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UNIQUELY PARTITIONABLE PLANAR GRAPHS WITH RESPECT TO PROPERTIES HAVING A FORBIDDEN TREE

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Abstract

Let $\mathcal{P}_1$, $\mathcal{P}_2$ be graph properties. A vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of a graph G is a partition $\{V_1, V_2\}$ of $V(G)$ such that for $i = 1, 2$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is defined to be the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2$ is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable if G has exactly one vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. In this note, we show the existence of uniquely partitionable planar graphs with respect to hereditary additive properties having a forbidden tree.

Keywords: uniquely partitionable planar graphs, forbidden graphs.

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1. Introduction

Let us denote by $\mathcal{I}$ the class of all finite undirected graphs without loops and multiple edges. If $\mathcal{P}$ is a proper isomorphism closed subclass of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$. We shall use the terms *set of graphs* and *property of graphs* interchangeably.

A property $\mathcal{P}$ is said to be *hereditary* if, whenever $G \in \mathcal{P}$ and H is a subgraph of G, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have the property $\mathcal{P}$ it follows that $G \in \mathcal{P}$, too.

For every hereditary property $\mathcal{P}$ there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$ called the *completeness* of $\mathcal{P}$. For example $c(\mathcal{O}) = 0$, $c(\mathcal{D}_1) = 1$, $c(\mathcal{T}_2) = 2$, $c(\mathcal{T}_3) = 3$, where $\mathcal{O}$ is the class of all totally disconnected graphs, $\mathcal{D}_1$ is the class of acyclic graphs, $\mathcal{T}_2$ is the class of outerplanar graphs and $\mathcal{T}_3$ is the class of planar graphs.

Any hereditary property $\mathcal{P}$ is uniquely determined by the set

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} | G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}$$

of its minimal forbidden subgraphs.

Let $\mathcal{P}_1, \mathcal{P}_2$ be arbitrary hereditary properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2)$*-partition* of a graph G is a partition $\{V_1, V_2\}$ of $V(G)$ such that for $i = 1, 2$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$.

A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is defined to be the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is additive and hereditary, too.

A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2$ is said to be *uniquely* $(\mathcal{P}_1, \mathcal{P}_2)$*-partitionable* if G has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. For the concept of uniquely partitionable graphs we refer the reader to [1]. Basic properties of uniquely partitionable graphs are discussed in [1] and [4].

Proposition 1 [1]**.** *Let $\mathcal{P}$ be an additive hereditary property. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable graph G if and only if $\mathcal{P} \neq \mathcal{O} \circ \mathcal{Q}$.*

The proof used non-planar graphs. The constructions of uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar and planar graphs were presented in [2]. The following results have been proved:

Proposition 2 [2]**.** *Let $\mathcal{P}$ be an additive hereditary property of completeness* 1*. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar graph G if and only if there is a tree T which is forbidden for $\mathcal{P}$.*

Proposition 3 [2]**.** *Let $\mathcal{P}$ be an additive hereditary property of completeness* 1*. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graph G if and only if either some odd cycle C_{2q+1} has property $\mathcal{P}$ or there is a bipartite planar graph H which is forbidden for $\mathcal{P}$.*

Our first result shows that the restriction on the completeness is not necessary for the existence of uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graphs.

Theorem 1. *Let $\mathcal{P}$ be an additive hereditary property. If there is a tree $T \in \boldsymbol{F}(\mathcal{P})$, then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graph.*

Furthermore, let us consider $(\mathcal{D}_1, \mathcal{D}_1)$-partitions of planar graphs. The following result is presented in [3]:

Proposition 4 [3]**.** *There are no uniquely $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable planar graph.*

In this note, we shall show that the property $\mathcal{D}_1 \circ \mathcal{D}_1$ is in some sence "a minimal property" having no uniquely partitionable planar graphs. More precisely, we will prove the following result:

