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STRONGLY MULTIPLICATIVE GRAPHS

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Abstract

A graph with p vertices is said to be strongly multiplicative if its vertices can be labelled $1, 2, \ldots, p$ so that the values on the edges, obtained as the product of the labels of their end vertices, are all distinct. In this paper, we study structural properties of strongly multiplicative graphs. We show that all graphs in some classes, including all trees, are strongly multiplicative, and consider the question of the maximum number of edges in a strongly multiplicative graph of a given order.

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1. Introduction

Graph labellings, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems, but they are also of interest in their own right. An enormous body of literature has grown around the subject, especially in the last thirty years or so, and even to mention the variety of problems that have been studied would take us too far afield here.

Most interesting graph labelling problems have three ingredients:

(i) a set of numbers S from which vertex labels are chosen;
(ii) a rule that assigns a value to each edge;
(iii) a condition that these values must satisfy.

Arguably two of the most interesting labelling problems are *gracefulness* and *harmoniousness*. Graceful labellings were introduced under the guise of β-valuations by Rosa [7], and much of their original interest lay in their connection with decompositions of complete graphs, in particular, into trees. (See Bloom [2] for a discussion of this topic.) In a graceful labelling of a graph with q edges, the labels are chosen as distinct values from $\{0, 1, \ldots, q\}$, each edge is given the absolute value of the labels on its vertices, and the requirement is that all edge labels be different.

Harmonious labellings were introduced by Graham and Sloane [6] and have connections with error-correcting codes. In a harmonious labelling, the vertices have distinct values from $\{1, 2, \ldots, q\}$, an edge is given the sum modulo q of the labels on its vertices, and, again, all edge labels must be different. Gallian [5] has written an extensive survey, updated periodically, in which results on many variations of these two types of labelling are compiled. Before working on labelling problems, readers would be well advised to consult Gallian's work. (Note that it does not consider another important labelling problem, the so-called band-width problem. For a survey of this, see Chung [3].)

In this paper, we consider a labelling that has much the same flavor as graceful and harmonious labellings in its simplicity of definition and its requirement that all of the edge labels be different. However, it uses products rather than sums or differences. The property, which we call "strong multiplicativity" (not a catchy name) is this:

Can the vertices of a graph be labelled $1, 2, \ldots, p$ *in such a way that the resulting products on the edges are all different* ?

After giving formal definitions and some elementary general results in Section 2, we turn in Section 3 to proving that the graphs in certain families are strongly multiplicative, and in Section 4 to the number-theoretic question of the greatest number of edges that a strongly multiplicative graph of order p can have.

2. Definitions and Basic Results

We begin with some definitions and background results. A *labelling* f of a graph G is an assignment of distinct positive integers to its vertices; that is, an injective function $f : V(G) \to \mathbf{N}$ (the set of natural numbers). (We note that in different situations other ranges might be appropriate.) The *product function* $f^{\times}$ assigns to each edge of G the product of the values on its two ends; $f^{\times} : E(G) \to \mathbf{N}$ with

$$f^{\times}(e) = f(v)f(w)$$

if e joins v and w.

The following result on product functions was observed by Acharya and Hegde [1].

Theorem 2.1. *Let $G = (V, E)$ be a graph and let $d(v)$ denote the degree of vertex v. Then for any labelling f,*

$$\prod_{e \in E} f^{\times}(e) = \prod_{v \in V} (f(v))^{d(v)}.$$

A labelling f of a graph G is called a *multiplicative labelling* of G if $f^{\times}$ is injective; that is, if all of the edges receive different product values.

For example, in Figure 1, we show two labellings of the same graph. The first labelling is not multiplicative, but the second is.

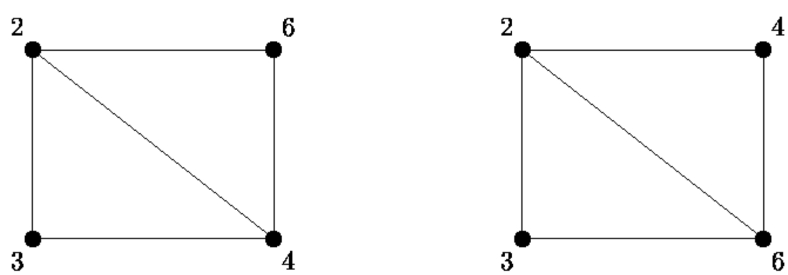

Figure 1

Theorem 2.2. *Every graph has a multiplicative labelling.*

Proof. Given a graph G, label its vertices with distinct primes. This labelling is clearly multiplicative. ■

