ON THE COMPLETENESS OF DECOMPOSABLE PROPERTIES OF GRAPHS

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Abstract

Let \( P_1, P_2 \) be additive hereditary properties of graphs. A \((P_1, P_2)\)-decomposition of a graph \( G \) is a partition of \( E(G) \) into sets \( E_1, E_2 \) such that induced subgraph \( G[E_i] \) has the property \( P_i, i = 1, 2 \). Let us define a property \( P_1 \oplus P_2 \) by \( \{ G : G \) has a \((P_1, P_2)\)-decomposition\}.

A property \( D \) is said to be decomposable if there exists nontrivial additive hereditary properties \( P_1, P_2 \) such that \( D = P_1 \oplus P_2 \). In this paper we determine the completeness of some decomposable properties and we characterize the decomposable properties of completeness 2.

Keywords: decomposition, hereditary property, completeness

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1 Introduction and Notation

We consider finite undirected simple graphs. In general, we follow the notation and terminology of [4, 6]. Let us denote by \( \mathcal{I} \) the class of all simple finite graphs. A graph property \( \mathcal{P} \) is any isomorphism-closed nonempty subclass of \( \mathcal{I} \). \( \mathcal{P} \) will also denote the property that a graph is a member of \( \mathcal{P} \). A property \( \mathcal{P} \) is said to be hereditary if \( G \in \mathcal{P} \) and \( H \subseteq G \) (\( H \) is a subgraph of \( G \)) implies \( H \in \mathcal{P} \). A property \( \mathcal{P} \) is called additive if for each graph \( G \) all
of whose components have property \( P \) it follows that \( G \in P \), too. The set \( \mathbb{L}^n \) of all hereditary and additive properties of graphs, partially ordered by set inclusion forms a complete distributive lattice. We will denote by \( \langle Q_1, Q_2 \rangle \) the interval between \( Q_1 \) and \( Q_2 \) in the lattice \( \mathbb{L}^n \).

Every hereditary property \( P \) is uniquely determined by the set

\[
F(P) = \{ G \in I : G \notin P \text{ but each proper subgraph of } G \text{ belongs to } P \}
\]

of its minimal forbidden subgraphs. By the property \( \{ H_1, \ldots, H_k \} \) we mean the property \( P \) with \( F(P) = \{ H_1, \ldots, H_k \} \).

**Example.** We list some important additive hereditary properties, using partially the notation of [2, 4].

\[
\begin{align*}
\mathcal{O} &= \{ G \in I : G \text{ is edgeless, i.e., } E(G) = \emptyset \}, \\
\mathcal{O}_k &= \{ G \in I : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \}, \\
\mathcal{S}_k &= \{ G \in I : \text{the maximum degree } \Delta(G) \leq k \}, \\
\mathcal{W}_k &= \{ G \in I : \text{the length of the longest path in } G \text{ is at most } k \}, \\
\mathcal{D}_k &= \{ G \in I : G \text{ is } k\text{-degenerate,} \\
& \quad \text{i.e., the minimum degree } \delta(H) \leq k \text{ for each } H \subseteq G \}, \\
\mathcal{T}_k &= \{ G \in I : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or } \\
& \quad K_{[\frac{k+3}{2}, \frac{k+3}{2}]} \}, \\
\mathcal{I}_k &= \{ G \in I : G \text{ does not contain } K_{k+2} \}, \\
\mathcal{E}_k &= \{ G \in I : \text{each component of } G \text{ has at most } k \text{ edges} \}, \\
\mathcal{L}_k &= \{ G \in I : G \in D_1 \land G \in S_2 \}, \\
\mathcal{S}_k &= \{ G \in I : \text{each component of } G \text{ is a star} \}.
\end{align*}
\]

An additive hereditary property \( P \) is said to be nontrivial if \( P \neq \mathcal{O} \) and \( P \neq I \). Let \( P \) be a nontrivial additive hereditary property. Then there is a nonnegative integer \( c(P) \) such that \( K_{c(P) + 1} \in P \) but \( K_{c(P) + 2} \notin P \); it is called the completeness of \( P \). Obviously

\[
c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{W}_k) = c(\mathcal{D}_k) = c(\mathcal{T}_k) = c(\mathcal{I}_k) = c(\mathcal{E}_k) = k,
\]

\[
c(\mathcal{E}_k) = \left\lfloor \frac{1}{2} (-1 + \sqrt{1 + 8k}) \right\rfloor
\]

and for additive properties \( c(P) = 0 \) if and only if \( P = \mathcal{O} \).

