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A NOTE ON A NEW CONDITION IMPLYING PANCYCLISM*

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Abstract

We first show that if a 2-connected graph G of order n is such that for each two vertices u and v such that $\delta = d(u)$ and $d(v) < n/2$ the edge uv belongs to $E(G)$, then G is hamiltonian. Next, by using this result, we prove that a graph G satysfying the above condition is either pancyclic or isomorphic to $K_{n/2,n/2}$.

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1 Notation, Terminology and Introduction

We shall consider only finite, undirected graphs, without loops or multiple edges. For a graph G we denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. If A is a subgraph or a subset of vertices, $|A|$ is the number of vertices in A.

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A graph of order n is said to be *hamiltonian* if it contains a cycle of length n and is said to be *pancyclic* if it contains cycles of all lengths from 3 to n. Let $P = [a, b]$ be a path in a graph $G = (V, E)$ with ends a and b. If x precedes y on P according to the orientation, we denote by $x \overrightarrow{P} y$ the sequence of consecutive vertices on P from x to y and by $y \overleftarrow{P} x$ the sequence of the same vertices but in reverse order. We use also the notation x^+ and x^- for (if it exists) the successor and the predecessor of x on P with respect to the orientation. Analogous notations are used for cycles.
Other notation and terminology can be found in [3].

Various sufficient conditions for a graph to be hamiltonian have been given in term of the vertex degrees. Recall some of them.

Theorem 1 (Ore [8]). *If a graph $G = (V, E)$ on n vertices is such that for any pair of nonadjacent vertices x and y we have*

$$d(x) + d(y) \geq n,$$

then G is hamiltonian.

The following theorem is an immediate consequence of this result.

Theorem 2 (Dirac [5]). *Let $G = (V, E)$ be a graph on n vertices. If $\delta(G) \geq n/2$, then G is hamiltonian.*

Theorem 3 (Zhu [11]). *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ. If for all nonadjacent vertices x and y we have*

$$d(x) + d(y) \geq n/2 + \delta,$$

then G is hamiltonian.

Remark. The last result was proved independently in [12].

The purpose of this note is to study a new (as far as we know) and rather natural condition on vertex degrees. Namely, for a graph $G = (V, E)$ on n vertices with minimum degree δ consider the following condition:

$$(*) \qquad G \text{ is 2-connected and}$$

$$\forall x, y \in V \text{ such that } \delta = d(x),\ d(y) < n/2, \text{ we have } xy \in E.$$

It is easy to see that condition (∗) is weaker than both, Ore's and Zhu's condition mentioned above and is independent of the well known Chvátal's [4] and Fan's [6] conditions (see Figure 1). Considering Condition (∗), we obtain the following result that clearly improves Theorems 1 and 3. The proof is given in next section.

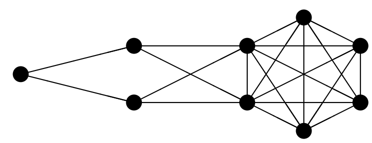

Figure 1

Theorem 4. *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ and such that for every pair $x, y \in V$ such that $\delta = d(x),\ d(y) < n/2$ we have $xy \in E$. Then G is hamiltonian.*

Bondy suggested the interesting "metaconjecture" in [1] that almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs). Various sufficient conditions for a graph to be hamiltonian have been given and many of them (especially in term of the vertex degrees) have been shown to imply pancyclism. For example, Bondy proved the following result concerning Ore's condition.

Theorem 5 [1]**.** *If a graph G is such that the degree sum of any pair of non-adjacent vertices is at least the order of G, then G is pancyclic or isomorphic to the complete bipartite graph.*

The problem of pancyclism of the graphs satisfying Zhu's condition has been considered in [12] where the authors prove that for such graphs, Bondy's "metaconjecture" is satisfied. This result is in fact a corollary of the following theorem dealing with Condition (∗).

Theorem 6. *If a graph G of order n satisfies (∗), then G is either pancyclic or isomorphic to $K_{n/2,n/2}$.*

Corollary 7 [12]**.** *Let $G = (V, E)$ be a 2-connected graph of order n with minimum degree δ. If for all pairs x, y of nonadjacent vertices we have*

$$d(x) + d(y) \geq n/2 + \delta,$$

then G is pancyclic or isomorphic to $K_{n/2,n/2}$.