Theorem 1 suggests an interesting parameter for a graph G of order p: the least integer n for which there is a multiplicative numbering of G using values from $\{1, 2, \dots, n\}$. In this paper, we study only those graphs for which the value of this parameter is p. More formally, let $\mathcal{M}(G)$ be the set of multiplicative numberings of G; and for $f \in \mathcal{M}(G)$, let

$$f_{\max}(G) = \max\{f(v) : v \in V\},$$
$$\theta(G) = \min\{f_{\max}(G) : f \in \mathcal{M}(G)\}.$$

A graph G is called *strongly multiplicative* if $\theta(G) = | G |$, and any multiplicative labelling f for which $f_{\max}(G) = | G |$ is called a *strong multiplier* for G. Figure 2 shows a strong multiplier (provided by Tony Evans of Wright State University) for the Petersen graph.

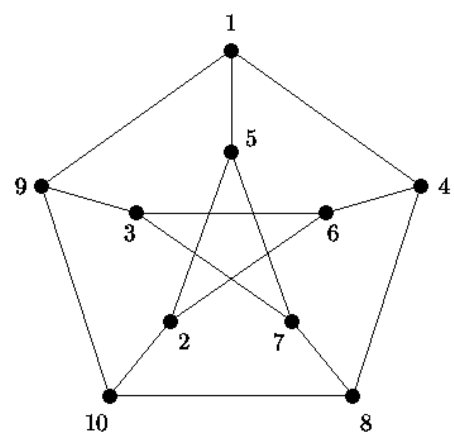

Figure 2

Not all graphs are strongly multiplicative however; for example, if the vertices of the complete graph K_6 are labelled $1, 2, \ldots, 6$, then one pair of edges have an induced label of 6, and another pair have label 12.

In our next result, we make some observations about two types of subgraphs of graphs and the strongly multiplicative property.

Theorem 2.3.

(a) *Every spanning subgraph of a strongly multiplicative graph is strongly multiplicative.*

(b) *Every graph is an induced subgraph of a strongly multiplicative graph.*

Proof. (a) This follows at once from the definition of strongly multiplicative.

(b) Let G be a graph of order p. By Theorem 2.2, G has a multiplicative labelling f. If $f_{\max}(G) = n$, add $n - p$ isolated vertices to G and label them with the labels less than n not used in G. The result is clearly a strongly multiplicative labelling of a graph having G as an induced subgraph. ∎

3. Families of Strongly Multiplicative Graphs

In this section we consider the question of whether the graphs in certain well-known and much-studied families are strongly multiplicative. We show that while only a few complete graphs and regular complete bipartite graphs are strongly multiplicative, all cycles, grids, wheels, and trees are.

Theorem 3.1. *The complete graph K_p is strongly multiplicative if and only if $p \leq 5$.*

Proof. It is straightforward to check that all ten products of two different numbers no greater than 5 are distinct; in other words, K_5 is strongly multiplicative. On the other hand, since $2 \cdot 3 = 1 \cdot 6$, no larger complete graph has a multiplicative labelling. ∎

Theorem 3.2. *The complete bipartite graph $K_{r,r}$ is strongly multiplicative if and only if $r \leq 4$.*

Proof. Figure 3 shows a strongly multiplicative labelling of $K_{4,4}$, and it contains as a subgraph such a labelling of $K_{3,3}$. It therefore follows that $K_{r,r}$ is strongly multiplicative for $r \leq 4$.

Suppose that for some $r \geq 5$, $K_{r,r}$ has a strongly multiplicative labelling. Let A and B be the sets of labels on the two partite sets. Note that A and B form a partition of the set $\{1, 2 \ldots, 2r\}$ into two sets of order r. Clearly the following two statements must hold:

(a) If $x \neq y$ and $\{x, y\} \subset A$, then $\{2x, 2y\} \not\subset B$, and vice versa.

(b) If $x \neq y$ and $\{x, 2x\} \subset A$, then $\{y, 2y\} \not\subset B$, and vice versa.

Note that it is impossible for each of three pairs $\{x, 2x\}$, $\{y, 2y\}$, and $\{z, 2z\}$ to be split between A and B. For suppose they are. Then, by (a) and (b), we may assume that $\{x, 2y\} \subset A$ and $\{y, 2x\} \subset B$. But then it is not possible for z and $2z$ to be split.

