \( w \), and \( x \) cannot be adjacent to both \( u \) and \( w \), \( xd \) is not a dominating edge for \( G + xd \). Then \( xs \mapsto d \) or \( ds \mapsto x \). If \( xs \mapsto d \), then \( d(x, y) = 2 \), a contradiction. If \( ds \mapsto x \), then \( s \) is adjacent to both \( u \) and \( w \), contradicting the fact that \( d(u, w) = 3 \). Hence \( \{x, Y\} \) is the unique diametrical pair of \( G \).

### 3. Characterisation

In the rest of this paper we characterise the 3t-critical graphs with diameter three. We introduce more notation to simplify the characterisation. Let \( G \) be a graph with \( \text{diam} \ G = 3 \) and let \( \{\{x\}, Y\} \) be a maximal diametrical pair of \( G \). Let \( A = N(x) \), \( B = \{b \mid b \notin Y \text{ and } b \succ Y\} \), and \( C = V - (A \cup B \cup Y \cup \{x\}) \).

Note that at least one of \( B \) and \( C \) is not empty. Let \( F \) be the family of all graphs \( G \) with \( \text{diam} \ G = 3 \) and the maximal diametrical pair \( \{\{x\}, Y\} \). Then \( F = F_1 \cup F_2 \cup F_3 \cup F_4 \), where

- \( G \in F_1 \) if \( C = \emptyset \) and \( |Y| \geq 2 \),
- \( G \in F_2 \) if \( C = \emptyset \) and \( |Y| = 1 \),
- \( G \in F_3 \) if \( B = \emptyset \),
- \( G \in F_4 \) if \( B \neq \emptyset \) and \( C \neq \emptyset \).

To characterise the 3t-critical graphs with diameter 3, we characterise the 3t-critical graphs in each family \( F_i \), \( 1 \leq i \leq 4 \). We begin with a lemma.
Lemma 11. Let $G \in \mathcal{F}$ be $3_t$-critical with $|Y| \geq 2$. If either $B = \emptyset$ or $C = \emptyset$, then $\langle A \rangle$ is complete.

Proof. Let $G \in \mathcal{F}$ with $|Y| \geq 2$ and suppose that $\langle A \rangle$ is not complete. First assume that $C = \emptyset$. Let $u, v \in A$ with $uv \notin E(G)$. Consider $\{u, y\}$ for some vertex $y \in Y$. Since neither $u$ nor $y$ is adjacent to $v$, $uw \rightarrow y$ or $yw \rightarrow u$ for some vertex $w$. If $uw \rightarrow y$, then $w \in A \cup \{x\}$ since $w \notin N(y)$. But then $Y - \{y\}$ is not dominated by $\{u, w\}$, a contradiction. If $yw \rightarrow u$, then $d(x, y) \leq 2$, again a contradiction. Next assume that $B = \emptyset$. Since $\{u, v\} \notin G$, we may assume, without loss of generality, that $uw \rightarrow v$. But this implies that $w > Y$, contradicting the fact that $B = \emptyset$.

Lemma 11 requires that the graph $G$ has a diametrical set $Y$ with cardinality greater than one. (See Figure 4(b)). The graph in Figure 4(a) is an example of a graph with a diametrical set $Y$ such that $|Y| = 1$ and $\langle A \rangle$ complete. However, the condition of the lemma is necessary as can be seen by the $3_t$-critical graph in Figure 5 that has $|Y| = 1$ and $\langle A \rangle$ is not complete.

![Figure 4](image1.png)

(a) $|X| = |Y| = 1$  (b) $|X| = 1$ and $|Y| > 1$

Figure 4. Two $3_t$-critical graphs with diameter three

![Figure 5](image2.png)

$|X| = |Y| = 1$

Figure 5. A $3_t$-critical graph with $\langle A \rangle$ not complete
We first characterise the 3₁-critical graphs $G \in \mathcal{F}_1$.

**Theorem 12.** A graph $G \in \mathcal{F}_1$ is 3₁-critical if and only if the following conditions hold:

1. $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y \rangle$ is complete.
2. $\langle A \rangle$ is complete.
3. For every nonadjacent pair $u, v \in B$, there is a vertex $a \in A$ such that $ua \mapsto v$. Also, no pair of adjacent vertices dominates $G$.
4. For every vertex $b \in B$, there is a vertex $d \in B \cup Y$ such that $bd \mapsto x$.
5. For every pair $\{a, b\}$ of nonadjacent vertices where $a \in A$ and $b \in B$, $\{a, b\} \succ G$ or $aw \mapsto b$ for some $w \in B$.

**Proof.** Let $G \in \mathcal{F}_1$ be 3₁-critical. By Theorem 10, $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y \rangle$ is complete.