Let \( P_1, P_2, \ldots, P_n \) be arbitrary hereditary properties of graphs. A vertex \((P_1, P_2, \ldots, P_n)\)-partition of a graph \( G \) is a partition of \( V(G) \) into sets \( V_1, V_2, \ldots, V_n \) such that for each \( i = 1, 2, \ldots, n \), the induced subgraph \( G|V_i| \)
has the property \( P_i \) (for convenience, the empty set \( \emptyset \) will be regarded as
the set inducing the subgraph with any property \( P \)).

A property \( R = P_1 \circ \cdots \circ P_n \) is defined as the set of all graphs having
a vertex \((P_1, P_2, \ldots, P_n)\)-partition. It is easy to see that if \( P_1, P_2, \ldots, P_n \) are
additive and hereditary, then \( R = P_1 \circ \cdots \circ P_n \) is additive and hereditary,
too. If \( P_1 = P_2 = \cdots = P_n = P \), then we write \( P^n = P_1 \circ P_2 \circ \cdots \circ P_n \).
Thus, \( O_k \), \( k \geq 2 \) denotes the class of all \( k \)-colourable graphs.

An hereditary property \( R \) is said to be reducible if there exist hereditary properties \( P, Q \) such that \( R = P \circ Q \) and irreducible, otherwise.

A \((P_1, P_2, \ldots, P_n)\)-decomposition of a graph \( G \) is a partition of \( E(G) \)
into sets \( E_1, E_2, \ldots, E_n \) such that for each \( i = 1, 2, \ldots, n \), the subgraph \( G[E_i] \) has the property \( P_i \) (for convenience, the empty set \( \emptyset \) will be regarded
as the set inducing the subgraph with any property \( P \)).

A property \( D = P_1 \oplus P_2 \oplus \cdots \oplus P_n \) is defined as the set of all graphs having
a \((P_1, P_2, \ldots, P_n)\)-decomposition. It is easy to see that if \( P_1, P_2, \ldots, P_n \) are
additive and hereditary, then \( D = P_1 \oplus P_2 \oplus \cdots \oplus P_n \) is additive
and hereditary, too. If \( P_1 = P_2 = \cdots = P_n = P \), then we write \( nP = P_1 \oplus P_2 \oplus \cdots \oplus P_n \).

A hereditary property \( D \) is said to be decomposable if there exist non-trivial hereditary properties \( P, Q \) such that \( D = P \oplus Q \) and indecomposable.
otherwise.

The Ramsey number \( r(m, n) \) is the smallest integer for which every graph
of order \( r(m, n) \) contains either a clique of size \( m \) or an independent set of
size \( n \).

Throughout this article, all properties we deal with are hereditary and
additive.

## 2 Completeness

There is an easy formula to determine the completeness of any reducible
property \( R = P \circ Q \), namely, \( c(R) = c(P) + c(Q) + 1 \) (see [8]). The calculation
of the completeness of decomposable properties is much more difficult. It is
easy to see that:

\[
\max\{ c(P), c(Q) \} \leq c(P \oplus Q) \leq c(I_{c(P)} + I_{c(Q)}) = r(c(P) + 2, c(Q) + 2) - 2.
\]

and hence the problem is related to the problem of determining the Ramsey
numbers.
Obviously, there is only one decomposable property of completeness 1, the property \( O_1 \oplus O_1 \). The next result characterize the decomposable properties of completeness equals 2.

**Theorem 1.** Let \( \mathcal{P}, \mathcal{Q} \) be nontrivial additive hereditary properties. Then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) if and only if \( \mathcal{P} \) and \( \mathcal{Q} \) satisfy at least one of the following conditions:

(i) \( \mathcal{P} = O_1 \) and \( \mathcal{Q} \in \langle \mathcal{E}_2, -\{C_4\} \rangle \),

(ii) \( \mathcal{P} = \mathcal{E}_2 \) and \( \mathcal{Q} \in \langle O_1, -\{\forall, C_4\} \rangle \),

(iii) \( \mathcal{P} = O_2 \) and \( \mathcal{Q} \in \langle O_1, -\{C_4\} \land S_2 \rangle \),

(iv) \( \mathcal{P} \in \langle \mathcal{E}_2, S_2 \land -\{C_3, C_4\} \rangle \) and \( \mathcal{Q} \in \langle O_1, W_2 \rangle \),

(v) \( \mathcal{P} \in \langle \mathcal{E}_2, \mathcal{S} \mathcal{F} \rangle \) and \( \mathcal{Q} \in \langle O_1, -\{C_3, C_4\} \rangle \).

