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THE CHROMATICITY OF A FAMILY OF 2-CONNECTED 3-CHROMATIC GRAPHS WITH FIVE TRIANGLES AND CYCLOMATIC NUMBER SIX

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Abstract

In this note, all chromatic equivalence classes for 2-connected 3-chromatic graphs with five triangles and cyclomatic number six are described. New families of chromatically unique graphs of order n are presented for each $n \geq 8$. This is a generalization of a result stated in [5]. Moreover, a proof for the conjecture posed in [5] is given.

Keywords: chromatically equivalent graphs, chromatic polynomial, chromatically unique graphs, cyclomatic number.

1991 Mathematics Subject Classification: 05C15.

1. Introduction

The graphs which we consider here are finite, undirected, simple and loopless. Let G be a graph, $V(G)$ its vertex set, $E(G)$ its edge set, $\chi(G)$ its chromatic number and $P(G, \lambda)$ its chromatic polynomial. Two graphs G and H are said to be *chromatically equivalent*, or in short χ-equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is said to be *chromatically unique*, or in short χ-unique, if for any graph H satisfying $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G. A family of all nonisomorphic chromatically equivalent graphs is called a χ-equivalence class.

A *wheel* W_n is a graph of order n, $n \geq 4$, obtained by the join of K_1 and C_{n-1}. Any edge incident with the central vertex in W_n is called a *spoke* of the wheel. For any two integers n, k with $n \geq 4$ and $n - 1 \geq k \geq 1$, let $W(n, k)$ denote the graph of order n obtained from a wheel W_n by deleting all but k consecutive spokes. It is known that the graphs $W(n,1)(n \geq 4)$ and $W(n,2)$ $(n \geq 4)$ are χ-unique. Chao and Whitehead [1] showed that the graphs $W(n,3)$ $(n \geq 5)$ and $W(n,4)$ $(n \geq 6)$ are χ-unique, while $W(7,5)$ is not. Then Koch and Teo [3] showed that all graphs $W(n,5)(n \geq 8)$ are χ-unique. Recently Li and Whitehead [5] showed that all graphs $W(n,6)$ $(n \geq 8)$ are χ-unique. This is a solution to one of the problems stated in [2] (see Problem 2 [2]). They also decribed two additional families of chromatically unique graphs. The family of graphs they studied consists of 2-connected 3-chromatic graphs with five triangles and cyclomatic number six. In this paper, all classes of χ-equivalent graphs of order at least 8 for this family are described. In particular, a complete characterization of chromatically unique graphs for the family is presented. Also a proof for the conjecture posed in [5] is given.

2. Known Results

In computing chromatic polynomials, we make use of Whitney's reduction formula given in [6]. The formula is

$$P(G,\lambda) = P(G_{-e},\lambda) - P(G/_e,\lambda)$$

or equivalently

$$P(G_{-e},\lambda) = P(G,\lambda) + P(G/_e,\lambda)$$

where G_{-e} is the graph obtained from G by deleting an edge e and $G/_e$ is the graph obtained from G by contracting the edge e.

We also make use of the overlaping formula given in [6]. The formula is

$$P(G,\lambda) = P(H,\lambda)P(F,\lambda)/P(K_p,\lambda)$$

where G is a gluing of two disjoint graphs H and F over their complete subgraph K_p for $p \geq 1$.

Moreover, we shall use the known results for χ-equivalent graphs presented in Lemma 1. For a graph F, let $I_G(F)$ denote the number of induced subgraphs of G which are isomorphic to F.

Lemma 1 [3]. *Let G and H be two χ-equivalent graphs. Then*

(i) $\mid V(G) \mid = \mid V(H) \mid$,

(ii) $\mid E(G) \mid = \mid E(H) \mid$,

(iii) $\chi(G) = \chi(H)$,

(iv) $I_G(C_3) = I_H(C_3)$,

(v) $I_G(C_4) - 2I_G(K_4) = I_H(C_4) - 2I_H(K_4)$,
(vi) *G is connected iff H is connected,*
(vii) *G is 2-connected iff H is 2-connected.*

3. Results

Next we consider the following 2-connected pairwise nonisomorphic graphs $X_i(n)$, shortly denoted by X_i, each of order n, $n \geq 8$, presented in Figure 1.