We shall need the following well-known result related to the notion of n-closure of a graph G.

Theorem 8 (Bondy-Chvátal [2]). *Let G be a graph on n vertices such that the edge $e = xy$ does not belong to $E(G)$ and $d(x) + d(y) \geq n$. Then, the graph G is hamiltonian if and only if the graph $G + e$ is hamiltonian.*

We note that most of the results on pancyclism are proved by starting with a hamiltonian cycle and by considering two consecutive vertices on the hamiltonian cycle. Our proof follows the same approach and we shall use the two following results which became now classic.

Lemma 9 (Bondy [1]). *Let G be a hamiltonian graph of order n with hamiltonian cycle $x_1x_2...x_nx_1$ such that $d(x_1) + d(x_n) \geq n + 1$. Then G is pancyclic.*

Theorem 10 (Schmeichel-Hakimi [9]). *If G is a hamiltonian graph of order $n \geq 3$ with hamiltonian cycle $x_1, x_2, ..., x_n, x_1$ such that $d(x_1) + d(x_n) \geq n$, then G is either*

- *pancyclic,*
- *bipartite, or*
- *missing only an $(n-1)$-cycle.*

Moreover, in the last case we have $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$.

Remark. Actually, the Schmeichel-Hakimi result gives some more information about possible adjacency structure near the vertices x_1 and x_n but the above version is sufficient for our proof.

Mention by the way that some closely related results can be found in [7].

2 Proofs

2.1 Hamiltonicity

Let $G = (V, E)$ be a graph on n vertices, 2-connected with minimum degree δ. We suppose that if x, y are two vertices such that $\delta = d(x) < n/2$ and $d(y) < n/2$, then the vertices x, y are adjacent. We shall prove that G contains a hamiltonian cycle.

Observe that if $\delta \geq n/2$ then G is hamiltonian by Dirac's Theorem. So, we may suppose that $\delta < n/2$ and denote by S the set of all vertices of

degree δ. Put $|S| = s$. It is easy to see that by $(*)$ the vertices of the set S form a clique.
Denote by B the set of vertices of degree greater than or equal to $n/2$. Observe that applying Theorem 8 we may assume that the vertices of the set B form a clique.

Denote by M the set of remaining vertices and put $|M| = m$. So we have $V = S \cup B \cup M$. If $m = 0$ then our graph consists of two cliques joined, by 2-connectivity, by two independent edges, and is evidently hamiltonian. Assume then $m > 0$. By definition of M, the vertices of M are of degree less that $n/2$. So, by $(*)$ they are joined to all vertices of the set S. That implies, in particular, that the minimum degree δ is at least equal to $s - 1 + m$. Let now x be a vertex of M. Since x is not in S we have

(i) $$s + m \leq d(x) < n/2.$$

Let $\mathcal{P} = (P_1, P_2, \ldots, P_p)$ be a cover of the set M by paths with the minimum number of paths. Denote by x_i, y_i the ends of the path P_i, $i = 1, 2, \ldots, p$. Consider first the case when $p \geq 2$. By the minimality of the path cover $\mathcal{P}$ the end x_i of the path P_i is not adjacent to other ends of the paths of $\mathcal{P}$, except, maybe, for the end y_i. In other words, the ends of the paths are not adjacent to at least $2(p-1)$ vertices in M. Therefore, by (i),

(ii) the ends of the paths of $\mathcal{P}$ send at least $2(p-1)+1$ edges to the set B.

Denote now by z_1, z_2 two vertices of the set S. We choose $z_1 \neq z_2$, if $|S| \geq 2$ and $z_1 = z_2$, otherwise. We define $2(p-1)$ vertices of the set B as follows: the vertices $u_3, u_4, \ldots, u_p$ are the neighbours of the vertices $x_3, x_4, \ldots, x_p$ and the vertices $v_1, v_2, \ldots, v_p$ are the neighbours of the vertices $y_1, y_2, \ldots, y_p$, respectively. Note, that by (ii) the choice of the neighbours can be made in such a way that all these $2(p-1)$ vertices are distinct (see Figure 2).