**Proof.** By the definition of the completeness if \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) then \( K_4 \not\in \mathcal{P} \oplus \mathcal{Q} \). Let \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \). Since \( O_1 \subseteq \mathcal{P}, \mathcal{Q} \), then \( C_4 \not\in \mathcal{P} \) and \( C_4 \not\in \mathcal{Q} \) (because \( K_4 \in \langle K_2 \cup K_2 \rangle \oplus C_4 \)).

To prove the theorem let us consider the following cases:

**Case 1.** Let \( K_2 \in \mathcal{P} \) and \( P_3 \not\in \mathcal{P} \). Then \( C_4 \not\in \mathcal{Q} \).

Conversely, if \( \mathcal{P} = O_1 \) and \( \mathcal{Q} \in \langle \mathcal{E}_2, -\{C_4\} \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (i).

**Case 2.** Let \( P_3 \in \mathcal{P} \), \( K_3 \not\in \mathcal{P} \), \( K_{1,3} \not\in \mathcal{P} \) and \( P_4 \not\in \mathcal{P} \).

Then \( C_4 \not\in \mathcal{Q} \) and \( \forall \not\in \mathcal{Q} \). Conversely, if \( \mathcal{P} = \mathcal{E}_2 \) and \( \mathcal{Q} \in \langle O_1, -\{C_4, \forall\} \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (ii).

**Case 3.** Let \( P_3 \in \mathcal{P} \), \( K_3 \in \mathcal{P} \), \( K_{1,3} \not\in \mathcal{P} \) and \( P_4 \not\in \mathcal{P} \).

Then \( C_4 \not\in \mathcal{Q} \) and \( K_{1,3} \not\in \mathcal{Q} \). Conversely, if \( \mathcal{P} = O_2 \) and \( \mathcal{Q} \in \langle O_1, -\{C_4, K_{1,3}\} \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (iii).

**Case 4.** Let \( P_3 \in \mathcal{P} \), \( K_3 \not\in \mathcal{P} \), \( K_{1,3} \in \mathcal{P} \) and \( P_4 \not\in \mathcal{P} \).

Then \( C_4 \not\in \mathcal{Q} \) and \( C_3 \not\in \mathcal{Q} \). Conversely, if \( \mathcal{P} \in \langle \mathcal{E}_2, \mathcal{S} \mathcal{F} \rangle \) and \( \mathcal{Q} \in \langle O_1, -\{C_4, C_3\} \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (v).

**Case 5.** Let \( P_3 \in \mathcal{P} \), \( K_3 \not\in \mathcal{P} \), \( K_{1,3} \in \mathcal{P} \) and \( P_4 \not\in \mathcal{P} \).

Then \( C_4 \not\in \mathcal{Q} \), \( C_3 \not\in \mathcal{Q} \) and \( K_{1,3} \not\in \mathcal{Q} \). Conversely, if \( \mathcal{P} \in \langle \mathcal{E}_2, W_2 \rangle \) and \( \mathcal{Q} \in \langle O_1, S_2 \land -\{C_4, C_3\} \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (iv).

**Case 6.** Let \( P_4 \in \mathcal{P} \), \( K_3 \not\in \mathcal{P} \), \( K_{1,3} \not\in \mathcal{P} \). Then \( P_4 \not\in \mathcal{Q} \).

Conversely, if \( \mathcal{P} \in \langle \mathcal{E}_2, S_2 \land -\{C_4, C_3\} \rangle \) and \( \mathcal{Q} \in \langle O_1, W_2 \rangle \), then \( c(\mathcal{P} \oplus \mathcal{Q}) = 2 \) and we have (iv).
Case 7. Let $P_1 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$. Then $P_1 \notin \mathcal{Q}$ and $K_{1,3} \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, S_2 \wedge \neg \{C_4\} \rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{O}_2 \rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (iii).

