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NOTE ON THE WEIGHT OF PATHS IN PLANE TRIANGULATIONS OF MINIMUM DEGREE 4 AND 5

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Abstract

The weight of a path in a graph is defined to be the sum of degrees of its vertices in entire graph. It is proved that each plane triangulation of minimum degree 5 contains a path P_5 on 5 vertices of weight at most 29, the bound being precise, and each plane triangulation of minimum degree 4 contains a path P_4 on 4 vertices of weight at most 31.

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Throughout this paper we consider connected graphs without loops or multiple edges. Let P_r (C_r) denote a path (cycle) on r vertices (an r-path and r-cycle, in the sequel). A vertex of degree m is called an m-vertex, a vertex of degree at least (at most) m is called a $+m$-vertex ($-m$-vertex).

The *weight* of the subgraph H in the graph G is defined to be the sum of the degrees of the vertices of H in G, $w(H) = \sum_{v \in V(H)} \deg_G(v)$. For a family $\mathcal{G}$ of graphs having a subgraph isomorphic to H, define the number $w(H, \mathcal{G}) = \max_{G \in \mathcal{G}} \min_{H \subseteq G} w(H)$.

The exact value of $w(H, \mathcal{G})$ is known only for a few graphs and families of graphs. For $\mathcal{G}(3)$ the family of all 3-connected plane graphs, Ando, Iwasaki and Kaneko [1] proved that $w(P_3, \mathcal{G}(3)) = 21$. From the result of Fabrici and Jendrol' [5] it follows that $w(P_k, \mathcal{G}(3)) \leq 5k^2$ for $k \geq 1$; also, they gave a lower bound for this number as a function of order $O(k \log(k))$, see [6]. Recently, the upper bound $5k^2$ was improved to $\frac{5}{2}k(k+1)$ for $k \geq 4$, see [11]. For $PHam$ the class of all hamiltonian plane graphs, Mohar [12] proved the exact value $w(P_k, PHam) = 6k-1$. For $\mathcal{G}(5)$ and $\mathcal{T}(5)$ the families of all connected plane graphs/triangulations of minimum degree 5 and subgraphs other than a path, the known exact values are $w(C_3, \mathcal{G}(5)) = 17$ ([2]), $w(K_{1,3}, \mathcal{G}(5)) = 23$ ([9]), $w(K_{1,4}, \mathcal{G}(5)) = 30$, $w(C_4, \mathcal{T}(5)) = 25$, $w(C_5, \mathcal{T}(5)) = 30$ ([4]).

In the following we deal with the weight of paths P_k in the graphs of the families $\mathcal{T}(4)$ and $\mathcal{T}(5)$ (plane triangulations of minimum degree 4 and 5). It is known that $w(P_2, \mathcal{G}(5)) = 11$ ([13]), $w(P_3, \mathcal{G}(5)) = 17$ ([8]), $w(P_4, \mathcal{G}(5)) = 23$ ([9]), $w(P_3, \mathcal{G}(4)) = 17$ ([1, 3]), $w(P_4, \mathcal{T}(4)) \leq 4 \cdot 15 = 60$ ([7]). The aim of this paper is to improve the best known upper bound for $w(P_k, \mathcal{T}(4)), w(P_k, \mathcal{T}(5))$ for small values of k, showing the following

Theorem 1. *$w(P_5, \mathcal{T}(5)) = 29$.*

Theorem 2. *$27 \leq w(P_4, \mathcal{T}(4)) \leq 31$.*

***Proof of Theorem* 1.** To prove first the inequality $w(P_5, \mathcal{T}(5)) \leq 29$ suppose that there exists a graph $G \in \mathcal{T}(5)$ in which every path P_5 has a weight $w(P_5) > 29$. We will use the Discharging method. According to the consequence of the Euler formula,

$$\sum_{x \in V(G)} (\deg_G(x) - 6) = -12$$

assign to each vertex $x \in V(G)$ the initial charge $\varphi(x) = \deg_G(x) - 6$. Thus $\sum_{x \in V(G)} \varphi(x) = -12$.

Now, we define a local redistribution of charges in a way such that the sum of the charges after redistribution remains the same. This redistribution is performed by the following

Rule. Each k-vertex x, $k \geq 6$, sends the charge $\frac{k-6}{m(x)}$ to each adjacent 5-vertex, where $m(x)$ is the number of 5-vertices adjacent to x. If $m(x) = 0$, no charge is transferred.