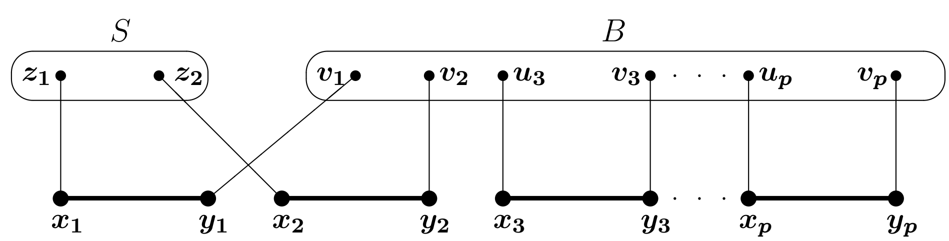

Figure 2

Now it is easy to define a hamiltonian cycle of G in the following way:

$$z_1x_1 \overrightarrow{P_1}\, y_1v_1u_3x_3 \overrightarrow{P_3}\, y_3v_3u_4x_4 \ldots x_p \overrightarrow{P_p}\, y_pv_pBv_2y_2 \overleftarrow{P_2}\, x_2z_2Sz_1$$

where v_pBv_2 (z_2Sz_1) denotes a path joining v_p with v_2 (z_2 with z_1) and passing throw all remaining vertices of the set B (S), respectively.

Suppose now that M is covered by one path P having the ends x and y. By (ii) each of these vertices has at least one neighbour in B. Suppose first that these neighbours are distinct and denote them by u and v, respectively. If, as above, z_1, z_2 are two vertices of the set S we can form the following hamiltonian cycle:

$$z_1xuBvy \overleftarrow{P}\, x^+z_2Sz_1.$$

Finally, consider the case where the vertices x and y have only one neighbour in B, say u. Then all other vertices of P are adjacent to the vertex u which is their unique neighbour in B, for otherwise $xy \in E$ and the path P could be replaced by a path with ends having two distinct neighbours in B. By (i), this implies, in particular, that the vertices of M form a clique. Therefore, our graphs consists of two cliques, B and $S \cup M$ and some edges between them. Since G is 2-connected this set of edges have to contain at least two independent edges. So, in consequence, G have to contain a hamiltonian cycle. This finishes the proof of Theorem 4.

Remark. Skupień [10] noticed that actually one can prove that the n-closure of a graph satisfying $(*)$ is a complete graph.

2.2 Pancyclism

Let $G = (V, E)$ be a graph on n vertices, 2-connected with minimum degree δ such that the following condition holds: if x, y are two vertices such that $\delta = d(x) < n/2$ and $d(y) < n/2$, then the edge $xy \in E$. Suppose, that G is not pancyclic.

By Theorem 4, G contains a hamiltonian cycle. Denote it by C and choose one of its orientations, say $\overrightarrow{C}$. Let u be a vertex of G such that $d(u) = \delta < n/2$.

We **claim** that in this case G cannot be bipartite. Suppose that G is bipartite. Since G is hamiltonian, it has to be a balanced bipartite graph i.e., $G = (L, R, E)$ with $|L| = |R| = n/2$. Without loss of generality we may suppose that $u \in L$. Since $\delta < n/2$, there exists a vertex $b \in R$ nonadjacent to u. But then $d(b) < n/2$ and by $(*)$ the edge ub has to belong to E, a contradiction.

Suppose now that there are two consecutive (with respect to the orientation $\overrightarrow{C}$) vertices x, y such that neither ux nor uy belongs to E. By $(*)$ we have $d(x) \geq n/2$ and $d(y) \geq n/2$.

If $d(x) + d(y) \geq n + 1$ then G in pancyclic by Lemma 9. Thus we have

$$d(x) = d(y) = n/2.$$

Since, as claimed above, G is not bipartite, by applying Theorem 10 we conclude that, in particular, the degrees of the vertices $x^{--}, x^{-}, y^{+}, y^{++}$ are less than $n/2$. Therefore, the condition $(*)$ ensures that the edges $ux^{--}, ux^{-}, uy^{+}, uy^{++}$ belong to E. A simple counting argument shows that the degree of u has to be at least $n/2$. This contradiction finishes the proof of Theorem 6.

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