Let $C = \{1, 2, \ldots, r\} \cup \{2, 4, \ldots, 2r\}$, and consider the r pairs $\{x, 2x\}$, for $1 \leq x \leq r$. From the above observations, it follows that at least $r - 2$

of these pairs must lie in one of the sets A or B and the other pairs may be split. Hence one of the sets, say A, contains at most two elements of C. However, for $r \geq 5$, C contains at least $r+3$ elements, so $\mid B \mid \geq r+1$, which is impossible. ■

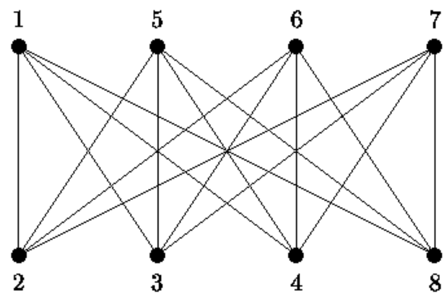

Figure 3

The (square lattice) grid graph $L_{r,s}$ is the product $P_r \times P_s$ of paths of length $r-1$ and $s-1$. A strongly multiplicative labelling of the general grid graph is shown in Figure 4, which proves the following result.

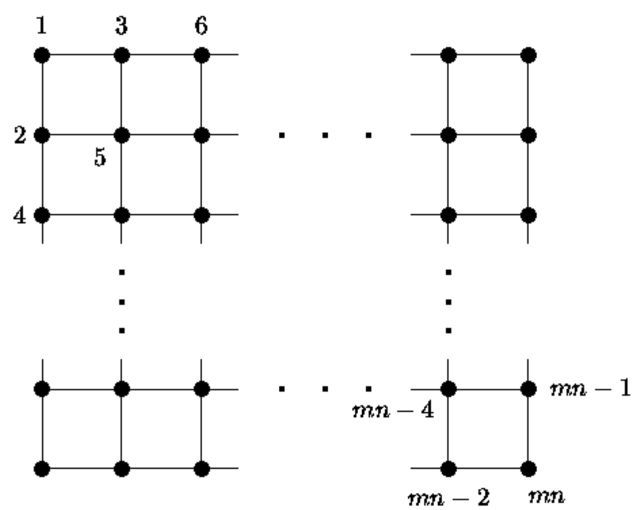

Figure 4

Theorem 3.3. *Every grid graph is strongly multiplicative.* ■

Theorem 3.4. *For all $p \geq 3$, the cycle C_p is strongly multiplicative.*

Proof. The result is easily shown for $3 \leq p \leq 6$. Let $p \geq 7$ and let $C = v_1v_2 \cdots v_pv_1$. First assume that there is no value of k for which $p = k(k+1)$. Then the labelling $f(v_i) = i$ results in the products on the edges being $1 \cdot 2$, $2 \cdot 3, \cdots, (p-1)p, p \cdot 1$, which are all clearly different. Now suppose that $p = k(k+1)$ and consider the labelling with $f(v_{k-1}) = k$, $f(v_k) = k-1$, and $f(v_i) = i$ otherwise. It is again straightforward to verify that the resulting labels on the edges are distinct. Hence, every cycle is strongly multiplicative. ∎

We next consider wheels. In Figure 5 we give a labelling showing that W_{12} is strongly multiplicative.

This graph is exceptional in that it does not readily fit into the general pattern in our proof. There we label the central vertex of W_{n+1} with the highest power $t = 2^k$ not exceeding $n+1$.

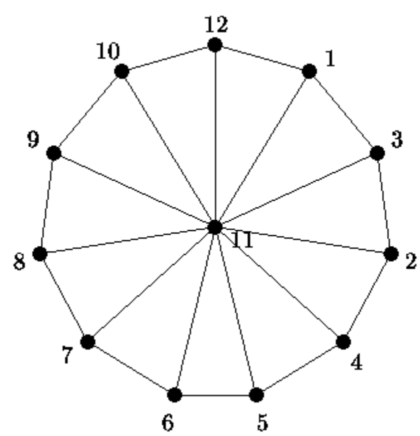

Figure 5

Theorem 3.5. *Every wheel is strongly multiplicative.*

Proof. We let $n \geq 5$ since the result is easily shown for smaller values. Consider the wheel W_{n+1} whose rim is the cycle $v_1v_2 \cdots v_nv_1$ and whose hub is the vertex w. Let $t = 2^k$ be the greatest power of 2 not exceeding $n+1$. Note that then $t > (n+1)/2$. We consider two cases.

