

## SHORT CYCLES OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5

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### Abstract

In this note, precise upper bounds are determined for the minimum degree-sum of the vertices of a 4-cycle and a 5-cycle in a plane triangulation with minimum degree 5:  $w(C_4) \leq 25$  and  $w(C_5) \leq 30$ . These hold because a normal plane map with minimum degree 5 must contain a 4-star with  $w(K_{1,4}) \leq 30$ . These results answer a question posed by Kotzig in 1979 and recent questions of Jendrol' and Madaras.

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The *weight* of a subgraph in a plane map  $M$  is the sum of the degrees (in  $M$ ) of its vertices. By  $w(S)$ , we denote the minimum weight of a subgraph isomorphic to  $S$  in  $M$ . By  $M_5$  or  $T_5$  we mean a connected plane map with minimum degree 5 and each face having size at least 3 (that is, a normal plane map) or exactly 3 (that is, a triangulation), respectively. As conjectured by Kotzig [4] for each  $T_5$  and proved in [1] for each  $M_5$ ,  $w(C_3) \leq 17$ , and this bound is precise. Also, Kotzig [5] announced that  $25 \leq w(C_4) \leq 26$  for each  $T_5$ . Jendrol' and Madaras [3] proved that  $w(C_4) \leq 35$ ,

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$w(C_5) \leq 45$  and  $w(K_{1,4}) \leq 39$  for each  $T_5$  and  $w(K_{1,3}) \leq 23$ , which bound is best possible, and  $w(K_{1,4}) \leq 45$  for each  $M_5$ .

Our main result is:

**Theorem 1.** *Each normal plane map with minimum degree 5 contains a 4-star with weight at most 30 with a 5-vertex as its centre.*

This clearly implies:

**Corollary 2.** *Each plane triangulation with minimum degree 5 contains a 4-cycle with weight at most 25 and a 5-cycle with weight at most 30.*

The bounds in Theorem 1 and Corollary 2 are all precise, as the following examples show. Take any polyhedron in which every vertex is of type  $5.6^2$  or  $6^3$ , such as the Archimedean solid in which every vertex is incident with a 5-face and two 6-faces. Truncate all the vertices to obtain a graph in which every vertex has type  $3.10.12$  or  $3.12^2$ . Cap each 10-face and 12-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with minimum degree 5 in which the neighbours of every 5-vertex  $v$  have degrees (in cyclic order round  $v$ )  $(5, 5, 10, 5, 12)$  or  $(5, 5, 12, 5, 12)$ . This graph clearly has  $w(C_4) = 25$  and  $w(C_5) = w(K_{1,4}) = 30$ .

It follows that our results above completely solve the problems raised by Kotzig [5] and Jendrol' and Madaras [3]. In the proof below, we use some ideas from our unpublished manuscript [2].

We shall use the following terminology. The number of edges incident with a vertex  $v$  or  $r(f)$  respectively, and  $v_1, \dots, v_{d(v)}$  denote the neighbours of  $v$ , in cyclic order round  $v$ . If  $d(v_i) = 5$  then  $v_i$  is a *strong*, *semiweak* or *weak* neighbour of  $v$  according as none, one or both of  $v_{i-1}, v_{i+1}$  have degree 5, and  $v_i$  is *twice weak* if  $d(v_j) = 5$  whenever  $|j-i| \leq 2$  (modulo  $d(v)$ ). A *k-vertex* is a vertex  $v$  with  $d(v) = k$ , and a *>k-vertex* has  $d(v) > k$ , etc.

**Proof of Theorem 1.** It suffices to prove the theorem for triangulations, since adding an extra edge to a normal plane map with minimum degree 5 cannot create a new 4-star with a 5-vertex as its centre, nor can it reduce the weight of any existing 4-star. So suppose that  $G = (V, E, F)$  is a triangulation that is a counterexample to Theorem 1. Since  $G$  is a triangulation,  $2|E| = 3|F|$ , and so Euler's formula  $|V| - |E| + |F| = 2$  implies

$$(1) \quad \sum_{v \in V} (d(v) - 6) = -12.$$

Assign a charge  $\mu(v) = d(v) - 6$  to each vertex  $v \in V$ , so that only 5-vertices have negative charge. Using the properties of  $G$  as a counterexample, we define a local redistribution of charges, preserving their sum, such that the new charge  $\mu'(v)$  is non-negative for all  $v \in V$ . This will contradict the fact that the sum of the new charges is, by (1), equal to  $-12$ . The technique of discharging is often used in solving structural and colouring problems on plane graphs.

Our discharging rules are as follows.

**Rule 1.** (a) Each vertex  $v$  of degree 7 sends  $\frac{1}{3}$  to each strong neighbour and  $\frac{1}{6}$  to each semiweak neighbour.

(b) Each vertex  $v$  with degree 8, 9 or  $\geq 12$  first gives a “basic” contribution of  $\frac{\mu(v)}{d(v)} = \frac{d(v)-6}{d(v)}$  to each neighbouring vertex  $v_i$ . Then each neighbour  $v_i$  with  $d(v_i) > 5$  shares the charge just received equally between  $v_{i-1}$  and  $v_{i+1}$ .

