

## SOME NEWS ABOUT THE INDEPENDENCE NUMBER OF A GRAPH

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### Abstract

For a finite undirected graph  $G$  on  $n$  vertices some continuous optimization problems taken over the  $n$ -dimensional cube are presented and it is proved that their optimum values equal the independence number of  $G$ .

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## 1 Introduction and Results

Let  $G$  be a finite simple and undirected graph on  $V(G) = \{1, 2, \dots, n\}$  with its edge set  $E(G)$ . A subset  $I$  of  $V(G)$ , such that the subgraph of  $G$  induced by  $I$  is edgeless, is called an *independent set* of  $G$ , and the maximum cardinality of an independent set of  $G$  is named the *independence number*  $\alpha(G)$  of  $G$ .  $N(i)$  and  $d_i$  denote the set and the number of neighbours of  $i \in V(G)$  in  $G$ , respectively, and let  $\Delta(G) = \max\{d_i \mid i \in V(G)\}$  and  $C^n = \{(x_1, x_2, \dots, x_n) \mid 0 \leq x_i \leq 1, i = 1, 2, \dots, n\}$ . For events  $A$  and  $B$  and for a random variable  $Z$  of an arbitrary random space,  $P(A)$ ,  $P(A|B)$ , and  $\mathcal{E}(Z)$  denote the probability of  $A$ , the conditional probability of  $A$  given  $B$ , and the expectation of  $Z$ , respectively. Since the computation of  $\alpha(G)$  is difficult (INDEPENDENT SET is an NP-complete problem; see [6]), much work was done to establish bounds on  $\alpha(G)$  (e.g., see [1, 3, 4, 5, 8, 10, 12, 13, 14, 15, 16, 17]), to find efficient algorithms forming a large independent set of  $G$  (e.g., see [2, 7, 8, 9, 10, 12]), or to replace the combinatorial optimization problem to determine  $\alpha(G)$  by a continuous one

(e.g., see [9, 11]). The last approach leads to bounds on  $\alpha(G)$  as well as to efficient algorithms (e.g., see [8, 9]). In the present paper some new continuous optimization problems taken over  $C^n$  are presented and it is proved that their optimum values equal  $\alpha(G)$ . Theorem 1 gives a remarkable result of T.S. Motzkin and E.G. Straus [11] and Theorem 2 is proved in [9].

**Theorem 1.**

$$\alpha(G) = \max_{(0,0,\dots,0) \neq (x_1, x_2, \dots, x_n) \in C^n} \frac{\left( \sum_{i \in V(G)} x_i \right)^2}{\sum_{i \in V(G)} x_i^2 + 2 \sum_{ij \in E(G)} x_i x_j}.$$

**Theorem 2.**  $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} \sum_{i \in V(G)} (x_i \prod_{j \in N(i)} (1 - x_j))$ .

A classical lower bound on  $\alpha(G)$  due to Y. Caro and V.K. Wei [3, 17] is given by the following theorem.

**Theorem 3.**  $\alpha(G) \geq \sum_{i \in V(G)} \frac{1}{1+d_i}$ .

The next Theorems 4, 5, 6, and 7 are the main results of the present paper.

**Theorem 4.**  $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} e_G(x_1, x_2, \dots, x_n)$ , where

$$e_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left( \frac{x_i}{1 + \sum_{j \in N(i)} x_j} + \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right).$$

**Theorem 5.**  $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} f_G(x_1, x_2, \dots, x_n)$ , where

$$f_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left( x_i + \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right) - \sum_{ij \in E(G)} x_i x_j.$$

The following Theorem 6 looks more "complicate", but it is "stronger" than Theorem 4 and Theorem 5 (see Remark 1).

**Theorem 6.**  $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} g_G(x_1, x_2, \dots, x_n)$ , where

$$g_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left( \left( x_i + \frac{1-x_i}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right) \prod_{j \in N(i)} (1-x_j) \right)$$

$$+ \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j} \text{ and } V' = \left\{ i \in V(G) \mid \sum_{j \in N(i)} x_j > 0 \right\}.$$

A "weaker" (see Remark 1), but a more "transparent" and (see Remark 2) an "algorithmically realizable" version of Theorem 5 is the following one.

**Theorem 7.**  $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} h_G(x_1, x_2, \dots, x_n)$ , where  
 $h_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j$ .