Theorem 2. *Let $\mathcal{P}$, $\mathcal{Q}$ be the additive hereditary properties of graphs with completeness* 1*. If there is a tree $T \in \boldsymbol{F}(\mathcal{P})$, then there exists a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable planar graph.*

2. Proofs of the Main Results

***Proof of Theorem* 1.** Let T be a forbidden tree for a property $\mathcal{P}$. As every connected bipartite planar graph is uniquely $(\mathcal{O}, \mathcal{O})$-partitionable, we can assume that T has at least 3 vertices. Then T contains a path wuv_1, where v_1 is an end vertex of T. Denote by $T^{'}$ the graph which we obtain from T by adding the edge wv_1. $T^{'}$ is outerplanar and so the join $K_1 + T'$ is a planar graph. Let $G(T, 1)$ be the graph which we obtain from $K_1 + T^{'}$ by deleting the edge av_1, where a denotes the vertex of K_1. Evidently, $G(T, 1)$ may be embedded on the plane such that the vertices a and v_1 lie in the exterior face (see Figure 1).

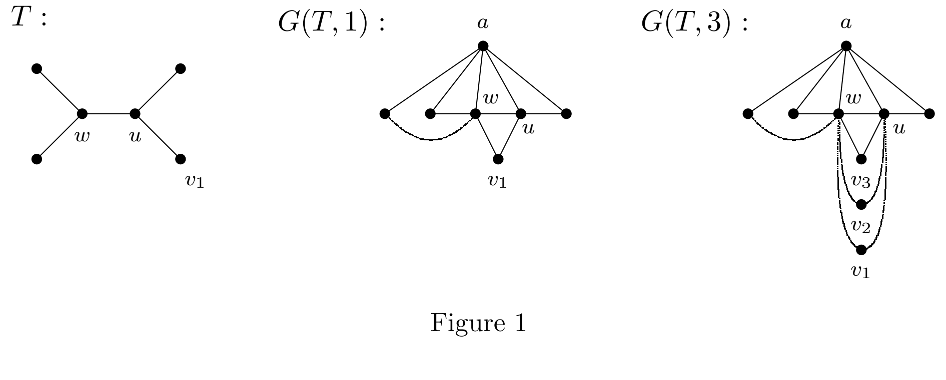
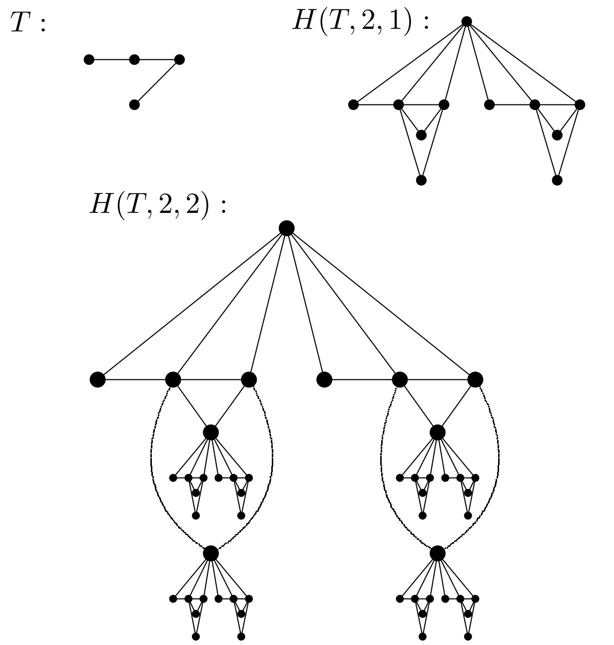


Figure 1

$G(T,k)$, for $k>1$, is a planar graph which we obtain from $G(T,1)$ by adding the vertices $v_2, v_3, \dots, v_k$ and edges $uv_2, uv_3, \dots, uv_k, wv_2, wv_3, \dots wv_k$. The vertex a is called the *root* of $G(T,k)$ and vertices $v_1, v_2, \dots, v_k$, are called *leaves* of $G(T,k)$. Moreover, for every leaf v_i we define its successor $s(v_i)$ by $s(v_1) = a$ and $s(v_i) = v_{i-1}$, if $i = 2, 3, \dots, k$. Obviously, $G(T,k)$ may be embedded on the plane such that both vertices v_i and $s(v_i)$ lie in a common face (see Figure 1).