(c) Each 10-vertex or 11-vertex  $v$  first gives a “basic”  $\frac{2}{5}$  to each neighbour. Then, whenever  $d(v_i) > 5$ ,  $v_i$  transfers  $\frac{1}{10}$  of  $v$ ’s donation to each 5-vertex in  $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$ .

**Rule 2.** If  $d(v) = 11$  then  $v$  gives a “supplementary”  $\frac{1}{10}$  to each twice weak neighbour.

**Rule 3.** If  $v$  is 5-vertex adjacent to an 11-vertex  $w$ , say  $w = v_5$ , and if  $d(v_1) = d(v_4) = 5$ , then  $v$  gives back to  $v_5$  the following:

- (a)  $\frac{1}{2}$  if both  $v_2$  and  $v_3$  have degree  $\geq 9$ ;
- (b)  $\frac{1}{4}$  if at least one of  $v_2, v_3$  has degree exactly 8.

We must prove that  $\mu'(v) \geq 0$  for each vertex  $v$ . If  $d(v) \notin \{5, 7, 11\}$ , then, by Rule 1 (b) and (c),  $v$  distributes its own original charge of  $\mu(v) = d(v) - 6$  to its neighbours in equal shares, and possibly participates in transferring the others’ charges, so that  $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{d(v)-6}{d(v)} = 0$ . We deal with the remaining values of  $d(v)$  in three cases.

*Case 1.*  $d(v) = 11$ . Then  $\mu(v) = d(v) - 6 = 5$ . If  $v$  has a neighbour  $v_i$  with  $d(v_i) \geq 6$ , then none of  $v_{i-2}, \dots, v_{i+2}$  is twice weak and so none of them receives a supplementary  $\frac{1}{10}$  from  $v$  by Rule 2. Thus  $\mu'(v) \geq 5 - 11 \times \frac{2}{5} - 6 \times \frac{1}{10} = 0$ . So we may assume that all neighbours of  $v$  have degree 5.

Each edge  $v_i v_{i+1}$  lies in two triangles, say  $v_i v_{i+1} v$  and  $v_i v_{i+1} w_i$ . If  $d(w_i) = 8$  for some  $i$ , then  $v$  receives  $\frac{1}{4}$  by Rule 3(b) from each of  $v_i$  and

$v_{i+1}$ , so that  $\mu'(v) \geq 5 + 2 \times \frac{1}{4} - 11 \times \frac{1}{2} = 0$ . So we may assume that  $d(w_i) \neq 8$ , for each  $i$ .

If  $d(w_{i-1}) \geq 9$  and  $d(w_i) \geq 9$  for some  $i$ , then  $v_i$  gives back  $\frac{1}{2}$  to  $v$  by Rule 3(a), and we are done. Also, it is impossible that  $d(w_{i-1}) \leq 7$  and  $d(w_i) \leq 7$  for any  $i$ , since by hypothesis there is no 4-star with weight  $\leq 30$  centered at  $v_i$ . Therefore, for each  $i$ , one of  $d(w_{i-1})$  and  $d(w_i)$  is at most 7 and the other is at least 9. But this cannot hold for all  $i$  modulo 11, since 11 is odd.

*Case 2.*  $d(v) = 7$ . Then  $\mu(v) = d(v) - 6 = 1$ . By Rule 1(a), no weak neighbour receives anything from  $v$ , and so there are at most four receivers. If there are exactly four, then at least two are semiweak and so receive  $\frac{1}{6}$  each, with a total expenditure by  $v$  of at most  $2 \times \frac{1}{6} + 2 \times \frac{1}{3} = 1$ . Otherwise,  $v$  gives at most  $3 \times \frac{1}{3} = 1$ .

neighbour:	strong	semiweak	weak
7:	1/3	1/6	0
8:	1/2	3/8	1/4
9:	2/3	1/2	1/3
10:	$\geq 3/5$	$\geq 1/2$	$\geq 2/5$
11:	$\geq 3/5$	$\geq 1/2$	1/2
$\geq 12$ :	$\geq 1$	$\geq 3/4$	$\geq 1/2$

Table 1. Donations to 5-vertices by Rules 1 and 2

*Case 3.*  $d(v) = 5$ . Then  $\mu(v) = d(v) - 6 = -1$ . The amounts of charge received by  $v$  from its neighbours by Rules 1 and 2 are summarized in Table 1. However,  $v$  may give back charge to some 11-vertices by Rule 3.

Suppose Rule 3(a) applies to  $v$ , so that  $v$ 's neighbours  $v_1, \dots, v_5$  have degrees  $(5, \geq 9, \geq 9, 5, 11)$ . Then  $v$  is a semiweak neighbour of each of  $v_2$  and  $v_3$ , so that it receives at least  $\frac{1}{2}$  from each of them by Table 1, and gives nothing back to either of them by Rule 3. It also receives at least  $\frac{1}{2}$  from  $v_5$  by Table 1, and gives back exactly  $\frac{1}{2}$  to  $v_5$  by Rule 3(a). We deduce that  $\mu'(v) \geq 0$ .