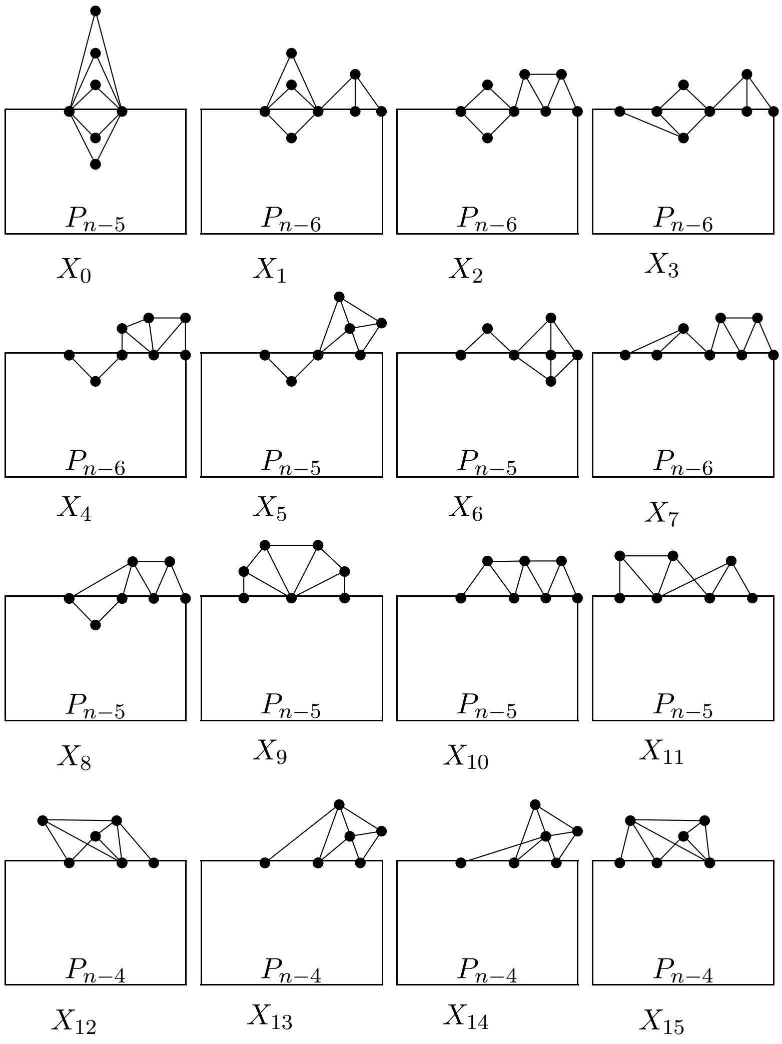


Figure 1

Thin lines denote here paths, filled circle — vertices, and bold lines — edges of a graph. Checking the degree sequences of these graphs one can easy note that they are pairwise nonisomorphic.

First we prove the following lemma.

Lemma 2. *$X_{15} \sim X_{14}$, and $X_i \not\sim X_j$ for other pairs $i, j = 0, ..., 13$ and $i \neq j$.*

Proof. By using Whitney's reduction formula we have:
$P(X_0, \lambda) = (\lambda - 2)^5 P(C_{n-5}, \lambda)$,
$P(X_1, \lambda) = (\lambda - 2)^4 (P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda))$,
$P(X_2, \lambda) = (\lambda - 2)^3 ((\lambda - 3)P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda))$,
$P(X_3, \lambda) = (\lambda - 2)^3 (P(C_{n-3}, \lambda) - 2P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda))$,
$P(X_4, \lambda) = (\lambda - 2)^2 ((\lambda^2 - 5\lambda + 7)P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda))$,
$P(X_5, \lambda) = (\lambda - 2)^2 (\lambda^2 - 5\lambda + 7)P(C_{n-4}, \lambda)$,
$P(X_6, \lambda) = (\lambda - 2)^2 ((\lambda - 2)P(C_{n-3}, \lambda) - (2\lambda - 5)P(C_{n-4}, \lambda))$,
$P(X_7, \lambda) = (\lambda - 2)^2 ((\lambda - 3)P(C_{n-3}, \lambda) - (\lambda - 4)P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda))$,
$P(X_8, \lambda) = (\lambda - 2)^2 ((\lambda - 2)P(C_{n-3}, \lambda) - (2\lambda - 5)P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda))$,