Case 8. Let $P_1 \in \mathcal{P}$, $K_3 \notin \mathcal{P}$, $K_{1,3} \in \mathcal{P}$. Then $P_1 \notin \mathcal{Q}$ and $K_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \neg \{C_4, C_3\} \rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{S}_\mathcal{F} \rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (v).

Case 9. Let $P_1 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$ and $\nabla \notin \mathcal{P}$. Then $P_1 \notin \mathcal{Q}$, $K_{1,3} \notin \mathcal{Q}$ and $K_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \neg \{C_4, \nabla\} \rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{E}_2 \rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (ii).

Case 10. Let $\nabla \in \mathcal{P}$.
Then $P_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \neg \{C_4\} \rangle$ and $\mathcal{Q} = \mathcal{O}_1$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (i).

Because all possible $(\mathcal{P}, \mathcal{Q})$-decomposition were considered and taking into consideration fact that $K_4 \notin \mathcal{P} \oplus \mathcal{Q}$, the proof is complete. □

**Theorem 2.** $\mathcal{D}_2$ is indecomposable.

**Proof.** It is easy to check that the graphs $G_i$ in Figure 1, belongs to $\mathcal{D}_2$, for $i = 1, \ldots, 4$ and $G_1 \notin \mathcal{O}_1 \oplus \neg \{C_4\}$, $G_2 \notin \mathcal{E}_2 \oplus \neg \{\nabla, C_4\}$, $G_3 \notin \mathcal{O}_2 \oplus \neg \{C_4\} \wedge \mathcal{S}_2$, $G_2 \notin \mathcal{W}_2 \oplus \neg \{C_3, C_4\} \wedge \mathcal{S}_2$ and $G_4 \notin \mathcal{S}_\mathcal{F} \oplus \neg \{C_3, C_4\}$.

![Figure 1](image)

Hence, it follows: $\mathcal{D}_2 \notin \mathcal{O}_1 \oplus \neg \{C_4\}$, $\mathcal{D}_2 \notin \mathcal{E}_2 \oplus \neg \{\nabla, C_4\}$, $\mathcal{D}_2 \notin \mathcal{O}_2 \oplus \neg \{C_4\} \wedge \mathcal{S}_2$, $\mathcal{D}_2 \notin \mathcal{W}_2 \oplus \neg \{C_3, C_4\} \wedge \mathcal{S}_2$ and $\mathcal{D}_2 \notin \mathcal{S}_\mathcal{F} \oplus \neg \{C_3, C_4\}$ and by Theorem 1 $\mathcal{D}_2$ is indecomposable. □

**Theorem 3.** Every reducible property of completeness 2 is indecomposable.

**Proof.** It is easy to see that the graph $G$ in Figure 2, belongs to $\mathcal{O} \circ \mathcal{O}_1$ and $G \notin \mathcal{O}_1 \oplus \neg \{C_4\}$, $G \notin \mathcal{E}_2 \oplus \neg \{\nabla, C_4\}$, $G \notin \mathcal{O}_2 \oplus \neg \{C_4\} \wedge \mathcal{S}_2$, $G \notin \mathcal{W}_2 \oplus \neg \{C_3, C_4\} \wedge \mathcal{S}_2$ and $G \notin \mathcal{S}_\mathcal{F} \oplus \neg \{C_3, C_4\}$.

![Figure 2](image)
Hence, it follows: \( O \circ O \mathcal{C} O_{1} \oplus - \{ C_{4} \}, O \circ O \mathcal{C} E_{2} \oplus - \{ Y, C_{1} \}, O \circ O \mathcal{C} O_{2} \oplus - \{ C_{4} \} \wedge S_{2}, O \circ O \mathcal{C} W_{2} \oplus - \{ C_{3}, C_{4} \} \wedge S_{2} \) and \( O \circ O \mathcal{S} F \oplus - \{ C_{3}, C_{4} \} \). Thus, since \( O \circ O \mathcal{C} \) is the smallest reducible property of completeness 2, any reducible property \( \mathcal{R} \) of completeness 2 is indecomposable.

Now we can reformulate as examples some well-known results in Ramsey Theory using our notations.