Case 1. $n + 1 = 2^k$. In this case, label the central vertex with t and each vertex v_i with i. If this labelling is not multiplicative, then either (a) two rim edges are labelled the same, or (b) a spoke has the same label as a rim edge. If (a) occurs, it must be that $n \cdot 1 = r(r+1)$ for some r. This is impossible since n is odd and $r(r+1)$ even. Hence, we suppose that (b) holds, so for some positive integers r and s, $2^k \cdot s = r(r+1)$. Since one of the factors on the right side is odd, the other must be a multiple of 2^k. Since $r < n < 2^k$, this is impossible.

Case 2. $2^k \leq n$. First label the central vertex w with t. If there is no integer r for which $r(r+1) = n+1$, label v_i with i for $i < t$ and with $i+1$ for $i \geq t$. If there is such an integer r, use the same labelling except for v_{r-1} and v_r, whose labels are switched. In either case, all edges on the rim have different labels. Suppose that some rim edge has the same label as a spoke. Then for some s, we have $s2^k$ equalling $q(q+1)$ for some q, or $st = r(r-2), (r-1)(r+1)$, or $(t-1)(t+1)$. Clearly, the last of these is impossible. Suppose $s2^k = q(q+1)$. As before, since either q or $q+1$ of these is odd, the other, call it x, must be divisible by 2^k, say $x = 2^k d$. Since $x = t$, $d > 1$, but since $x \leq n+1 < 2^{k+1}$, $d < 2$; a contradiction.

Having $r(r+1) = n+1$, suppose now that one of the products $r(r-2)$ or $(r-1)(r+1)$ equals st for some s. Since these products are both less than $r(r+1)$ and since $n+1 < 2t$, it follows that $s = 1$. Consequently, $r(r-2) = 2^k$ or $(r+1)(r-1) = 2^k$. This implies that $k = 3$ and $r = 3$ or 4. But if $r = 4$, then $n = 19$, and this forces k to be 4, which is impossible. Hence $r = 3$. But then $n = 11$, and since W_{12} was already shown to be strongly multiplicative, the result follows. ■

Figure 6 shows a tree with a strongly multiplicative labelling. Note that the labelling corresponds to a breadth-first search. Such labellings can be used to show not only that every tree is strongly multiplicative, but also that other properties can be satisfied. The following result gives one possibility.

Theorem 3.6. *Every tree has a strongly multiplicative labelling in which an arbitrary vertex is labelled* 1.

Proof. Let T be a tree and let v be any vertex of T. Embed T in the plane with v as root, and label the vertices in succession using a breadth-first search. To see that this labelling is strongly multiplicative, let d and e be two edges and assume that the ends of d are labelled i and j with $i < j$ and those of e are k and l with $k < l$. Without loss of generality, assume also that $j < l$. From the breadth-first property of the labelling, it follows that $i \leq k$, so $ij < kl$. Hence all of the edge labels are distinct. ■

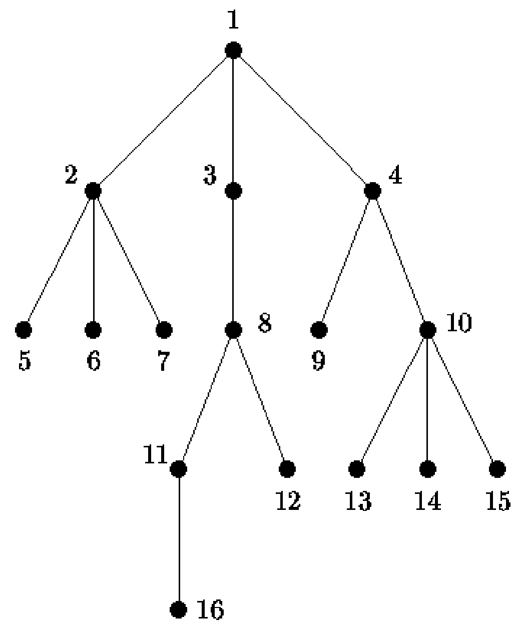

Figure 6

4. Numbers of Edges

In this section, we consider the question of how many edges there can be in a strongly multiplicative graph of order p. We denote this maximum number by $\lambda(p)$. One way to find $\lambda(p)$ is to label the vertices of the complete graph K_p with the integers $1, 2, \ldots, p$ and then successively delete edges whose product label is duplicated on another edge.