and the chromatic polynomials for other graphs G of the lemma are of the following form : $P(G, \lambda) = (\lambda - 1)(\lambda - 2)Q(G, \lambda)$, where the factor $Q(G, \lambda)$ is presented in Table 1 and $(\lambda - 2)^2 \nmid P(G, \lambda)$.

Table 1

G	$Q(G, \lambda)$
X_9	$[(\lambda-2)^3 - (\lambda-2)^2 + (\lambda-2) - 1]\ [(\lambda-1)^{n-5} + (-1)^{n-4}] +$ $+ (\lambda-1)^{n-6} + (-1)^{n-5}$
X_{10}	$(\lambda-2)[(\lambda-1)^{n-3} - 2(\lambda-1)^{n-4} - (\lambda-4)(\lambda-1)^{n-5}$ $-(-1)^n(\lambda-7)] - [(\lambda-1)^{n-5} - (\lambda-1)^{n-6} + 2(-1)^n]$
X_{11}	$(\lambda-2)\{(\lambda-1)^{n-3} - (\lambda-1)^{n-4} + (\lambda-1)^{n-5}$ $+3(-1)^{n-2} - 2(\lambda-2)[(\lambda-1)^{n-5} + (-1)^{n-4}]\}$ $-[(\lambda-1)^{n-5} - (\lambda-1)^{n-6} + 2(-1)^{n-4}]$
X_{12}	$(\lambda-2)[(\lambda-1)^{n-3} + (-1)^{n-2}] - \{(\lambda-3)[(\lambda-1)^{n-4}$ $+(-1)^{n-3}] + (\lambda^2 - 5\lambda + 7)[(\lambda-1)^{n-5} + (-1)^{n-4}]\}$
X_{13}	$(\lambda^2 - 6\lambda + 9)[(\lambda-1)^{n-4} + (-1)^{n-3}]$ $+(2\lambda-5)[(\lambda-1)^{n-5} + (-1)^{n-4}]$
X_{14}, X_{15}	$(\lambda^2 - 5\lambda + 7)[(\lambda-1)^{n-5}(\lambda-2) - 2(-1)^n]$

Since $P(C_n,\lambda) = (\lambda-1)((\lambda-1)^{n-1}+(-1)^n)$, we get the following properties:
$(\lambda-2)^5 \mid P(X_0,\lambda)$;
$(\lambda-2)^4 \mid P(X_1,\lambda)$ and $(\lambda-2)^5 \nmid P(X_1,\lambda)$;
$(\lambda-2)^3 \mid P(X_i,\lambda)$ and $(\lambda-2)^4 \nmid P(X_i,\lambda)$ for $i=2,3$;
$(\lambda-2)^2 \mid P(X_i,\lambda)$ and $(\lambda-2)^3 \nmid P(X_i,\lambda)$ for $4 \le i \le 8$;
$(\lambda-2)^2 \nmid P(X_i,\lambda)$ for $9 \le i \le 15$;

Evidently graphs X_{14}, X_{15} are χ-equivalent. Looking at the above properties and checking the values of chromatic polynomials for $\lambda = 2,3$ or 4, we calculate that other pairs of the graphs are not χ-equivalent. This completes the proof. ■

Theorem 3. *For each $n \ge 8$, a 2-connected 3-chromatic graph of order n with five triangles and cyclomatic number six is χ-equivalent to one of the graphs $X_i(n), i = 0,...,14$ presented in Figure 1.*

Proof. Let R be a 2-connected 3-chromatic graph of order $n \ge 8$ with five triangles and cyclomatic number six.

Suppose that there exists a graph $G \not\cong R$ and such that $G \sim R$.