From now on, we may assume that Rule 3(a) does not apply to  $v$ . Suppose Rule 3(b) applies. Because  $v$  is not the centre of a 4-star with weight  $\leq 30$ ,  $v$ 's neighbours have degrees  $(5, 8, \geq 8, 5, 11)$ . Thus  $v$  is a semiweak neighbour of  $v_2$  and  $v_3$  and so it receives at least  $\frac{3}{8}$  from each of them by Table 1, and gives nothing back. It also receives at least  $\frac{1}{2}$  from  $v_5$  by Table 1, and gives  $\frac{1}{4}$  back. Thus  $\mu'(v) \geq 0$ .

So we may suppose that Rule 3 does not apply to  $v$  at all, and the amount that  $v$  receives from its neighbours is at least that given in Table 1. Because of the absence of 4-stars with weight  $\leq 30$ , the degree-sequence of  $v$ 's neighbours, in nondecreasing order, must be one of the following.

(5, 5, 5,  $\geq 11$ ,  $\geq 11$ ): Then each  $\geq 11$ -vertex gives  $\geq \frac{1}{2}$  to  $v$  by Table 1.

(5, 5, 6,  $\geq 10$ ,  $\geq 10$ ): If each of the two  $\geq 10$ -neighbours gives  $\geq \frac{1}{2}$  to  $v$ , we are done.

Suppose there is a 10-vertex, say  $v_1$ , giving  $\frac{2}{5}$  to  $v$ . Then  $v$  must be a twice weak neighbour of  $v_1$  by Rule 1(c). W.l.o.g., suppose that  $d(v_2) = d(v_5) = 5$  and  $d(v_3) = 6$ . If  $d(v_4) \geq 12$  then  $v_4$  gives  $\geq \frac{3}{4}$  to  $v$  by Table 1, so that  $\mu'(v) \geq -1 + \frac{2}{5} + \frac{3}{4} > 0$ . So we may assume  $10 \leq d(v_4) \leq 11$ ; note that  $v$  is not a twice weak neighbour of  $v_4$ . Let  $u$  be the vertex (other than  $v$ ) adjacent to  $v_4$  and  $v_5$ . Since  $v_5$  has two 5-neighbours other than  $u$  (because  $v$  is a twice weak neighbour of  $v_1$ ), and also has a 10-neighbour  $v_1$ , it follows that  $d(u) > 5$ . Then Rule 1(c) ensures that  $v$  receives  $\frac{1}{10}$  from  $v_4$  via each of  $v_3$  and  $u$ , so that  $\mu'(v) \geq -1 + 2 \times \frac{2}{5} + 2 \times \frac{1}{10} = 0$ .

(5, 5, 7,  $\geq 9$ ,  $\geq 9$ ): If  $v$  is weak for neither of the  $\geq 9$ -neighbours then each of them gives  $\geq \frac{1}{2}$ , and we are done by Table 1. Otherwise,  $v$  is weak for a  $\geq 9$ -neighbour, giving  $\geq \frac{1}{3}$ , and semiweak for the other two neighbours of degree 7 and  $\geq 9$ , giving  $\geq \frac{1}{6} + \frac{1}{2}$  in total.

(5, 5,  $\geq 8$ ,  $\geq 8$ ,  $\geq 8$ ): If  $v$  is weak for none of the three  $\geq 8$ -neighbours, then  $v$  receives  $\geq 3 \times \frac{3}{8} > 1$  in total. Otherwise,  $v$  is weak for one of them and semiweak for the other two, so that receives  $\geq \frac{1}{4} + 2 \times \frac{3}{8} = 1$  in total.

(5, 6, 6,  $\geq 9$ ,  $\geq 9$ ): Each  $\geq 9$ -neighbour gives  $\geq \frac{1}{2}$ .

(5, 6,  $\geq 7$ ,  $\geq 8$ ,  $\geq 8$ ): For each of the three  $\geq 7$ -neighbours,  $v$  is semiweak or strong; for at least one of them,  $v$  is strong. By Table 1,  $v$  thus receives either  $\geq \frac{1}{6} + \frac{3}{8} + \frac{1}{2} > 1$  or  $\geq \frac{3}{8} + \frac{3}{8} + \frac{1}{3} > 1$ .

(5,  $\geq 7$ ,  $\geq 7$ ,  $\geq 7$ ,  $\geq 7$ ): For at least two  $\geq 7$ -neighbours,  $v$  is strong; for the others, semiweak. Thus,  $\mu'(v) \geq -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0$ .

( $\geq 6$ ,  $\geq 6$ ,  $\geq 6$ ,  $\geq 8$ ,  $\geq 8$ ):  $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$ .

( $\geq 6$ ,  $\geq 6$ ,  $\geq 7$ ,  $\geq 7$ ,  $\geq 7$ ):  $\mu'(v) \geq -1 + 3 \times \frac{1}{3} = 0$ .

Thus we have proved  $\mu'(v) \geq 0$  for every  $v \in V$  and  $f \in F$ , which contradicts (1) and completes the proof of Theorem 1. ■

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