Since $C = \emptyset$, it follows that $\{x, y\} \succ G$ for every $y \in Y$. From Lemma 11 we have that $\langle A \rangle$ is complete. Furthermore, since $(\{x\}, Y)$ is a maximal diametrical pair, each $b \in B$ is adjacent to at least one vertex $a \in A$. If there is a vertex $b \in B$ that dominates $B$, then $\{a, b\} \succ G$ for an $a \in A$, contradicting the fact that $\gamma_1(G) = 3$. Let $u, v \in B$ with $uv \notin E(G)$. Obviously, $\{u, v\} \nRightarrow x$, so without loss of generality, there is a vertex $a \in A$ such that $au \mapsto v$. Since $\gamma_1(G) = 3$, no pair of adjacent vertices dominates $G$. To show that (4) holds, let $b$ be any vertex in $B$. Since there is at least one vertex in $B$ not adjacent to $b$, $\{x, b\} \nRightarrow G$. No vertex in $N[x]$ dominates $Y$, so $bd \mapsto x$ for some $d \in B \cup Y$. Condition (5) follows directly from Observation 4 and the fact that if $bw \mapsto a$, then $w \in A$ to dominate $x$; hence $w \succ a$ since $\langle A \rangle$ is complete, a contradiction.

Conversely, let $G \in \mathcal{F}_1$ such that the stated properties hold. Since no pair of adjacent vertices dominates $G$, $\gamma_l(G) \geq 3$. Further, $\{a, b, y\}$ is a $\gamma_l$-set for every $a \in A$, $b \in B$, $y \in Y$ where $ab \in E(G)$, implying that $\gamma_l(G) \leq 3$. Hence $\gamma_l(G) = 3$. To show that $G$ is 3₁-critical we consider first $\{x, y\}$ for $y \in Y$. Since $C = \emptyset$, $\{x, y\} \succ G$. Similarly, $\{a, y\} \succ G$ for every $a \in A$. We next consider $\{x, b\}$. Since condition (4) holds, there is a vertex $d \in B \cup Y$ such that $bd \mapsto x$. We also consider $\{a, b\}$, $a \in A$ and $b \in B$. Property (5) implies that either $\{a, b\} \succ G$ or there is a vertex $w \in B$ such that $aw \mapsto b$. Finally we consider $\{b, c\}$, where $b, c \in B$. Since condition (3) holds, there is a vertex $a \in A$ such that $ab \mapsto c$. Thus $G$ is 3₁-critical. ■
Note that \( \{x, y\} \not\rightarrow G \) for every \( y \in Y \). We state this result as a corollary.

**Corollary 13.** If \( G \in F_1 \) is \( 3_t \)-critical, then \( \gamma(G) = 2 \).

We now give a more descriptive characterisation of the \( 3_t \)-critical graphs \( G \in F_1 \) with \( \delta(G) = 2 \). We first show that if \( \delta(G) = 2 \), then \( \deg(x) = 2 \).

**Proof.** Let \( G \in F_1 \) be \( 3_t \)-critical. Since \( G \) has no cutvertices (Theorem 3), \( |A|, |B| \geq 2 \). Every vertex \( b \in B \) is adjacent to some vertex \( a \in A \) and to every vertex \( y \in Y \). Thus \( \deg(b) \geq 3 \) for every \( b \in B \), since \( |Y| \geq 2 \).

By Theorem 10, \( \langle Y \rangle \) is complete. Therefore \( \deg(y) \geq 3 \) for each \( y \in Y \).

Finally, every vertex \( a \in A \) has at least one neighbour in \( A \), implying that \( \deg(a) \geq 3 \).

We use the following notation for the characterisation. Let \( A = N(x) = \{x_1, x_2\} \) and \( B_1 = (N(x_1) \cap N(x_2)) - \{x\}, B_2 = N(x_1) - (B_1 \cup \{x, x_2\}), \) and \( B_3 = N(x_2) - (B_1 \cup \{x, x_1\}) \). Recall that \( C = \emptyset \) and hence \( B = B_1 \cup B_2 \cup B_3 \).

We need the following lemmas for the characterisation. To simplify notation we refer to the \( 3_t \)-critical graphs \( G \in F_1 \) with \( \delta(G) = 2 \) as family \( G_2 \).

**Lemma 14.** If \( G \in F_1 \) and \( G \) is \( 3_t \)-critical with \( \delta(G) = 2 \), then \( \deg(x) = 2 \) and \( \deg(v) \geq 3 \) for all \( v \in V(G) - \{x\} \).

**Proof.** Let \( G \in F_1 \) be \( 3_t \)-critical. Since \( G \) has no cutvertices (Theorem 3), \( |A|, |B| \geq 2 \). Every vertex \( b \in B \) is adjacent to some vertex \( a \in A \) and to every vertex \( y \in Y \). Thus \( \deg(b) \geq 3 \) for every \( b \in B \), since \( |Y| \geq 2 \).

By Theorem 10, \( \langle Y \rangle \) is complete. Therefore \( \deg(y) \geq 3 \) for each \( y \in Y \).

Finally, every vertex \( a \in A \) has at least one neighbour in \( A \), implying that \( \deg(a) \geq 3 \).