Lemma 1 implies $\mid V(G) \mid = n, \mid E(G) \mid = n+5, \chi(G) = 3, I_G(K_3) = 5$, $I_G(K_4) = 0$ and G is a 2-connected graph. Let H be a subgraph of G induced by the edges of the five triangles in G, and let $\mid V(H) \mid = h$. So

$$6 \le h \tag{1}$$

Now we define some parameters, which will be useful for the description of all possible candidates for H with h vertices and five triangles, and its supergraph G. Let

$$\begin{aligned}
\alpha &= 2(\mid E(H) \mid -h),\\
\beta' &= \mid \{x \in V(H) \mid d_H(x) = 2 \text{ and } d_G(x) = 3\} \mid,\\
\beta'' &= \mid \{x \in V(H) \mid d_H(x) = 2 \text{ and } d_G(x) \ge 4\} \mid,\\
\gamma &= \mid \{x \in V(H) \mid d_H(x) \ge 3 \text{ and } d_G(x) = d_H(x)+1\} \mid,\\
\gamma' &= \mid \{x \in V(H) \mid d_H(x) \ge 3 \text{ and } d_G(x) \ge d_H(x)+2\} \mid,\\
\delta &= \mid \{x \in V(G)-V(H) \mid d_G(x) \ge 3\} \mid .
\end{aligned} \tag{2}$$

We have

$$\begin{aligned}
2(n+5) \ge \textstyle\sum(d_H(x) \mid \ x \in V(H)) + \beta' + 2\beta'' + 2(n-h) + \gamma + 2\gamma' + \delta\\
= \alpha + \beta' + 2\beta'' + \gamma + 2\gamma' + \delta + 2n\,.
\end{aligned}$$

This implies that

$$\alpha + \beta' + 2\beta'' + \gamma + 2\gamma' + \delta \leq 10\,. \tag{3}$$

Let c be the number of connected components of the graph H. Evidently each connected component contains at least one triangle. Since G is a 2-connected graph and H has five triangles and it does not contain K_4, then the cyclomatic number of H is equal to 5 if H is disconnected, and it is equal to 5 or 6 if H is connected. So by (2) we get

$$\alpha = 10 - 2c \text{ if } c > 1 \tag{4}$$

and

$$\alpha \geq 8 \text{ if } c = 1\,. \tag{5}$$

The list of all possible candidates for H with h vertices and five triangles will be described by considering the following five cases. Three of them are very simple. For the cases 4 and 5 we use the known theorem of Erdös and Gallai on characterization of degree sequences (see [4], Theorem 6.2). All resulting graphs are presented in Figures 2–3 if $H \not\simeq G$ and in Figure 4 if $H \simeq G$.

Case 1. Suppose that $c = 5$, and let $H_i,\ i = 1,2,3,4,5$ be connected components of H. Evidently each of H_i is isomorphic to K_3.

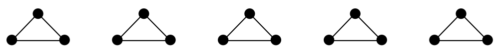
Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 0,\ \beta' = 10$, $\beta'' = 0, \gamma = 0, \gamma' = 0, \delta = 0$.

Case 2. Suppose that $c = 4$, and let $H_i,\ i = 1,2,3,4$ be connected components of H. Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 2, \beta' + \gamma = 8, \beta'' = 0, \gamma' = 0, \delta = 0$. Evidently each of $H_i, i = 1,2,3$ is isomorphic to K_3 and H_4 is isomorphic to $2K_1 + K_2$ or $K_1 + 2K_2$.

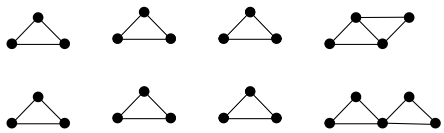
Case 3. Suppose that $c = 3$, and let $H_i,\ i = 1,2,3$ be connected components of H. Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 4, \beta' + \gamma = 6, \beta'' = 0, \gamma' = 0, \delta = 0$. Moreover, if a graph H_i has a cut vertex x, then the graph $H_i - x$ has exactly two connected components.

Thus if two graphs of $H_i, i = 1,2,3$ are isomorphic to K_3 then, the other one is isomorphic to the last graph presented in lines 1–5 of Figure 2 and if exactly one of $H_i, i = 1,2,3$ is isomorphic to K_3, then the other two are isomorphic to a graph $2K_1 + K_2$ or $K_1 + 2K_2$ (see lines 6–8 of Figure 2).