Now we construct a planar graph $H(T,k,d)$ using the induction on d. $H(T,k,1)$ is a graph which we obtain from k copies of $G(T,k)$ by identifying their roots. The vertex arisen by the identification is called the *root* of $H(T,k,1)$. The *leaves* of copies of $G(T,k)$ are leaves of $H(T,k,1)$. Similary, the successor of a leaf in $H(T,k,1)$ is equal to the successor of this leaf in the corresponding copy of $G(T,k)$. For $d>1$, $H(T,k,d)$ is a planar graph which we obtain from $H(T,k,1)$ and k^2 copies of $H(T,k,d-1)$ by identifying each leaf of $H(T,k,1)$ with the root of a copy of $H(T,k,d-1)$. Evidently, a copy of $H(T,k,d-1)$ can be inserted into a face of $H(T,k,1)$ which contains a corresponding leaf x of $H(T,k,1)$ and its successor $s_1(x)$ in $H(T,k,1)$ (see Figure 2).



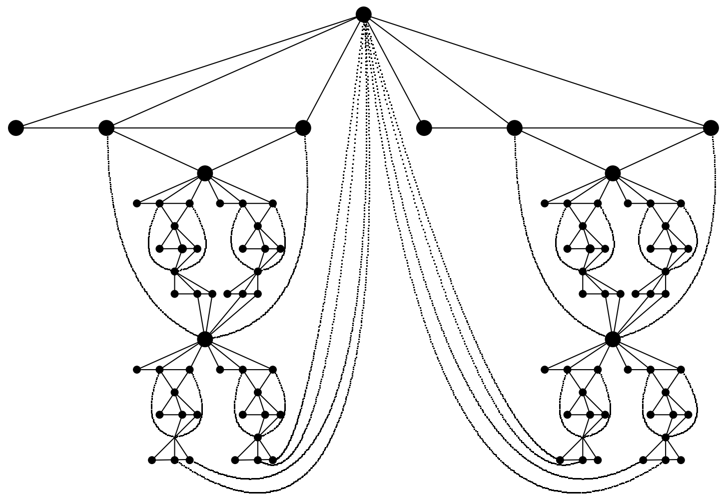


Figure 2

The root of $H(T,k,d)$ is the root of $H(T,k,1)$ and the leaves of $H(T,k,d)$ are leaves of copies of $H(T,k,d-1)$. Denote by $s_d(y)$ and $s_{d-1}(y)$ the successor of a leaf y in $H(T,k,d)$ and in a corresponding copy of $H(T,k,d-1)$. Then

$$s_d(y) = \begin{cases} s_1(x), & \text{if } s_{d-1}(y) \text{ was identified with } x, \\ s_{d-1}(y), & \text{otherwise.} \end{cases}$$

Finaly, $H^*(T,k,d)$ is a planar graph which we obtain from $H(T,k,d)$ such that we connect each leaf of $H(T,k,d)$ with its successor by a copy of $G(T,1)$ identifying the leaf with the root of $G(T,1)$ and the successor with the leaf of $G(T,1)$ (see Figure 2).

Put $V_1 = \{x \in V(H^*(T,k,d)) | d(r,x) \equiv 0 \pmod 2\}$, where r denotes the root of $H(T,k,d)$ and $d(y,z)$ is the length of the shortest path between y and z in $H(T,k,d)$. The vertices belonging to V_1 are depicted by white in Figure 2. It is easy to see that V_1 is an independent set of $H^*(T,k,d)$. Moreover, the set $V_2 = V(H^*(T,k,d)) - V_1$ induces a subgraph of $H^*(T,k,d)$ each of whose components is isomorphic to $T - v_1$. So, $\{V_1, V_2\}$ is a vertex $(\mathcal{O},\mathcal{P})$-partition of $H^*(T,k,d)$.