Proposition. *Each +8-vertex sends at least $\frac{1}{2}$ to each adjacent 5-vertex; each 7-vertex sends at least $\frac{1}{4}$ to each adjacent 5-vertex.*

Proof. Consider a 7-vertex x. Then x is adjacent to at most four 5-vertices (otherwise two pairs of adjacent 5-vertices are found in the neighbourhood of x, hence there exists a path P_5 of weight 27, a contradiction). From the similar reason, a 8-vertex (9-vertex) is adjacent to at most four (five) 5-vertices. Since none five consecutive vertices in the neighbourhood of a k-vertex, $k \geq 6$, can be 5-vertices, every 10-vertex and every 11-vertex is adjacent to at most eight 5-vertices. Then computing $\frac{k-6}{m(x)}$ yields the desired values of charge. A +12-vertex always sends at least $\frac{1}{2}$. ■

We will show that, after redistribution of charges, the new charges $\widetilde{\varphi}(x)$ are non-negative for all $x \in V(G)$. This will contradict the fact that $\sum_{x \in V(G)} \widetilde{\varphi}(x) = \sum_{x \in V(G)} \varphi(x) = -12$. To this end, several cases have to be considered.

Case 1. x is a 5-vertex. Then x is adjacent to at least two +7-vertices (otherwise, it is adjacent to at least four −6-vertices and there exists a path P_5 with $w(P_5) \leq 5 + 4 \cdot 6 = 29$, a contradiction); denote them u, v. If u, v are both +8-vertices, then $\widetilde{\varphi}(x) \geq -1 + 2 \cdot \frac{1}{2} = 0$ by Proposition. Otherwise consider the following possibilities:

Case 1a. u is a +8-vertex, v is a 7-vertex, all other neighbours are 6-vertices. Observe that x is the only 5-neighbour of v (otherwise, a 5-path of weight at most $2 \cdot 5 + 2 \cdot 6 + 7 = 29$ is found). Thus $\widetilde{\varphi}(x) \geq -1 + 1 + \frac{1}{2} > 0$.

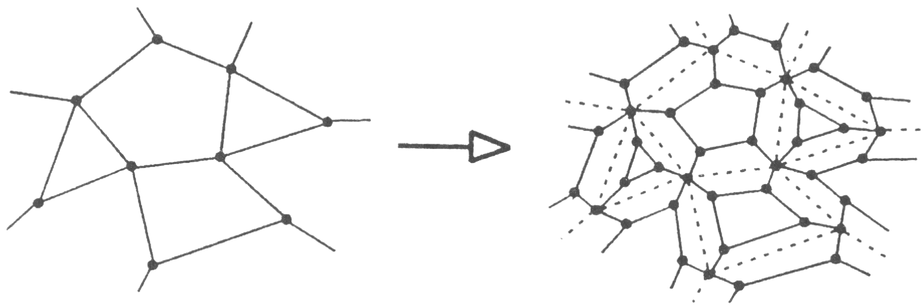
Case 1b. u, v are 7-vertices, all other neighbours are 6-vertices. As above, x is the only 5-neighbour of u, v, thus $\widetilde{\varphi}(x) \geq -1 + 2 \cdot 1 > 0$.

Case 1c. Three of the neighbours of x are 7-vertices, the other ones are −6-vertices. Observe that, for at least one 7-vertex, x is its only 5-neighbour; thus $\widetilde{\varphi}(x) \geq -1 + 1 + 2 \cdot \frac{1}{4} > 0$.

Case 1d. At least four of the neighbours of x are 7-vertices. Then $\widetilde{\varphi}(x) \geq -1 + 4 \cdot \frac{1}{4} = 0$.

Case 2. x is a k-vertex, $k \geq 6$. If x is adjacent to a 5-vertex, then $\widetilde{\varphi}(x) = k - 6 - m(x) \cdot \frac{k-6}{m(x)} = 0$; otherwise $\widetilde{\varphi}(x) = \varphi(x) = k - 6 \geq 0$.

To prove that the upper bound is best possible consider the so called *edge-hexagon substitution* by which a given plane map G is transformed into the following plane map G': Let every $x \in V(G)$ be also a vertex of G'. Assign to every incident pair (x, α) of a vertex x and a face α of G a new vertex of G'. Connect two vertices $x'_1, x'_2 \in V(G')$ by an edge iff either x'_1, x'_2 are assigned to $(x_1, \alpha_1), (x_2, \alpha_2)$ with $(x_1, x_2) \in E(G)$ and with $\alpha_1 = \alpha_2$, or if x'_1 is assigned to a pair (x_1, α_1) where $x'_2 = x_1$, see Figure (cf. [10]):



Consider a graph of the Archimedean polytope $(6, 6, 5)$ and on each its edge apply the edge-hexagon substitution. Into each face of the obtained graph insert a new vertex and join it with new edges to the vertices of the face boundary. In the resulting graph, every 5-path is of the weight of at least 29. ■

***Proof of Theorem* 2.** To prove the upper bound suppose that there exists a counterexample G in which every 4-path has a weight of at least 32.