We use the following notation for the characterisation. Let \( A = N(x) = \{x_1, x_2\} \) and \( B_1 = (N(x_1) \cap N(x_2)) - \{x\}, B_2 = N(x_1) - (B_1 \cup \{x, x_2\}), \) and \( B_3 = N(x_2) - (B_1 \cup \{x, x_1\}) \). Recall that \( C = \emptyset \) and hence \( B = B_1 \cup B_2 \cup B_3 \).

**Lemma 15.** If \( G \in G_2 \) and \( B_i \neq \emptyset \), then \( \langle B_i \rangle \) is complete for \( i \in \{1, 2, 3\} \).

**Proof.** Let \( G \in G_2 \) and assume that \( B_i \neq \emptyset \). Suppose that \( u, v \in B_i \) and \( uv \not\in E(G) \). Since neither \( u \) nor \( v \) dominates \( x \), without loss of generality, \( uw \leftrightarrow v \). Then \( w \in N(u) \cap N(x) \). But since \( u \) and \( v \) are in \( B_i \), \( v \in N(w) \), contradicting that \( uw \leftrightarrow v \).

**Lemma 16.** If \( G \in G_2 \) and \( B_1 \neq \emptyset \), then each vertex in \( B_1 \) dominates exactly \( |B_i| - 1 \) vertices in \( B_i \) for \( i \in \{2, 3\} \).

**Proof.** It is easy to see that no vertex \( b \in B_1 \) dominates \( B_2 \) or \( B_3 \). Suppose, without loss of generality, a vertex \( b \in B_1 \) is not adjacent to two vertices in \( B_2 \), say \( u \) and \( v \), and consider \( \{b, u\} \). Since neither \( b \) nor \( u \) dominates \( x \), \( \{b, u\} \not\in G \). Furthermore, \( ux_1 \not\in b \) since \( x_1 \in N(b) \). Hence \( bx_2 \not\in u \), implying that \( v \in B_3 \), a contradiction.

**Lemma 17.** If \( G \in G_2 \), then \( |B_i| \geq 2 \) for \( i \in \{2, 3\} \).
Proof. Let $G \in \mathcal{G}_2$. Since $(\{x\}, Y)$ is a maximal diametrical pair, each $a \in A$ is adjacent to some $b \in B$. Hence $B_1 \cup B_i \neq \emptyset$ for $i \in \{2, 3\}$. If $B_2 = \emptyset$ (or $B_3 = \emptyset$, respectively), then $\{x_2, b_3\} \succ_1 G$ for $b_3 \in B_1 \cup B_3$ (if $b_2 \in B_1 \cup B_2$, respectively). Hence neither $B_2$ nor $B_3$ is empty. Suppose without loss of generality that $|B_2| = 1$, say $B_2 = \{b_2\}$. By Lemma 16, $b_2$ is not adjacent to any vertex in $B_1$. Also, $b_2$ is not adjacent to any vertex in $B_3$, for otherwise $\{x_2, b_3\} \succ_1 G$ for some $b_3 \in B_3 \cup N(b_2)$. Now consider $\{b_2, x\}$. Since $\{b_2, x\} \nRightarrow B_3 \neq \emptyset$ and $\{x, x_i\} \nRightarrow Y$, there exists a vertex $w$ such that $b_2w \mapsto x$. But no vertex adjacent to $b_2$ dominates $x_2$ as well as $B_3$, a contradiction. Hence $|B_i| \geq 2$ for $i \in \{2, 3\}$.

Lemma 18. If $G \in \mathcal{G}_2$, then $\langle B_2 \cup B_3 \rangle$ is the disjoint union of non-trivial stars.

Proof. Note that $\langle B_2 \cup B_3 \rangle$ has no isolates, for if $u \in B_2$ (say) dominates $B_3$, then $\{u, x_1\} \succ_1 G$, contradicting the fact that $\gamma_t(G) = 3$. Assume without loss of generality that a vertex $u \in B_2$ is not adjacent to vertices $b_1, \ldots, b_k \in B_3$, where $k \geq 2$ and where $b_1$ (say) is not adjacent to $v \in B_2$, $v \neq u$. Since $\{u, b_1\} \nRightarrow x$, we may assume without loss of generality that $uw \mapsto b_1$ for some vertex $w$. Then $w = x_1$ to dominate $x$, but $\{u, x_1\} \nRightarrow b_2$, a contradiction. The result follows since $\langle B_i \rangle$ is complete for $i = 2, 3$.

Theorem 19. A graph $G \in \mathcal{G}_2$ if and only if the following conditions hold:

1. $(\{x\}, Y)$ is the unique maximal diametrical pair and $(Y)$ is complete.
2. $deg(x) = 2$ and $(A)$ is complete.
3. $B_1 = \emptyset$ or $\langle B_1 \rangle$ is complete.
4. $|B_i| \geq 2$ and $\langle B_i \rangle$ for $i \in \{2, 3\}$ is complete.
5. $\langle B_2 \cup B_3 \rangle$ is the disjoint union of non-trivial stars.
6. If $B_1 \neq \emptyset$, then every vertex in $B_1$ dominates exactly $|B_i| - 1$ vertices in $B_i$ for $i \in \{2, 3\}$. Also, if $u \in B_2$ ($u \in B_3$, respectively) does not dominate $B_1$, then there is a vertex $v \in B_1 \cup B_3$ ($v \in B_1 \cup B_2$, respectively) such that $\{u, v\} \succ_1 B$.