It follows from this observation that all maximal strongly multiplicative graphs on p vertices have the same number $\lambda(p)$ of edges. However, for $p \geq 6$, not all of these graphs are isomorphic. For example, both of the graphs obtainable from K_6 by deleting two edges are strongly multiplicative. If we remove from a labelled K_6 one edge with product label 6 and another with label 12, the result is strongly multiplicative (and maximal). It is possible for these edges to be adjacent ($6 \cdot 1$ and $6 \cdot 2$) or independent ($6 \cdot 1$ and $4 \cdot 3$), so both graphs are possible. We do not pursue further the question of how many graphs of a given order are maximal strongly multiplicative, but return to the question of the number of edges, $\lambda(p)$.

At its heart, this problem is purely number-theoretical: For a given integer p, determine how many different numbers are the product of two positive integers, each at most p. In other words,

$$\lambda(p) = | \{rs : 1 \leq r < s \leq p\} | .$$

Erdös [4] determined the asymptotic behavior of this parameter.

Theorem 4.1. *Let* $c = 1 - (1 + \ln\ln 2)/\ln 2$. *Then*

$$\lambda(p) \sim \frac{p^2}{(\ln p)^{c+o(1)}}.$$

■

Beyond this, we are interested in specific values of $\lambda(p)$ as well as in bounds. Table 1 shows all of the distinct products $r \cdot s$ for $1 \leq r < s \leq 10$. What remains unstricken in the nth row is the set of new products having n as one factor.

Table 1

p	Products									$\lambda(p)$
2	$2\cdot1$									1
3	$3\cdot1$	$3\cdot2$								3
4	$4\cdot1$	$4\cdot2$	$4\cdot3$							6
5	$5\cdot1$	$5\cdot2$	$5\cdot3$	$5\cdot4$						10
6	~~$6\cdot1$~~	~~$6\cdot2$~~	$6\cdot3$	$6\cdot4$	$6\cdot5$					13
7	$7\cdot1$	$7\cdot2$	$7\cdot3$	$7\cdot4$	$7\cdot5$	$7\cdot6$				19
8	~~$8\cdot1$~~	$8\cdot2$	~~$8\cdot3$~~	$8\cdot4$	$8\cdot5$	$8\cdot6$	$8\cdot7$			24
9	$9\cdot1$	~~$9\cdot2$~~	$9\cdot3$	$9\cdot4$	$9\cdot5$	$9\cdot6$	$9\cdot7$	$9\cdot8$		31
10	~~$10\cdot1$~~	~~$10\cdot2$~~	~~$10\cdot3$~~	~~$10\cdot4$~~	$10\cdot5$	$10\cdot6$	$10\cdot7$	$10\cdot8$	$10\cdot9$	36

In Table 2 we extend these values of $\lambda(p)$ to all $p \leq 20$.

Table 2

p	1	2	3	4	5	6	7	8	9	10
$\lambda(p)$	0	1	3	6	10	13	19	24	31	36

p	11	12	13	14	15	16	17	18	19	20
$\lambda(p)$	46	51	63	70	78	87	103	111	129	138

Getting an exact formula for $\lambda(n)$ seems unlikely, so we look first for an upper bound. To this end, let $\delta(p) = \lambda(p) - \lambda(p-1)$, the number of new products one can get by going from $p-1$ to p as the largest factor. Clearly, if p is prime, then $\delta(p) = p-1$, and conversely. On the other hand, if $p \equiv 2 \pmod 4$, say $p = 4k+2$, then all of the products $n \cdot i$ for $i = 1, 2, \ldots, 2k$ are repetitions (of $(2k+1)(2i)$), so $\delta(4k+2) \le (4k+1) - 2k = 2k+1$. Similarly, if $p \equiv 0 \pmod 4$, say $p = 4k$, then except possibly for $i = k$, there are repetitions for $i = 1, 2, \ldots, 2k-1$, so $\delta(4k) \le (4k-1) - (2k-2) = 2k+1$. We put these facts into a lemma.