The following propositions are easy to prove:

Proposition 1. *Each k-vertex with $7 \leq k \leq 16$ is adjacent to at most $\lfloor \frac{k}{2} \rfloor$ -5-vertices.*

Proposition 2. *Each k-vertex, $k \geq 17$, is adjacent to at most $\lfloor \frac{3k}{4} \rfloor$ -5-vertices.*

We use again the Discharging method. As before, the initial assignment of charges is $\mu(x) = \deg_G(x) - 6$ for each vertex $x \in V(G)$. The local redistribution of charges is based on the following rules:

Rule 1. Each k-vertex x, $k \geq 6$, sends the charge $\frac{k-6}{m(x)}$ to each adjacent -5-vertex; $m(x)$ is the number of -5-vertices adjacent to x. If $m(x) = 0$, no charge is transferred.

The following table shows the minimal charge sent by a k-vertex x, $k \geq 7$, to an adjacent -5-vertex, according to Rule 1 (the corresponding values $m(x)$ are computed due to Propositions 1 and 2):

k	7	8	9	10	11	12	13	14	15	16	17	18	19	20	≥ 21
min.charge	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{5}$	1	1	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{7}$	$\frac{5}{4}$	$\frac{11}{12}$	$\frac{12}{13}$	$\frac{13}{14}$	$\frac{14}{15}$	≥ 1

As seen from the table, the only cases when the minimal charge is less than 1 are those with $k \in \{7, 8, 9, 10, 17, 18, 19, 20\}$.

Let $\overline{\mu}$ denote the charge of a vertex after application of Rule 1. A vertex y is said to be *overcharged* if $\overline{\mu}(y) > 0$, and *undercharged* if $\overline{\mu}(y) < 0$.

Rule 2. Each overcharged -5-vertex x sends the charge $\frac{\overline{\mu}(x)}{\overline{m}(x)}$ to each adjacent undercharged 4-vertex; $\overline{m}(x)$ is the number of undercharged 4-vertices adjacent to x. If $\overline{m}(x) = 0$, no charge is transferred.

Let $\widetilde{\mu}$ be the charge of vertices after application of Rule 2. Note that $\overline{\mu}(y) \geq 0$ implies that $\widetilde{\mu}(y) \geq 0$. We will show that after redistribution of charges we have $\widetilde{\mu}(x) \geq 0$ for each vertex $x \in G$, a contradiction. To this end, several cases have to be considered.

Case 1. Let x be a k-vertex, $k \geq 6$. Then either all its charge is sent to adjacent -5-vertices ($\overline{\mu}(x) = 0$) or there is no transfer from x and $\overline{\mu}(x) = k - 6 \geq 0$.

Case 2. Let x be a 5-vertex. Then x is adjacent to at least three $+9$-vertices (otherwise it is adjacent to at least three -8-vertices and we can find a 4-path of weight of at most $8 \cdot 3 + 5 = 29 < 31$); hence $\overline{\mu}(x) \geq -1 + 3 \cdot \frac{3}{4} = \frac{5}{4} > 0$ (thus every 5-vertex is overcharged).

Case 3. Let x be a 4-vertex. Then x is adjacent to at least two $+10$-vertices (otherwise it is adjacent to at least three -9-vertices and we can find a 4-path of weight of at most $9 \cdot 3 + 4 = 31$). If x is adjacent to at least three $+10$-vertices then $\overline{\mu}(x) \geq -2 + 3 \cdot \frac{4}{5} = \frac{2}{5} > 0$; so, suppose that x is adjacent to exactly two $+10$-vertices u, v. If both u, v are $+21$-vertices, or one of them is $+21$-vertex and the degree of another one is between 11 and 16, or both their degrees are between 11 and 16, then u and v send 1 to x (see Table) and $\overline{\mu}(x) \geq -2 + 2 \cdot 1 = 0$. Hence (without loss of generality) it is enough to consider the following possibilities for degrees of u, v (denote y, z the remaining neighbours of x):

Case 3.1. Both u, v are 10-vertices. Then both y, z are +8-vertices (otherwise a 4-path of weight of at most $4 + 2 \cdot 10 + 7 = 31$ is found) and $\overline{\mu}(x) \geq -2 + 2 \cdot \frac{4}{5} + 2 \cdot \frac{1}{2} > 0$.

Case 3.2. u is 10-vertex, v is +11-vertex. Then the sum of degrees of y, z is at least 18 (otherwise x, y, u, z form a 4-path of weight of at most $10 + 4 + 17 = 31$); hence, one of them has to be a +9-vertex. Thus $\overline{\mu}(x) \geq -2 + \frac{4}{5} + \frac{11}{12} + \frac{3}{4} > 0$.