Suppose that $\{U_1, U_2\}$ is a vertex $(\mathcal{O},\mathcal{P})$-partition of $H^*(T,k,d)$. Consider two cases:

Case 1. $U_1 \cap V_1 \neq \emptyset$. Let $x \in U_1 \cap V_1$ and let y be any vertex of $V_1 - \{x\}$. From the construction of $H^*(T,k,d)$ it can easily be seen that there exists a sequence $x = x_1, x_2, \ldots, x_t = y$ satisfying: For every $i = 1, \ldots, t-1$, there is a subgraph G_i of $H^*(T,k,d)$ isomorphic to $G(T,k)$ (or $G(T,1)$), where x_i is its root and x_{i+1} is its leaf. As x_1 belongs to U_1, all vertices of G_1 adjacent to x_1 belong to U_2. However, these neighbours of x_1 together with x_2 induce a subgraph of G_1 containing T. Therefore, $x_2 \in U_1$, and by induction, $y \in U_1$. Since y is any vertex of $V_1 - \{x\}$, $V_1 \subseteq U_1$. The set V_1 is a domination set of $H^*(T,k,d)$, and so, $V_1 = U_1$, i.e., $\{U_1, U_2\} = \{V_1, V_2\}$.

Case 2. $V_1 \subseteq U_2$. It is easy to see that every block of $H(T,k,d)$ is a copy of $G(T,k)$, where the root and leaves of the copy belong to V_1. As the vertices of a block corresponding to u and w are adjacent, at least one of them belongs to U_2. Thus, vertices of a block belonging to U_2 induce a graph containing a star $K_{1,k+1}$. From the construction of $H(T,k,d)$ one can see that vertices of $H(T,k,d)$ belonging to U_2 induce a graph containing a complete k-ary tree with $2d+1$ levels. Therefore, for $k \geq \Delta(T)$ and $d \geq \frac{1}{2}rad(T)$, $H^*(T,k,d)[U_2]$ contains a subgraph isomorphic to T, a contradiction. Thus, for $k \geq \Delta(T)$ and $d \geq \frac{1}{2}rad(T)$, the graph $H^*(T,k,d)$ is uniquely $(\mathcal{O},\mathcal{P})$-partitionable. ■

***Proof of Theorem* 2.** To construct the planar graph $H_r(s)$, for $r \geq 1$, $s \geq 2$ we will use the induction on r. The first step is the construction of planar graph $H_1(s)$:

$H_1(s) = K_2 + \cup_{i=1}^{s} K_2$, where $V(K_2) = \{x_1, x_2\}$ and $V(\cup_{i=1}^{s} K_2) = \{y_{1i}, y_{2i} \mid i = 1, 2, \ldots, s\}$. The edge x_1x_2 of $H_1(s)$ we will call the "*major*" edge of $H_1(s)$ and edges $y_{1i}y_{2i}$, $i = 1, 2, \ldots, s$ we will call "*minor*" edges of $H_1(s)$. For the construction of $H_1(3)$ see Figure 3.

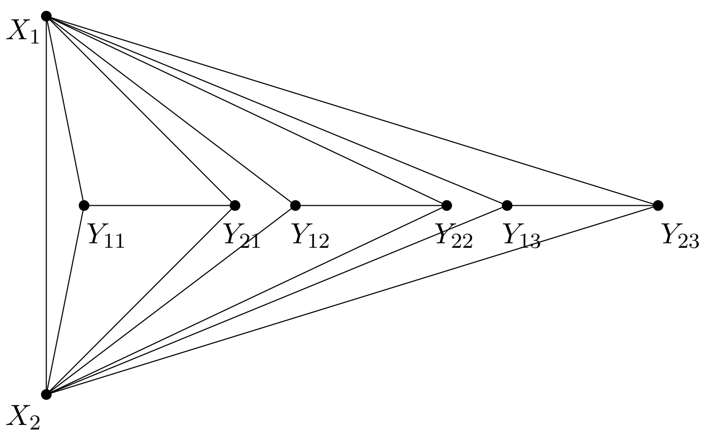


Figure 3. The graph $H_1(3)$

Let us construct the graph $H_{k+1}(s)$ in the following way:

We insert s copies of graph $H_k(s)$ to graph $H_1(s)$ such that we identify the "*major*" edges of copies of graphs $H_k(s)$ with "*minor*" edges of $H_1(s)$. For the construction of $H_2(3)$ see Figure 4.

It is easy to see from the construction, that $H_r(s)$ is a planar graph. Now we shall show, that if the maximum degree $\Delta(T)$ of the tree $T \in \boldsymbol{F}(\mathcal{P})$ is $\Delta(T) \leq s$ and radius $rad(T)$ of the tree T is $rad(T) \leq r$, then the planar graph $H_r(s)$ is uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable.

Let us distinquish two "possible" vertex partitions of the graph $H_r(s)$:
1. The end vertices x_1, x_2 of "*major*" edge of $H_1(s)$ belong to different classes of the vertex partition. From the fact that K_3 is forbidden for both properties $\mathcal{P}, \mathcal{Q}$, it follows that vertices of "*minor*" edges of $H_1(s)$ belong to different classes of the vertex partition, too. By induction on r in both classes of the partition, it grove the complete s-ary tree with $1 + r$ levels, which is, for $r \geq rad(T)$ and $s \geq \Delta(T)$, a supergraph of the forbidden tree T. It means, that it is not a $(\mathcal{P}, \mathcal{Q})$-partition of $H_r(s)$.

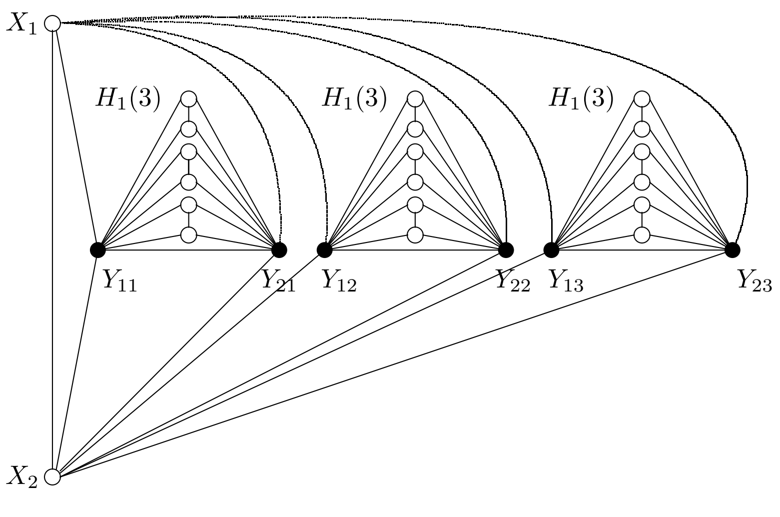


Figure 4. The graph $H_2(3)$

2. Hence the end vertices x_1, x_2 of "*major*" edge of the graph $H_1(s)$ have to belong to the same class of a vertex partition. From the fact that K_3 is forbidden for both properties $\mathcal{P}, \mathcal{Q}$, it follows, that vertices of "*minor*" edges of $H_1(s)$ have both to belong to the second class of the vertex partition. From

the construction of $H_r(s)$ and from the fact that K_3 is forbidden it is easy to see that the partition of $H_r(s)$ is a $(\mathcal{P}, \mathcal{Q})$-partition of $H_r(s)$. Thus $H_r(s)$, for $r \geq rad(T)$ and $s \geq \Delta(T)$ is a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph. ■

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