Lemma. *The number of new products $\delta(p)$ with p as a factor satisfies*

$$\begin{aligned} \delta(4k) &\le 2k+1, \\ \delta(4k+1) &\le 4k, \\ \delta(4k+2) &\le 2k+1, \\ \delta(4k+3) &\le 4k+2. \end{aligned}$$

■

From this lemma, we obtain the bounds in the following theorem.

Theorem 4.2. *The maximum number $\lambda(n)$ of edges in a strongly multiplicative graph of order n satisfies these inequalities:*

$$\begin{aligned} \lambda(4r) &\le 6r^2, \\ \lambda(4r+1) &\le 6r^2 + 4r, \\ \lambda(4r+2) &\le 6r^2 + 6r + 1, \\ \lambda(r4+3) &\le 6r^2 + 10r + 3. \end{aligned}$$

Proof. Each of the four cases is proved by a separate induction on r. We first note that

$$\lambda(p) = \lambda(p-4) + \delta(p-3) + \delta(p-2) + \delta(p-1) + \delta(p).$$

It follows from the lemma that

$$\begin{aligned} \lambda(4r) &\le \lambda(4r-4) + 12r - 6, \\ \lambda(4r+1) &\le \lambda(4r-3) + 12r - 2, \\ \lambda(4r+2) &\le \lambda(4r-2) + 12r, \\ \lambda(4r+3) &\le \lambda(4r-1) + 12r + 4. \end{aligned}$$

The inequalities of the theorem now follow by induction. ■

In the following corollary, we restate Theorem 4.2 in a slightly different form.

Corollary 4.2.1. *For all* p,

$$\lambda(p) \leq \begin{cases} 3p^2/8 & \text{if } p \equiv 0 \pmod 4, \\ (3p^2+2p-5)/8 & \text{if } p \equiv 1 \pmod 4, \\ (3p^2-4)/8 & \text{if } p \equiv 2 \pmod 4, \\ (3p^2+2p-9)/8 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

■

In obtaining our bounds, we have only taken into account duplications of products using even factors, and not all of these. Therefore, our bounds can be improved, at least a bit, by taking additional products into account. One can also start the induction at higher values of p. We give only one such improvement; it is chosen for its elegance as well as its generality.

Corollary 4.2.2. *For* $p \geq 12$, $\lambda(p) \leq \frac{3p^2}{8}$.

Proof. Begin with a value of p and add the next four numbers, $p+1$, $p+2$, $p+3$, and $p+4$. There are potentially $4p+6$ new products. We will show that at least p of these are duplications. As we observed earlier (see the lemma), the two even numbers combined will contribute at least $p-1$ duplications. In addition, at least one of the four numbers is divisible by 3, say it equals $3s$, $s \geq 5$. Then $3s \cdot 1$ is a duplicate of $3 \cdot s$, so there are at least p duplications, as claimed. Hence there are at most $3p+6$ new products. Now we use induction on p. The inequality holds for $p = 12, 13, 14$, and 15. Assume that it holds for $p = n$. Then

$$\lambda(n+4) \leq \frac{3n^2}{8} + 3n + 6 = \frac{3(n+4)^2}{8},$$

and the result follows. ■

We note that the only values for which the bound in Corollary 4.2.2 does not hold are $p = 5, 7, 9$, and 11, and in each case it is too small by 5/8. We also observe that if $p \equiv 2 \pmod 4$, the bound in Corollary 4.2.1 is slightly better.

The problem of a (nontrivial) general lower bound for $\lambda(p)$ remains open, and significant further results on $\delta(p)$ would therefore be interesting.

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References

[1] B.D. Acharya and S.M. Hegde, *On certain vertex valuations of a graph*, Indian J. Pure Appl. Math. **22** (1991) 553–560.

[2] G.S. Bloom, *A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful*, Ann. N.Y. Acad. Sci. **326** (1979) 32–51.

[3] F.R.K. Chung, *Labelings of graphs*, Selected Topics in Graph Theory 3 (Academic Press, 1988) 151–168.

[4] P. Erdös, *An asymptotic inequality in the theory of numbers*, Vestnik Leningrad. Univ. **15** (1960) 41–49.

[5] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Comb. **5** (1998) #DS6.

[6] R.L. Graham and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Algebraic Discrete Methods **1** (1980) 382–404.

[7] A. Rosa, *On certain valuations of the vertices of a graph*, in: Theory of Graphs, Internat. Symposium, Rome, July 1966 (Gordon and Breach, Dunod, 1967) 349–355.

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