Proof. Let $G \in \mathcal{G}_2$. By Theorem 12, $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$, $(Y)$ is complete, and $(A)$ is complete. By Lemma 14, $deg(x) = 2$. By Lemmas 15, 17, and 18, conditions (3), (4), and (5) hold. Assume without loss of generality that $u \in B_2$ does not dominate $B_1$. Since
\[ \{x, u\} \not\supset G \text{ and } \{x, x_i\} \not\supset Y, \] it follows that \(uv \mapsto x\) for some \(v\). To dominate \(x_2\) but not \(x, v \in B_1 \cup B_3\), and clearly \(\{u, v\} \succ_t B\). Thus by Lemma 16, condition (6) holds.

Conversely, let \(G\) be graph such that all the conditions of the theorem hold. There is no edge \(uv \in E(G)\) such that \(\{u, v\} \succ G\). Hence \(\gamma_t(G) \geq 3\). The path \(x_1, x_2, b_i\), for \(b_i \in B\), is a total dominating set. Therefore \(\gamma_t(G) = 3\).

To show that \(G\) is \(\gamma_t\)-critical we first consider \(\{x, y\}\) for any \(y \in Y\). Since \(C = \emptyset\), \(\{x, y\} \succ G\) for every \(y \in Y\). Next consider \(\{x, b\}\) for any \(b \in B_2\). Since \(b \succ A \cup Y\), \(by \mapsto x\) for any \(y \in Y\). Now consider \(\{x, u\}\) for any \(u \in B_2\). If \(u\) is not adjacent to any vertex in \(B_3\), then by (5), every \(c \in B_2 - \{u\} \neq \emptyset\) is adjacent to all vertices in \(B_3\), i.e., \(\{x_1, c\} \succ_t G\), a contradiction. So, if \(B_1 = \emptyset\) or \(u \succ B_1\), let \(v \in B_3\) be adjacent to \(u\). Clearly, \(uv \mapsto x\). If \(u \not\succ B_1\), then by (6) there is a vertex \(v \in B_1 \cup B_3\) such that \(\{u, v\} \succ_t B\) and it is easy to see that \(uv \mapsto x\). The set \(\{x, u\}\) for any \(u \in B_3\) is dealt with in exactly the same way. Further, it is easy to see that \(\{x_1, v\}\) and \(\{x_2, u\}\) dominate \(G\) for every \(v \in B_3\) and every \(u \in B_2\). Also, \(\{x_i, y\} \succ G\) for \(i = 1, 2\) and every \(y \in Y\). By Condition (6) a vertex \(b \in B_1\) dominates exactly \(|B_i| - 1\) vertices in \(B_i\), \(i = 2, 3\). Let \(u \in B_2\) be non-adjacent to \(b \in B_1\). Then \(bx_2 \mapsto u\). Similarly, \(bx_1 \mapsto v\), for \(v \in B_3\) and \(bv \notin E(G)\). Finally, we consider \(\{u, v\}\) with \(u \in B_2\) and \(v \in B_3\), where \(uv \notin E\). Since \((B_2 \cup B_3)\) is the disjoint union of non-trivial stars, we may assume without loss of generality that \(u\) has degree 1 in \((B_2 \cup B_3)\). Then \(ux_1 \mapsto v\). It now follows that \(G \in G_2\). 

For an example of a \(3_t\)-critical graph \(G \in G_2\), see Figure 6.

For \(3_t\)-critical graphs \(G \in F_1\), the cardinality of \(Y\) is greater than one. A necessary condition for these graphs is that \(\langle A\rangle\) is complete. However, when
the cardinality of \( Y \) is equal to one, this condition is no longer required. Figure 4(a) is an example of \( G \in \mathcal{F}_2 \) and \( 3_t \)-critical with \( \langle A \rangle \) complete and Figure 5 is an example of a graph \( G \in \mathcal{F}_2 \) and \( 3_t \)-critical with \( |Y| = 1 \) and \( (A) \) not complete.

**Theorem 20.** A graph \( G \in \mathcal{F}_2 \) is \( 3_t \)-critical if and only if the following conditions hold:

1. \( \{x, y\} \) is the unique diametrical pair of \( G \).
2. For each \( a \in A \) and \( b \in B \) with \( ab \in E(G) \) there exists a vertex \( w \notin N(a) \cup N(b) \).
3. For each \( a, a' \in A \), with \( aa' \notin E(G) \), there exists \( b' \in B \) such that \( ab' \iff a' \). A similar statement holds for each \( b, b' \in B \) with \( bb' \notin E(G) \).
4. For every \( a \in A \), \( \{a, y\} \succ G \) or there exists \( a' \in A \) such that \( aa' \iff y \). A similar statement holds for every \( b \in B \) and \( \{x\} \).
5. For each \( a \in A \) and \( b \in B \) with \( ab \notin E(G) \), \( \{a, b\} \succ G \) or, without loss of generality, there exists \( b' \in B \) such that \( ab' \iff b \).