**Theorem 4** [10]. \( c(P_{1} \oplus P_{2} \oplus \ldots \oplus P_{n}) \leq \frac{(\sum_{i=1}^{n} c(P_{i})+n)!}{\prod_{i=1}^{n} (c(P_{i})+1)!} - 2. \)

**Theorem 5** [7]. \( c(I_{k_{1}} \oplus I_{k_{2}} \oplus \ldots \oplus I_{k_{n}}) \leq c(I_{k_{1}-1} \oplus I_{k_{2}} \oplus \ldots \oplus I_{k_{n}}) + c(I_{k_{1}} \oplus I_{k_{2}-1} \oplus \ldots \oplus I_{k_{n}}) + \ldots + c(I_{k_{1}} \oplus I_{k_{2}} \oplus \ldots \oplus I_{k_{n}-1}) + n. \)

**Proposition 6.** \( c(I_{1} \oplus I_{1}) = 4, c(I_{1} \oplus I_{1} \oplus I_{1}) = 15. \)

**Theorem 7** [5].

\[
c(S_{k_{1}} \oplus S_{k_{2}} \oplus \ldots \oplus S_{k_{n}}) = \begin{cases} \sum_{i=1}^{n} k_{i}, & \text{when } \sum_{i=1}^{n} k_{i} \text{ is odd} \\ \sum_{i=1}^{n} k_{i} - 1, & \text{otherwise.} \end{cases}
\]

We found an upper bound for \( c(D_{p} \oplus D_{q}). \)

**Theorem 8.** \( c(D_{p} \oplus D_{q}) \leq p + q - 1 + \frac{1+\sqrt{1+8pq}}{2}. \)

**Proof.** For any graph \( G \in D_{p} \oplus D_{q} \), if \( K_{n} \subseteq G \) then \( K_{n} \in D_{p} \oplus D_{q}. \) Since the number of edges in a \( k \)-degenerate graph of order \( n \) is at most \( kn - \binom{k+1}{2} \), then \( \binom{n}{2} \leq mn - \binom{p+1}{2} + qn - \binom{q+1}{2}. \) By an easy computation we have \( n \leq p + q + \frac{1+\sqrt{1+8pq}}{2}. \)

**Corollary 9.** \( c(kD_{p}) \leq kp + \frac{-1+\sqrt{1+4p^{2}k(k-1)}}{2}. \)

**Proof.** For any graph \( G \in kD_{p} \), if \( K_{n} \subseteq G \) then \( K_{n} \in kD_{p}. \) Then \( \binom{n}{2} \leq k \left( mn - \binom{p+1}{2} \right). \) It implies \( n \leq kp + 1 + \frac{-1+\sqrt{1+4p^{2}k(k-1)}}{2}. \)

But we are expecting that the following conjectures are true.
Conjecture 10. \( c(D_p \oplus D_q) = p + q - 1 + \left\lfloor \frac{1+\sqrt{1+8pq}}{2} \right\rfloor. \)

Conjecture 11. \( c(kD_p) = kp + \left\lfloor -\frac{1+\sqrt{1+4p^2k(k-1)}}{2} \right\rfloor. \)

In the paper [3] the following upper bound is found
\[
c(D_{k_1} \oplus D_{k_2} \oplus \ldots \oplus D_{k_n}) \leq 2 \sum_{i=1}^{n} k_i - 1.
\]

In [9] has been proved that \( \mathcal{P} \oplus \mathcal{Q}^k = (\mathcal{P} \oplus \mathcal{Q})^k \). From this we have the following equality.

Corollary 12. \( c(O^2 \oplus \mathcal{P}) = 2c(\mathcal{P}) + 1. \)

Proposition 13. \( c(kL \mathcal{F}) = c(kD_1) = 2k - 1. \)

**Proof.** Beineke [1] proved that a complete graph \( K_{2k} \) can be decomposed into \( k \) spanning paths. Hence \( c(kL \mathcal{F}) \geq 2k - 1 \). Because \( |E(K_{2k+1})| > |E(G)| \), for any graph \( G \in kD_1 \), then \( c(kD_1) \leq 2k - 1 \). This establishes the formula \( c(kL \mathcal{F}) = c(kD_1) = 2k - 1. \)

Theorem 14. \( c(2I_1 \oplus \mathcal{P}) \geq 5c(\mathcal{P}) + 4. \)

Theorem 15. Let \( \mathcal{P}, \mathcal{Q} \) be nontrivial additive hereditary properties. Then \( c(\mathcal{P} \oplus \mathcal{Q}) = 1 \) if and only if \( \mathcal{P} = \mathcal{O}_1 \) and \( \mathcal{Q} = \mathcal{O}_1 \).

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**References**


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