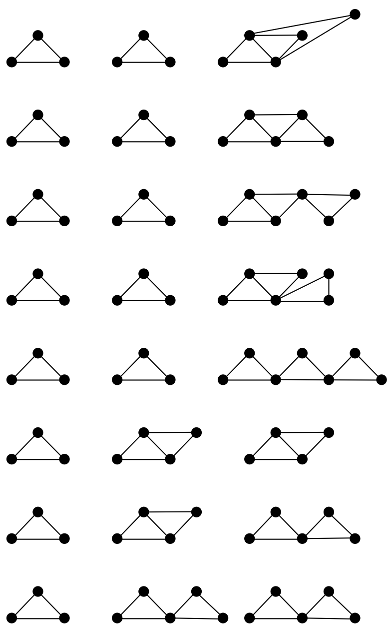
Case 4. Suppose that $c = 2$, and let $H_1,\ H_2$ be connected components of H. Then from (4) $\alpha \geq 6$ and from (3) $\beta' + \gamma = 4, \beta'' = 0, \gamma' = 0, \delta = 0$. So 2-connectivity of G implies that if a graph H_i has a cut vertex x, then the graph $H_i - x$ has two connected components. Thus we have to consider three cases for the second component H_2. Namely, $H_2 \cong K_3, K_1 + 2K_2, 2K_1 + K_2$. Considering simple degree conditions we get all available candidates for H_1 presented in Figure 2 (continued).



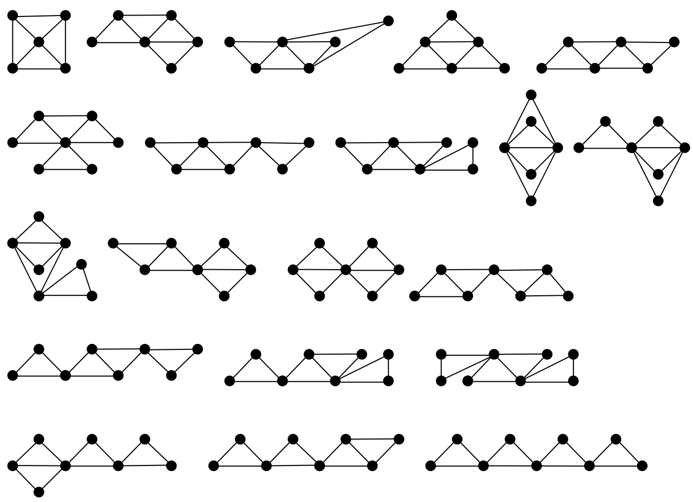
$c = 5$



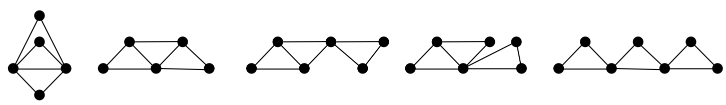
$c = 4$



$c = 3$



$c = 2,\ H_2 = K_3$



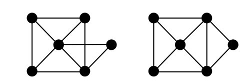
$c = 2,\ H_2 = K_1 + 2K_2$
or $H_2 = 2K_1 + K_2$

Figure 2

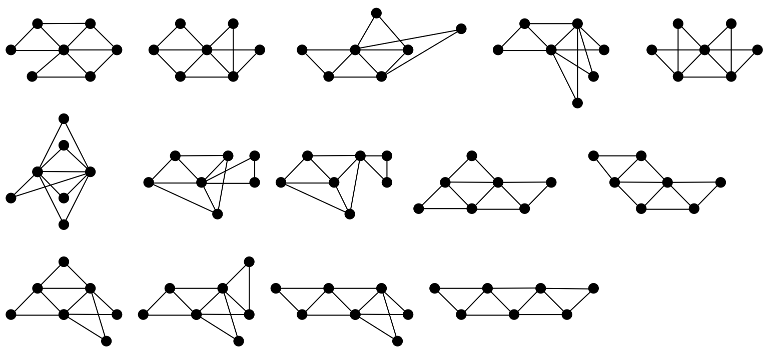
Case 5. Assume that $c = 1$. Then from (2) and (3) $\alpha \geq 8$ and $\beta' + \gamma \leq 2$. Moreover, $\beta'' + \gamma' + \delta = 0$. If $\beta' + \gamma = 0$ we get $H \simeq G$ and Figure 4 lists all such graphs (each edge belongs to a triangle). For the opposite case $\beta' + \gamma = 2$. This follows by 2-connectivity of G. Moreover, 2-connectivity of G implies that if a graph H has a cut vertex x, then the graph $H - x$ has two connected components. Since H has five triangles, we get $h \leq 11$. Considering $h = 6, ..., 11$ and keeping the inequalities (2) and (3) we get all available candidates for H presented in Figure 3.