**Proof.** Let \( G \in \mathcal{F}_2 \) be \( 3_t \)-critical. By Theorem 7 \( \{x, y\} \) is the unique diametrical pair of \( G \). Condition (2) follows from the fact that \( \gamma_t(G) = 3 \). Since \( \langle A \rangle \) and \( \langle B \rangle \) cannot both be complete, let \( a, a' \in A \) with \( aa' \notin E(G) \). Neither \( a \) nor \( a' \) is adjacent to \( y \). Therefore without loss of generality there exists \( b' \in B \) such that \( ab' \iff a' \). Let \( a \in A \) be an arbitrary vertex. If \( \{a, y\} \succ G \), then Condition (4) holds. Otherwise there exists \( w \) such that \( yw \iff a \) or \( aw \iff y \). If \( yw \iff a \), then \( x \in N(w) \) implying \( d(x, y) = 2 \), a contradiction. Hence \( aw \iff y \) for some \( w \in A \). A similar argument shows that for every \( b \in B \), \( \{b, x\} \succ G \) or there exists \( b' \in B \) such that \( bb' \iff x \). Let \( a \in A \) and \( b \in B \) with \( ab \notin E(G) \). If \( \{a, b\} \succ G \), then Condition (5) holds. Otherwise, without loss of generality, there exists \( b' \in B \) such that \( ab' \iff b \).

Conversely, let \( G \) be a graph such that the stated conditions hold. By Condition (2) there is no edge that dominates \( G \). Thus, \( \gamma_t(G) \geq 3 \). Consider \( \{a, y\} \) for any \( a \in A \). If \( \{a, y\} \succ G \), then with \( b \in N(a) \cap N(y) \), \( \{a, b, y\} \) is a total dominating set. If \( \{a, y\} \nsubseteq G \), then by Condition (4) there exists \( a' \in A \) such that \( aa' \iff y \). Again with \( b \in N(a) \cap N(y) \), \( \{a, a', b\} \) is a total dominating set, so \( \gamma_t(G) \leq 3 \). Hence \( \gamma_t(G) = 3 \). That \( G \) is \( \gamma_t \)-critical follows from the fact that \( \{x, y\} \succ G \) and from Conditions (2) through (5).

Two additional lemmas are needed for the remaining characterisations.
Lemma 21. If $G \in \mathcal{F}$ is 3$_t$-critical, then every vertex in $C$ is adjacent to exactly $|Y| - 1$ vertices in $Y$.

Proof. By definition, there is no vertex in $C$ that dominates $Y$. Suppose there is a vertex $c \in C$ that is not adjacent to at least two vertices in $Y$, say $u$ and $v$. Clearly, $\{c, u\} \not\in G$. Therefore $cw \leftrightarrow u$ or $uw \leftrightarrow c$ for some vertex $w$. If $cw \leftrightarrow u$, then $w \in N(x)$ and $w \succ v$, contradicting the fact that $d(x, v) = 3$. If $uw \leftrightarrow c$, then $w \succ x$, again contradicting that $d(x, u) = 3$. 

It was shown in Theorem 7 that $\langle Y \rangle$ is complete. We now consider $\langle C \rangle$.

Lemma 22. If $G \in \mathcal{F}$ is 3$_t$-critical and $C \neq \emptyset$, then $\langle C \rangle$ is complete.

Proof. Let $u, v \in C$ and $uv \notin E(G)$. Since $\{u, v\} \not\in G$, assume without loss of generality that $uw \leftrightarrow v$. By definition there is a vertex $y \in Y$ not adjacent to $u$. Therefore, $w \succ y$ and $w \succ x$. But this contradicts the fact that $d(x, y) = 3$.

We now characterise the 3$_t$-critical graphs in family $\mathcal{F}_3$.

Theorem 23. A graph $G \in \mathcal{F}_3$ is 3$_t$-critical if and only if the following conditions hold:

1. $\langle \{x\}, Y \rangle$ is the unique maximal diametrical pair of $G$ and $\langle Y \rangle$ is complete.
2. $\langle A \cup C \rangle$ is complete.
3. $|C| \geq 2$, $|Y| \geq 2$ and every vertex in $C$ is adjacent to exactly $|Y| - 1$ vertices in $Y$.

Proof. Let $G \in \mathcal{F}_3$ be 3$_t$-critical. From Theorem 7 we have that $\langle \{x\}, Y \rangle$ is the unique maximal diametrical pair and $\langle Y \rangle$ is complete.

By Lemmas 11 and 22, $\langle A \rangle$ and $\langle C \rangle$ are complete. We show that $\langle A \cup C \rangle$ is complete. Let $a \in A$ and $c \in C$ with $ac \notin E(G)$. Since there is at least one vertex in $Y$ not adjacent to $c$, $\{a, c\} \not\in G$. The only possibility is that $aw \leftrightarrow c$. Thus $w \succ Y$, contradicting the fact that $B = \emptyset$.