Case 3.3. The degrees of u, v are between 17 and 20. If some of y, z is a +7-vertex, then a simple calculation yields $\overline{\mu}(x) \geq -2 + 2 \cdot \frac{11}{12} + \frac{1}{3} > 0$; if some of them is a 5-vertex, the application of Rule 2 yields $\widetilde{\mu}(x) \geq -2+2 \cdot \frac{11}{12} + \frac{\frac{5}{4}}{2} > 0$. Now, suppose that y, z are 6- or 4-vertices; then we have to treat several cases:

Case 3.3a. y, z are 4-vertices forming a triangular face with x. Then u, v are 20-vertices. Consider the neighbourhood of the vertices u, v, y, z; then the vertices u, v have at least six +6-neighbours. Thus $\overline{\mu}(x) \geq -2 + 2 \cdot \frac{20-6}{20-6} = 0$.

Case 3.3b. y, z are 4-vertices not forming a triangular face with x. Then all their neighbours, except x, are +20-vertices and we have $\overline{\mu}(y) \geq -2 + 3 \cdot \frac{14}{15} = \frac{12}{15}$, $\overline{\mu}(z) \geq -2 + 3 \cdot \frac{14}{15} = \frac{12}{15}$. Hence y, z are overcharged and using Rule 2 we have $\widetilde{\mu}(x) \geq -2 + 2 \cdot \frac{14}{15} + 2 \cdot \frac{\frac{12}{15}}{1} > 0$.

Case 3.3c. y, z are 6-vertices. Considering that each of their neighbours except x has to be a +16-vertex, it is easy to see that u, v have at least six +6-neighbours, thus $\overline{\mu}(x) \geq -2 + 2 \cdot \frac{17-6}{17-6} = 0$.

Case 3.3d. y is a 4-vertex, z is a 6-vertex and they do not form a triangular face with x. Then each neighbour of z, except for x, is a +18-vertex, i.e., u, v are +18-vertices and, moreover, they have at least six +6-neighbours. Hence $\overline{\mu}(x) \geq -2 + 2 \cdot \frac{18-6}{18-6} = 0$.

Case 3.3e. y is a 4-vertex, z is a 6-vertex and they form a triangular face with x. Then u, v are +18-vertices. Let u be adjacent to y and v to z. Since every neighbour of z, except x and y, has to be a +18-vertex, v has at least six +6-neighbours and it sends at least 1 to x. If u is a 20-vertex, then it has also at least six +6-neighbours, thus $\overline{\mu}(x) \geq -2 + 2 \cdot 1 = 0$. So suppose that u is 18- or 19-vertex not having at least six +6-neighbours.

If u is a 19-vertex, then in consequence of Proposition 2 it has exactly five $+6$-neighbours and sends $\frac{19-6}{19-5} = \frac{13}{14}$ to x. Denote $v_1', v_2' \ldots v_l'$ the neighbours of v in the cyclical ordering such that $v_1' = z, v_2' = x, v_3' = u$. Due to the neighbourhood of u, v_4' has to be a 5-vertex and v_5' has to be $+17$-vertex. From this fact we obtain that v has at least seven $+6$-neighbours, so it sends at least $\frac{18-6}{18-7} = \frac{12}{11}$ to x. Hence $\overline{\mu}(x) \geq -2 + \frac{13}{14} + \frac{12}{11} = \frac{3}{154} > 0$.

If u is a 18-vertex, then every its neighbour, except x and y, has to be a $+6$-vertex (otherwise a 4-path of weight of at most $2 \cdot 4 + 18 + 5 = 31$ can be found), so u even sends at least 2 to x and clearly $\overline{\mu}(x) > 0$.

Case 3.4. The degree of u is between 17 and 20, the degree of v is either between 11 and 16, or is at least 21. According to the similarity to case 3.3 (note that v always sends at least 1 to x) it is enough to consider the cases when y or z are neither $+7$-vertices nor -5-vertices, that means, $(\deg_G(y), \deg_G(z)) \in \{(4,4),(4,6),(6,4),(6,6)\}$. In these cases, it is routine check to prove that u has at least 6 $+6$-neighbours, or we obtain a similar situation as in 3.3e, so $\overline{\mu}(x) \geq 0$.

Consider the graph of an icosahedron; into each its triangular face $[XYZ]$ insert a new triangle $[ABC]$ and add new edges $\{A,X\},\{A,Y\},\{B,Y\}$, $\{B,Z\},\{C,Z\},\{C,X\}$. In the resulting graph, every 4-path is of weight of at least $15 + 3 \cdot 4 = 27$. ■

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