## 2 Proofs

Throughout the proofs we will use the well-known fact that for a random subset  $M$  of a given finite set  $N$ ,

$$\mathcal{E}(|M|) = \sum_{y \in N} P(y \in M) = \sum_{k=0}^{|N|} k P(|M| = k).$$

Let  $I$  be a maximum independent set of  $G$  and let  $x_i^* = 1$  if  $i \in I$  and  $x_i^* = 0$  if  $i \notin I$ . Since  $(1 - x_i^*) \prod_{j \in N(i)} (1 - x_j^*) = 0$  for  $i \in V(G)$  and  $\sum_{ij \in E(G)} x_i^* x_j^* = 0$ , we obtain

**Lemma 1.**  $\alpha(G) = e_G(x_1^*, x_2^*, \dots, x_n^*) = f_G(x_1^*, x_2^*, \dots, x_n^*) = g_G(x_1^*, x_2^*, \dots, x_n^*) = h_G(x_1^*, x_2^*, \dots, x_n^*)$ .

With Lemma 1, it is clear that Theorem 7 follows from Theorem 5.

Now, let  $(x_1, x_2, \dots, x_n)$  be an arbitrary member of  $C^n$ . We form a set  $X \subseteq V(G)$  by random and independent choice of  $i \in V(G)$ , where  $P(i \in X) = x_i$ . Let  $H_1$ ,  $H_2$ , and  $H_3$  be the subgraph of  $G$  induced by the vertices of  $X$ , by the vertices  $i \in X$  with  $N(i) \cap X \neq \emptyset$ , and by the vertices  $i \notin X$  with  $N(i) \cap X = \emptyset$ , respectively. Furthermore, let  $Y$  be a smallest subset of  $V(H_2)$  covering all edges of  $H_2$ , i.e., the graph induced by  $V(H_2) - Y$  is edgeless, and let  $I_1$  and  $I_3$  be a maximum independent set of  $H_1$  and  $H_3$ , respectively. It can be seen easily that  $|Y| = |V(H_2)| - \alpha(H_2)$ ,  $|Y| \leq |E(H_2)|$  and that  $(X - Y) \cup I_3$  and  $I_1 \cup I_3$  are independent sets of  $G$ . Because of these remarks and the property of the expectation to be an average value, we have Lemma 2 as follows.

**Lemma 2.**  $\alpha(G) \geq \mathcal{E}(|X - Y|) + \mathcal{E}(\alpha(H_3))$ ,  $\alpha(G) \geq \mathcal{E}(\alpha(H_1)) + \mathcal{E}(\alpha(H_3))$ ,  
 $\mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|Y|) \geq \mathcal{E}(|X|) - \mathcal{E}(|E(H_2)|)$ , and  
 $\mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|V(H_2)|) + \mathcal{E}(\alpha(H_2))$ .

Lower bounds on  $\mathcal{E}(\alpha(H_1))$ ,  $\mathcal{E}(\alpha(H_2))$ , and  $\mathcal{E}(\alpha(H_3))$  are given in Lemma 3.

**Lemma 3.**  $\mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j}$ ,  
 $\mathcal{E}(\alpha(H_2)) \geq \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}$ , where  $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$ ,  
and  
 $\mathcal{E}(\alpha(H_3)) \geq \sum_{i \in V(G)} \frac{(1 - x_i) \prod_{j \in N(i)} (1 - x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1 - x_l)}$ .

**Proof.** For  $i \in V(G)$  define the random variable  $Z_i^1$  with  $Z_i^1 = \frac{1}{1+k}$  if  $i \in X$  and  $|N(i) \cap X| = k \geq 0$ , and  $Z_i^1 = 0$  if  $i \notin X$ . Using Theorem 3,

$$\begin{aligned} \mathcal{E}(\alpha(H_1)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^1\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^1) \\ &= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k) \\ &= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k) \\ &= \sum_{i \in V(G)} x_i \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k). \end{aligned}$$

For  $i \in V(G)$  we have  $\sum_{k=0}^{d_i} P(|N(i) \cap X| = k) = 1$ . With Jensen's inequality

$$\sum_{l=1}^m \tau_l \phi(y_l) \geq \phi\left(\sum_{l=1}^m \tau_l y_l\right) \text{ for any convex function } \phi \text{ and any } \tau_l \geq 0 \text{ for } l = 1, 2, \dots, m \text{ with } \sum_{l=1}^m \tau_l = 1,$$

$$\mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} x_i \frac{1}{1 + \sum_{k=0}^{d_i} k P(|N(i) \cap X| = k)} = \sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j}.$$

Now, let  $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$ . For  $i \in V(G)$  let  $Z_i^2$  be the random variable with  $Z_i^2 = \frac{1}{1+k}$  if  $i \in X$  and  $|N(i) \cap X| = k \geq 1$ , and  $Z_i^2 = 0$  otherwise. Then,