By Lemma 21, if $Y = \{y\}$ (say), then no vertex in $C$ is adjacent to $y$ and since $B = \emptyset$, it follows that $y$ is isolated in $G$, which is impossible. Hence $|Y| \geq 2$. Suppose that $|C| = 1$. Since $|Y| \geq 2$, there is a vertex $y \in Y$ that is not adjacent to a vertex of $C$. But then $diam(G) > 3$, a contradiction. Hence $|C| \geq 2$. 

For the necessity, let $G \in F_3$ and assume that the conditions of the theorem hold. It is easy to see that there is no edge $ac \in E(G)$ such that $\{a, c\}$ dominates $G$. Thus $\gamma_t(G) \geq 3$. On the other hand, every shortest $y-a$ path, $y \in Y$ and $a \in A$, is a total dominating set of cardinality three, implying that $\gamma_t(G) = 3$. We now show that $G$ is $3_t$-critical. First consider $\{x, c\}$, for any $c \in C$. Since $c \succ A \cup C$, $cy \mapsto x$ for any $y \in Y$ adjacent to $c$. Next, consider $\{x, y\}$, for any $y \in Y$. Here it is also easy to see that $yc \mapsto x$ for any $c \in N(y) \cap C$. For any $a \in A$ and $y \in Y$, $\{a, y\} \succ G$. Finally we consider $\{c, y\}$ with $cy \notin E$. Since $y$ is the only vertex in $Y$ not adjacent to $c$, $ca \mapsto y$ for any $a \in A$.

**Corollary 24.** If $G \in F_3$ is $3_t$-critical, then $\gamma(G) = 2$.

See Figures 2 and 4(b) for examples of $3_t$-critical graphs in $F_3$. Note that this family of $3_t$-critical graphs includes those graphs with minimum degree one characterised in Theorem 3 where $x$ is the endvertex of $G$.

Next we consider the family $F_4$. See Figure 7 for an example.

![Figure 7. A 3_t-critical graph $G \in F_4$](image)

We now characterise the $3_t$-critical graphs $G \in F_4$ using the same notation as before.

**Theorem 25.** A graph $G \in F_4$ is $3_t$-critical if and only if the following conditions hold:

1. $(x, Y)$ is the unique maximal diametrical pair of $G$ and $(Y)$ is complete.
2. $(C)$ is complete and each $c \in C$ dominates exactly $|Y| - 1$ vertices in $Y$. 
(3) If $|Y| \geq 2$, then for every $y \in Y$, \{x, y\} $\not\supset G$ or there exists $w \in B \cup C$ such that $yw \mapsto x$. If $|Y| = 1$ (say $Y = \{y\}$), then \{x, y\} $\not\supset G$ and there exists $y' \in B$ such that $y' \supset A \cup C$ or $x' \in A$ such that $x' \supset B \cup C$.

(4) For every $c \in C$, there exists $w \in B \cup C \cup Y$ such that $cw \mapsto x$.

(5) For every $b \in B$, \{x, b\} $\supset G$ or there exists $w \in B \cup C \cup Y$ such that $bw \mapsto x$.

(6) For every $a \in A$ and $y \in Y$, \{a, y\} $\supset G$ or there exists $w \in A \cup C$ if $Y = \{y\}$ (w $\in C$ if $|Y| \geq 2$) such that $aw \mapsto y$.

(7) For each $a \in A$ and $c \in C$ with $ac \not\in E(G)$, there exists $b \in B$ such that $ab \mapsto c$.

(8) For each $a \in A$ and $b \in B$ with $ab \not\in E(G)$, \{a, b\} $\supset G$ or there exists $a' \in A$ such that $a'b \mapsto a$ or $b' \in B$ such that $ab' \mapsto b$. For each $ab \in E(G)$ with $a \in A$ and $b \in B$, there exists $w \in A \cup B \cup C$ such that $w \not\in (N(a) \cup N(b))$.

(9) For each $b \in B$ and $c \in C$ with $bc \not\in E(G)$, there exists $a \in A$ such that $ab \mapsto c$.

(10) For each $c \in C$ and $y \in Y$ with $cy \not\in E(G)$, there exists $a \in A$ such that $ac \mapsto y$.

**Proof.** Let $G \in \mathcal{F}_4$ be 3$_r$-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, \{C\} is complete. By Lemma 21, each vertex in $C$ is adjacent to exactly $|Y| - 1$ vertices in $Y$.

