LONG INDUCED PATHS IN 3-CONNECTED PLANAR GRAPHS

Jorge Luis Arocha and Pilar Valencia
Instituto de Matemáticas, UNAM, Ciudad Universitaria
Area de la Investigación Científica, Circuito Exterior
México, D.F. 04510
e-mail: arocha@math.unam.mx
e-mail: pilarvalencia@hotmail.com

Abstract

It is shown that every 3-connected planar graph with a large number of vertices has a long induced path.

Keywords: Induced paths, 3-connected planar graphs.

1991 Mathematics Subject Classification: 05C38, 05C40.

Let $G$ be an undirected graph without loops and multiple edges. Denote by $p(G)$ the number of vertices in the longest induced path of $G$. Finding long induced paths in graphs is an interesting but difficult problem. However, it is easy to revise all the references devoted to related problems (see [1-7]).

Denote $p_n = \min \{p(G)\}$ where the minimum is taken over all triconnected planar graphs of order $n$. The purpose of this note is to prove the following.

Theorem. $\lim_{n \to \infty} p_n = \infty$

Proof. Denote by $G_n$ a fixed triconnected planar graph such that $p(G_n) = p_n$. Let $\Delta_n$ be the maximum degree of $G_n$ and let $v_n$ be a fixed vertex of maximum degree in $G_n$. It is easy to see that the diameter $d$ of any graph is large if it has a small maximum degree. In fact one can prove that $p_n \geq d(G_n) + 1 \geq \log \Delta_n n$. So if $\{\Delta_n\}$ is bounded, then we are done. Hence, we can suppose that $\{\Delta_n\}$ grows.

A well known theorem of Whitney states that, any triconnected planar graph has an unique embedding in the sphere. In this embedding the topological neighborhood of a vertex $v$ is an open disk bounded by a cycle $C_v$. 
of the graph which in general contains more vertices than the ones in the graphical neighborhood of the vertex.

Denote by $G'_n$ the graph obtained from $G_n$ by deleting $v_n$ and every other vertex not in $C_{v_n}$. Of course, any induced path in $G'_n$ is an induced path in $G_n$. We denote by $n'$ the order of $G'_n$. We know that $n' \geq \Delta_n$ and therefore \{n'\} is unbounded.

We can think on the graph $G'_n$ as drawn in the plane in such a way that the cycle $C_{v_n}$ bounds the infinite face. Let $D_n$ be the dual graph of $G'_n$, and let us delete from $D_n$ the vertex corresponding to the infinite face to obtain $D'_n$. Since every vertex of $G'_n$ lies in the boundary of the infinite face then, $D'_n$ is a tree.

Let us associate to each vertex of $D'_n$ a weight equal to the number of vertices of the corresponding face in $G'_n$ minus two. The weight of a path in $D'_n$ is by definition the sum of the weights of its vertices. Observe that a path of weight $w$ in $D'_n$ corresponds to a subgraph $P$ of $G'_n$ which is a path of faces separated by edges. It is easy to see that $P$ has exactly $w + 2$ vertices. Deleting a vertex from each of the two end faces of $P$ we split the boundary of $P$ into two paths. Again, the fact that every vertex of $G'_n$ lies in the boundary of the infinite face implies that these two paths are induced in $G'_n$ and one of them has at least $w/2$ vertices. Therefore, if we denote by $w_n$ the maximum weight of a path in $D'_n$, then, to prove the proposition we must show that \{w_n\} is unbounded.

Denote by $k = k(n)$ the size of the biggest interior face in $G'_n$ and by $m = m(n)$ the number of vertices in $D'_n$. If we triangulate all interior faces of $G'_n$, then the number of all interior triangles with respect to the cycle $C_{v_n}$ must be $n' - 2$, but in the interior of each face there are at most $k - 2$ triangles and so $m \geq \frac{n' - 2}{k - 2}$. Let $v$ be a vertex in $D'_n$ of eccentricity equal to the diameter $d = d(n)$ of $D'_n$ and denote by $V_i$ the set of vertices at distance $i$ from $v$.

It is clear that

$$\frac{n' - 2}{k - 2} \leq m = \sum_{i=0}^{d} |V_i| \leq \sum_{i=0}^{d} k^i \leq \frac{k^{d+1} - 2}{k - 2}$$

and therefore $\log_3 n' \leq (d + 1) \log_3 k$. Since any vertex has weight no less than one then $w_n \geq d + 1$. On the other hand, $w_n \geq k - 2 \geq \log_3 k$ for any $k \geq 3$. Hence, $w_n \geq \sqrt[3]{\log_3 n'}$ and the proof is completed.

**Remark.** The method in the proof of the proposition gives a lower bound $O(\log n)$ for maximal outerplanar graphs with $n$ vertices. However, this an
easier result that can be proved in several other ways. In this case the bound is asymptotically sharp. It is reached in the family \( \{S_i\} \) shown in the figure.

![Diagram of polygon triangulations](image)

Figure 1. Polygon triangulations with \( p = O(\log n) \).

References


Received 22 June 1999
Revised 1 October 1999