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^2\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^2) \\
&= \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k) \\
&= \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k) \\
&= \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k).
\end{aligned}$$

$P(|N(i) \cap X| = 0) + \sum_{k=1}^{d_i} P(|N(i) \cap X| = k) = 1$  for  $i \in V(G)$  and with  $\mu_i = P(|N(i) \cap X| = 0) = \prod_{j \in N(i)} (1 - x_j)$  and  $\sigma_{ik} = P(|N(i) \cap X| = k)$ ,

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V(G), \mu_i < 1} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} \\
&= \sum_{i \in V'} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V'} x_i (1 - \mu_i) \sum_{k=1}^{d_i} \frac{\sigma_{ik}}{(1+k)(1-\mu_i)}.
\end{aligned}$$

For  $\lambda_{ik} = \frac{\sigma_{ik}}{1-\mu_i}$  we have  $\lambda_{ik} \geq 0$ ,  $\sum_{k=1}^{d_i} \lambda_{ik} = 1$  if  $i \in V'$ , and again using Jensen's inequality,

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \sum_{i \in V'} x_i (1 - \mu_i) \frac{1}{1 + \sum_{k=1}^{d_i} k \lambda_{ik}} \\
&= \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{k=1}^{d_i} k P(|N(i) \cap X| = k)} = \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}.
\end{aligned}$$

Finally, let us consider the random variable  $Z_i^3$  with  $Z_i^3 = \frac{1}{1+k}$  if  $i \in V(H_3)$  and  $|N(i) \cap V(H_3)| = k \geq 0$ , and  $Z_i^3 = 0$  if  $i \notin V(H_3)$ . Then

$$\begin{aligned}
\mathcal{E}(\alpha(H_3)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^3\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^3) \\
&= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3) \text{ and } |N(i) \cap V(H_3)| = k) \\
&= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3)) P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)) \\
&= \sum_{i \in V(G)} \left( (1 - x_i) \prod_{j \in N(i)} (1 - x_j) \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)) \right)
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i \in V(G)} \left( (1-x_i) \prod_{j \in N(i)} (1-x_j) \frac{1}{1 + \sum_{k=0}^{d_i} k P(|N(i) \cap V(H_3)|=k \mid i \in V(H_3))} \right) \\
&= \sum_{i \in V(G)} \left( (1-x_i) \prod_{j \in N(i)} (1-x_j) \frac{1}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right),
\end{aligned}$$

and Lemma 3 is proved. ■

$$\begin{aligned}
&\text{Theorem 4, 5, and 6 follow with } \mathcal{E}(|X|) = \sum_{i \in V(G)} x_i, \mathcal{E}(|E(H_2)|) \\
&= \sum_{ij \in E(G)} x_i x_j, \mathcal{E}(|V(H_2)|) = \sum_{i \in V(G)} x_i (1 - \prod_{j \in N(i)} (1-x_j)), \text{ Lemma 1, 2, and 3.}
\end{aligned}$$

### 3 Remarks

For  $\phi, \psi \in \{e, f, g, h\}$  define  $\phi \leq \psi$  if  $\phi_G(x_1, x_2, \dots, x_n) \leq \psi_G(x_1, x_2, \dots, x_n)$  for every graph  $G$  on  $n$  vertices and for every  $(x_1, x_2, \dots, x_n) \in C^n$ . We write  $\phi \langle \rangle \psi$  if neither  $\phi \leq \psi$  nor  $\psi \leq \phi$ .

**Remark 1.**  $h \leq f \leq g, e \leq g$  and  $e \langle \rangle f$ .

**Proof.** We will use the following Lemma 4, which can be seen easily by induction on  $r$ .

**Lemma 4.** For an integer  $r \geq 1$  and  $a_1, a_2, \dots, a_r \in [0, 1]$ ,

$$\sum_{q=1}^r a_q + \prod_{q=1}^r (1 - a_q) \geq 1.$$

The inequality  $h \leq f$  is obvious. To see  $f \leq g$ , first notice that  $\sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j = \sum_{i \in V(G)} x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j)$ . If  $\sum_{j \in N(i)} x_j = 0$  for an  $i \in V(G)$  then  $x_i = x_i (\prod_{j \in N(i)} (1 - x_j)) = x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j)$ . Hence, with the abbreviation  $\mu_i = \prod_{j \in N(i)} (1 - x_j)$  and  $\rho_i = \sum_{j \in N(i)} x_j$  for  $i \in V(G)$  we have to show  $\sum_{i \in V'} (x_i (\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i})) \geq \sum_{i \in V'} (x_i (1 - \frac{1}{2} \rho_i))$ , where again  $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$ .