Consider arbitrary $y \in Y$. If \{x, y\} $\supset G$, then $|Y| \geq 2$ since $C \neq \emptyset$ and $y$ must dominate $C$. Hence Condition (3) holds in this case. Therefore we may assume that \{x, y\} does not dominate $G$. Since $G$ is 3$_r$-critical, $xw \mapsto y$ or $yw \mapsto x$. If $xw \mapsto y$, then $w \in A$ implying that $w \supset B \cup C$ and that $Y = \{y\}$. Thus if $|Y| \geq 2$, then $yw \mapsto x$ and we have shown that Condition (3) holds if $|Y| \geq 2$. Therefore we may assume that $|Y| = 1$. Now $G$ has the unique maximal diametrical pair (\{x\}, \{y\}) and neither $x$ nor $y$ dominates any vertex in $C$. Hence $xx' \mapsto y$ with $x' \in A$ or $yy' \mapsto x$ with $y' \in B$, and Condition (3) follows.

Condition (4) follows from the fact that each $c \in C$ dominates at most $|Y| - 1$ vertices in $Y$ and there is no $x' \in A$ such that $xx' \mapsto c$ for any $c \in C$.

Let $b$ be an arbitrary vertex in $B$. If \{x, b\} $\supset G$, then Condition (5) holds. Otherwise $xx' \mapsto b$ for $x' \in A$ or $bb' \mapsto x$ for $b' \in B \cup C \cup Y$. If $xx' \mapsto b$, then $x' \supset Y$ implying $d(x, y) = 2$, a contradiction. Hence $bb' \mapsto x$. 
If for \( a \in A \) and \( y \in Y \), \( \{a, y\} \succ G \), then Condition (6) holds. Otherwise \( yy' \mapsto a \) for \( y' \in N(y) \) or \( aa' \mapsto y \) for \( a' \in A \cup C \). If \( yy' \mapsto a \), then \( x \in N(y') \) implying \( d(x, y) < 3 \), a contradiction. Hence \( aa' \mapsto y \).

Consider \( \{a, c\} \) where \( a \in A \) and \( c \in C \) are not adjacent. Since neither \( a \) nor \( c \) dominates \( Y \), \( \{a, c\} \not\succ G \). Therefore, \( ca' \mapsto a \) with \( a' \in A \) (to dominate \( x \)) or \( ab' \mapsto c \) with \( b' \in B \) (to dominate \( Y \)). If \( ca' \mapsto a \), then \( c \succ Y \), contradicting that each \( c \in C \) dominates at most \( |Y| - 1 \) vertices in \( Y \). Hence \( ab' \mapsto c \) and Condition (7) holds.

Condition (8) follows directly from the definition of 3\(_t\)-critical graphs. If \( b \in B \) and \( c \in C \) with \( bc \not\in E(G) \), then \( \{b, c\} \not\succ G \) since neither \( b \) nor \( c \) is adjacent to \( x \). Since there is no \( c' \in N(c) \) such that \( cc' \mapsto b \), \( ba' \mapsto c \) with \( a' \in A \). Hence Condition (9) holds.

Finally we consider \( \{c, y\} \) with \( cy \not\in E(G) \). Again since neither \( c \) nor \( y \) is adjacent to \( x \), \( \{c, y\} \not\succ G \). Also, since \( y \) has no neighbour \( y' \) such that \( y' \succ x \), \( ca' \mapsto y \) with \( a' \in A \).

Let \( G \) be a graph such that the stated properties hold. By Condition (8) there is no \( ab \in E(G) \) with \( a \in A \) and \( b \in B \) such that \( \{a, b\} \succ G \), and since no other edge dominates \( G \), \( \gamma_t(G) \geq 3 \). By Condition (10), there is \( a \in A \) for every \( c \in C \) such that \( ac \mapsto y \) for some \( y \in Y \). Further, each \( a \in A \) is adjacent to some \( b \in B \) since \( \{x\}, Y \) is the unique maximal diametrical pair. Therefore, \( \{a, b, c\} \) is a total dominating set of \( G \), implying that \( \gamma_t(G) \leq 3 \). Hence \( \gamma_t(G) = 3 \). That \( G \) is \( \gamma_t \)-critical, follows from Conditions (3) through (10).

Finally we consider a subclass of the family \( \mathcal{F}_4 \).

**Lemma 26.** If \( G \in \mathcal{F}_4 \) is \( 3_t \)-critical and \( \langle A \rangle \) is not complete, then every \( y \in Y \) dominates at most \( |C| - 1 \) vertices in \( C \).

**Proof.** Let \( u, v \in A \) with \( uv \not\in E(G) \) and suppose there is a vertex \( y \in Y \) such that \( y \succ C \). Consider \( \{u, y\} \). Since \( \{u, y\} \not\succ G \) and there is no vertex \( c \in C \) such that \( uc \mapsto y \), there must be a vertex \( w \in N(y) \) such that \( yw \mapsto u \). But then \( d(y, x) \leq 2 \), contradicting \( \text{diam}(G) = 3 \).

**Lemma 27.** If \( G \in \mathcal{F}_4 \) is \( 3_t \)-critical and \( \langle A \rangle \) is not complete, then \( |C| \geq |Y| \).

