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LONG INDUCED PATHS IN 3-CONNECTED PLANAR GRAPHS

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Abstract

It is shown that every 3-connected planar graph with a large number of vertices has a long induced path.

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Let G be an undirected graph without loops and multiple edges. Denote by $p(G)$ the number of vertices in the longest induced path of G. Finding long induced paths in graphs is an interesting but difficult problem. However, it is easy to revise all the references devoted to related problems (see [1-7]).

Denote $p_n = \min\{p(G)\}$ where the minimum is taken over all triconnected planar graphs of order n. The purpose of this note is to prove the following.

Theorem. $\lim_{n\to\infty} p_n = \infty$

Proof. Denote by G_n a fixed triconnected planar graph such that $p(G_n) = p_n$. Let Δ_n be the maximum degree of G_n and let v_n be a fixed vertex of maximum degree in G_n. It is easy to see that the diameter d of any graph is large if it has an small maximum degree. In fact one can prove that $p_n \geq d(G_n) + 1 \geq \log_{\Delta_n} n$. So if $\{\Delta_n\}$ is bounded, then we are done. Hence, we can suppose that $\{\Delta_n\}$ grows.

A well known theorem of Whitney states that, any triconnected planar graph has an unique embedding in the sphere. In this embedding the topological neighborhood of a vertex v is an open disk bounded by a cycle C_v

of the graph which in general contains more vertices than the ones in the graphical neighborhood of the vertex.

Denote by G'_n the graph obtained from G_n by deleting v_n and every other vertex not in C_{v_n}. Of course, any induced path in G'_n is an induced path in G_n. We denote by n' the order of G'_n. We know that $n' \geq \Delta_n$ and therefore $\{n'\}$ is unbounded.

We can think on the graph G'_n as drawn in the plane in such a way that the cycle C_{v_n} bounds the infinite face. Let D_n be the dual graph of G'_n and let us delete from D_n the vertex corresponding to the infinite face to obtain D'_n. Since every vertex of G'_n lies in the boundary of the infinite face then, D'_n is a tree.

Let us associate to each vertex of D'_n a weight equal to the number of vertices of the corresponding face in G'_n minus two. The weight of a path in D'_n is by definition the sum of the weights of its vertices. Observe that a path of weight w in D'_n corresponds to a subgraph P of G'_n which is a path of faces separated by edges. It is easy to see that P has exactly $w+2$ vertices. Deleting a vertex from each of the two end faces of P we split the boundary of P into two paths. Again, the fact that every vertex of G'_n lies in the boundary of the infinite face implies that these two paths are induced in G'_n and one of them has at least $w/2$ vertices. Therefore, if we denote by w_n the maximum weight of a path in D'_n then, to prove the proposition we must show that $\{w_n\}$ is unbounded.

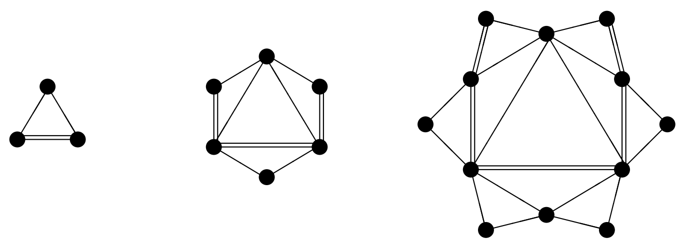
Denote by $k = k(n)$ the size of the biggest interior face in G'_n and by $m = m(n)$ the number of vertices in D'_n. If we triangulate all interior faces of G'_n, then the number of all interior triangles with respect to the cycle C_{v_n} must be $n'-2$, but in the interior of each face there are at most $k-2$ triangles and so $m \geq \frac{n'-2}{k-2}$. Let v be a vertex in D'_n of eccentricity equal to the diameter $d = d(n)$ of D'_n and denote by V_i the set of vertices at distance i from v.

It is clear that

$$\frac{n'-2}{k-2} \leq m = \sum_{i=0}^{d} |V_i| \leq \sum_{i=0}^{d} k^i \leq \frac{k^{d+1}-2}{k-2}$$

and therefore $\log_3 n' \leq (d+1)\log_3 k$. Since any vertex has weight no less than one then $w_n \geq d+1$. On the other hand, $w_n \geq k-2 \geq \log_3 k$ for any $k \geq 3$. Hence, $w_n \geq \sqrt{\log_3 n'}$ and the proof is completed. ■

Remark. The method in the proof of the proposition gives a lower bound $O(\log n)$ for maximal outerplanar graphs with n vertices. However, this an easier result that can be proved in several other ways. In this case the bound is asymptotically sharp. It is reached in the family $\{\mathbf{S}_i\}$ shown in the figure.



S_0 S_1 S_2

Figure 1. Polygon triangulations with $p = O(\log n)$.

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