Using Lemma 4, even  $\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i} \geq 1 - \frac{1}{2} \rho_i$  for all  $i \in V'$ .

To prove  $e \leq g$ , we have to show

$$\sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j} \leq \sum_{i \in V(G)} x_i \prod_{j \in N(i)} (1 - x_j) + \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}.$$

Since  $\frac{x_i}{1 + \sum_{j \in N(i)} x_j} = x_i \prod_{j \in N(i)} (1 - x_j)$  if  $\sum_{j \in N(i)} x_j = 0$ , it is sufficient to establish

$$\frac{1}{1 + \rho_i} \leq \mu_i + \frac{(1 - \mu_i)^2}{1 - \mu_i + \rho_i} \text{ if } \sum_{j \in N(i)} x_j > 0, \text{ what is verified easily.}$$

For a cycle  $C_n$  on  $n$  vertices  $e_{C_n}(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}) < f_{C_n}(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$ ,  $e_{C_n}(\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3}) > f_{C_n}(\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3})$  and Remark 1 is proved. ■

With  $h \leq f$  and  $e \ll f$  it is clear that  $e \leq h$  does not hold. It remains open, whether  $h \ll e$  or  $h \leq e$ .

Theorems 1, 2, 4, 5, 6, and 7 are of that type that the independence number  $\alpha(G)$  of a graph  $G$  on  $n$  vertices equals the optimum value of a continuous optimization problem  $O(G)$  to maximize a certain function  $\phi_G$  over  $C^n$ . Hence,  $\phi_G(x_1, x_2, \dots, x_n)$  is a lower bound on  $\alpha(G)$  for every  $(x_1, x_2, \dots, x_n) \in C^n$ . Let  $(x'_1, x'_2, \dots, x'_n) \in C^n$  be the solution of an arbitrary approximation algorithm for  $O(G)$ . How to find an independent set  $I$  of  $G$  in polynomial time such that  $|I| \geq \phi_G(x'_1, x'_2, \dots, x'_n)$ ? In [8] and [9] efficient algorithms forming  $I$  with  $|I| \geq \phi_G(x'_1, x'_2, \dots, x'_n)$  are given if  $O(G)$  is the optimization problem of Theorem 1 or of Theorem 2. Remark 2 shows that this is also possible if we consider the case  $\phi_G = h_G$ . In case  $\phi_G = e_G$ ,  $\phi_G = f_G$  or  $\phi_G = g_G$  the problem remains open, whether such an algorithm exists.

**Remark 2.** There is an  $\mathcal{O}(\Delta(G)n)$ -algorithm with

INPUT:  $(x_1, x_2, \dots, x_n) \in C^n$ ,

OUTPUT: an independent set  $I \subseteq V(G)$  with  $|I| \geq \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j$ .

**Proof.** First we give the Algorithm:

1. For  $i = 1$  to  $n$  do if  $\sum_{j \in N(i)} x_j < 1$  then  $x_i := 1$  else  $x_i := 0$ .
2. For  $i = 1$  to  $n$  do if  $(x_i = 1 \text{ and } \prod_{j \in N(i)} (1 - x_j) = 0)$  then  $x_i := 0$ .
3.  $I := \{i \in V(G) \mid x_i = 1\}$ .

STOP

It is obvious that the algorithm is an  $\mathcal{O}(\Delta(G)n)$ -algorithm. For the input vector  $(x_1, x_2, \dots, x_n) \in C^n$  set

$$\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j = a.$$

After step 1, the current  $(x_1, x_2, \dots, x_n)$  is a 0-1-vector and

$$\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \geq a$$

because

$$\frac{\partial}{\partial x_i} \left( \sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \right) = 1 - \sum_{j \in N(i)} x_j, \text{ i.e., } \sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j$$

is multilinear.

In step 2,  $\prod_{j \in N(i)} (1 - x_j) = 0$  if and only if there is at least one  $j \in N(i)$  such that  $x_j = 1$ . With  $x_i = 0$  instead of  $x_i = 1$  the sum  $\sum_{k \in V(G)} x_k$  decreases by 1 and the sum  $\sum_{kj \in E(G)} x_k x_j$  decreases by at least 1, hence  $\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j$  does not decrease.

After step 2,

$$x_k x_j = 0 \text{ for all } kj \in E(G), |I| = \sum_{k \in V(G)} x_k = \sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \geq a$$

and Remark 2 is proved. ■

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