**Proof.** Let \( |C| = k \) and \( |Y| = p \). Since every vertex in \( C \) is adjacent to exactly \( |Y| - 1 \) vertices in \( Y \), there are exactly \( k(p - 1) \) edges from \( C \) to \( Y \). By Lemma 26, every \( y \in Y \) dominates at most \( |C| - 1 \) vertices in \( C \).
Therefore there are at most \( p(k - 1) - s \) edges from \( Y \) to \( C \), \( s \geq 0 \). Thus
\[
p(k - 1) - s = k(p - 1),
\]

hence
\[
k - s = p
\]
and it follows that \( k \geq p \).

Restricting our attention to the graphs \( G \in \mathcal{F}_4 \) with \( \langle A \rangle \) not complete and \( |Y| = |C| \), we are able to obtain a more concise and descriptive characterisation than the one given for the family \( \mathcal{F}_4 \).

**Theorem 28.** Let \( G \) be a graph in \( \mathcal{F}_4 \) with \( \langle A \rangle \) not complete and \( |Y| = |C| \). Then \( G \) is 3\(_t\)-critical if and only if the following conditions hold:

1. \( \{x\}, Y \) is the unique maximal diametrical pair of \( G \) and \( \langle Y \rangle \) is complete.
2. \( \langle C \rangle \) is complete and \( \langle C \cup Y \rangle \) is complete minus a perfect matching between \( C \) and \( Y \).
3. Every vertex \( c \in C \) dominates \( A \cup B \).
4. For every \( ab \in E(\langle A \cup B \rangle) \), there is a vertex \( a_i \in A \) or \( b_j \in B \) not adjacent to \( a \) and \( b \) and if \( a_1, a_2 \) (\( b_1, b_2 \), respectively) are nonadjacent vertices in \( A \) (\( B \), respectively), then there is a vertex \( b \in B \) (\( a \in A \), respectively) such that \( a_1 b \mapsto a_2 \) (\( b_1 a \mapsto b_2 \), respectively). Also for every \( a \in A \) and \( b \in B \) that are not adjacent, \( \{a, b\} \succ G \) or there is a vertex \( w \) such that \( aw \mapsto b \) or \( bw \mapsto a \).

**Proof.** Let \( G \in \mathcal{F}_4 \) with \( \langle A \rangle \) not complete and \( |Y| = |C| \) be 3\(_t\)-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, \( \langle C \rangle \) is complete. By Lemmas 21 and 26, we have that each vertex in \( C \) is adjacent to \( |Y| - 1 \) vertices in \( Y \) and if \( A \) is not complete, then each vertex in \( Y \) is adjacent to at most \( |C| - 1 \) vertices is \( C \). Thus there are \( |C|(|Y| - 1) \) edges from \( C \) to \( Y \) and at most \( |Y|(|C| - 1) \) edges from \( Y \) to \( C \). Since \( |C| = |Y| \), there are exactly \( |Y|(|C| - 1) \) edges from \( Y \) to \( C \) and so every vertex in \( Y \) is adjacent to exactly \( |C| - 1 \) vertices in \( C \). Therefore, all edges minus a perfect matching are present between \( C \) and \( Y \).

To show that (3) holds, suppose that \( ac \notin E(G) \), \( a \in A \) and \( c \in C \). Consider \( \{a, y\} \) where \( y \in Y \) and \( cy \notin E(G) \). Obviously, \( \{a, y\} \notin G \). Since no vertex in \( N[y] \) dominates \( x \), it follows that \( az \mapsto y \) and \( z \in C \). But this contradicts condition (2). Hence \( c \succ A \) for each \( c \in C \).
Now suppose that $cb \notin E(G)$, $c \in C$ and $b \in B$, and consider $\{b, c\}$. Since
neither $b$ nor $c$ is adjacent to $x$, $\{c, b\} \notin G$. Therefore $cw \leftrightarrow b$ or $bw \leftrightarrow c$
for $w \in A$ (to dominate $x$). But if $cw \leftrightarrow b$, then $Y$ is not dominated, a
contradiction. And if $bw \leftrightarrow c$, then $w \notin N(c)$, contradicting the fact that
every vertex $c \in C$ dominates $A$. Hence $c \triangleright B$ for each $c \in C$. Thus,
condition (3) holds. Condition (4) follows from the fact that every $b \in B$
dominates $C \cup Y$ and every $a \in A$ dominates $C$.

Let $G \in \mathcal{F}_4$ with $\langle A \rangle$ not complete and $|Y| = |C|$ and assume that the
conditions of the theorem hold. Since no pair of adjacent vertices dominate
$G$, $\gamma_t(G) \geq 3$. Further, $\{a, b, c\}$, where $a \in A$, $b \in B$ and $c \in C$, is a total
dominating set, so $\gamma_t(G) = 3$. To show that $G$ is $3_t$-critical, we first consider
$\{x, y\}$ for $y \in Y$. Then $cy \leftrightarrow x$ where $c \in N(y)$. A similar argument holds
for $\{x, c\}$. Next consider $\{x, b\}$ for $b \in B$. Then $bc \leftrightarrow x$ for any $c \in C$. For
$\{a, y\}$, $ac \leftrightarrow y$ where $c \in C - N(y)$. It now follows from condition (4) that
$G$ is $3_t$-critical. ■

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