For each case of $c = 5, 4, 3, 2$ and 1 (if $H \not\simeq G$) each required 2-connected graph G is obtained from H by adding paths in such a way that exactly two vertices of each connected component of H are incident to an edge outside H. Looking at the graphs H and Lemma 2 we get all information on G presented in Table 1. The last column of Table 2 lists the graphs X_i that are χ-equivalent to respective graphs G. The column NB denotes a consecutive number of a graph H or H_1 for each respective group. This completes the proof. ■

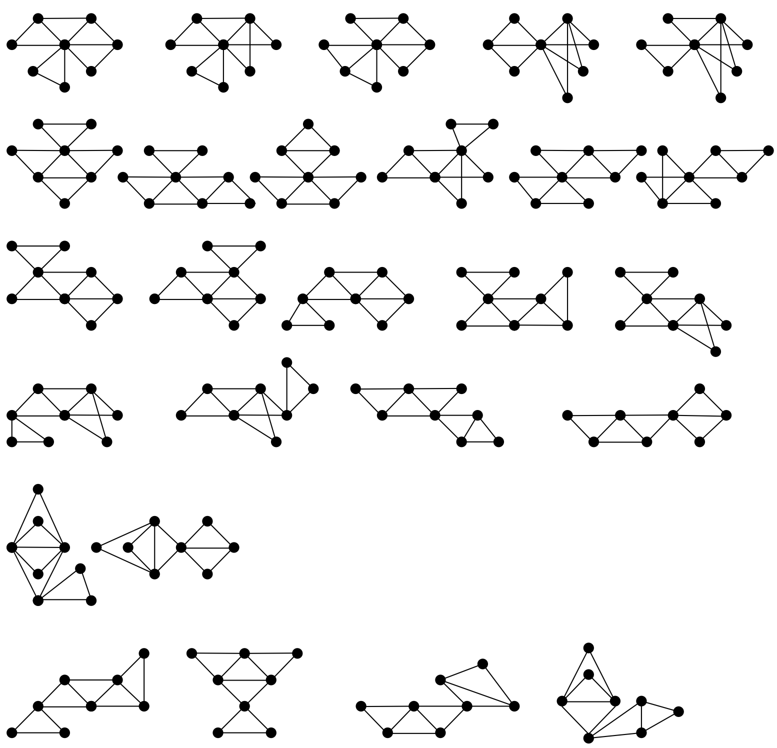
$c = 1,\ h = 6$



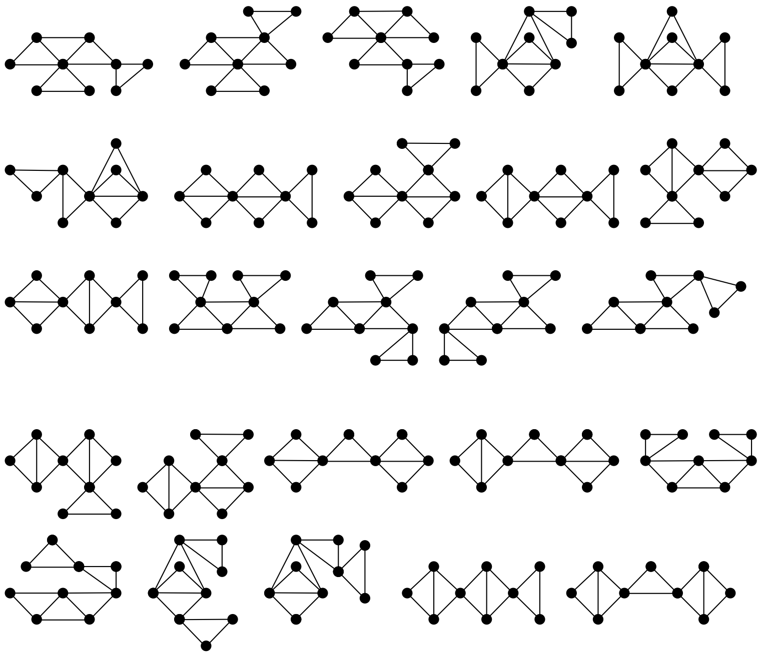
$c = 1,\ h = 7$



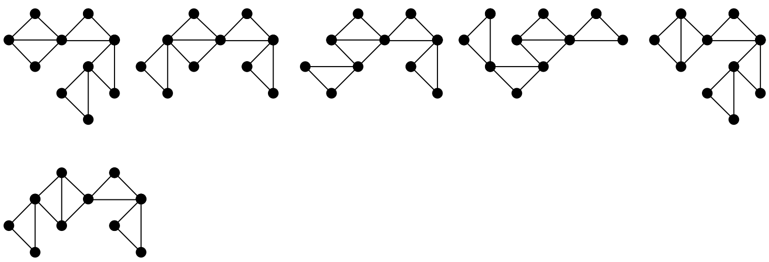
$c = 1,\ h = 8$



$c = 1,\ h = 9$



$h = 10$



$h = 11$

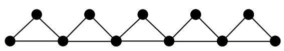


Figure 3

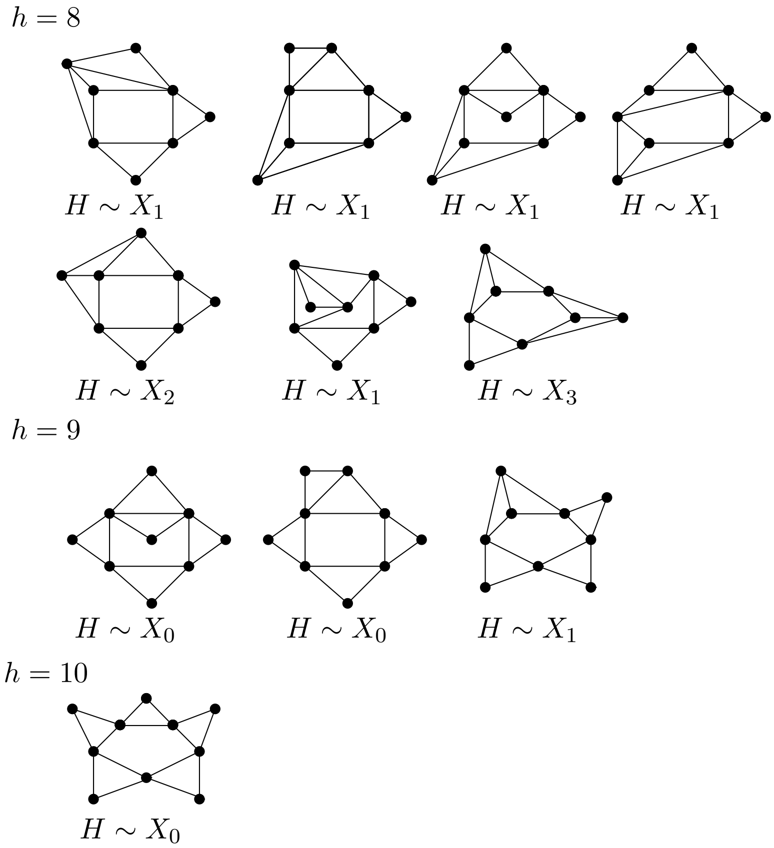


Figure 4

Immediately from the proof of Theorem 3 and Lemma 2 we get the following three results.

Corollary 4. *Each* $X_i(n)$ *for* $9 \leq i \leq 13$ *and* $n \geq 8$ *is a* χ*-unique graph of order* n.

The new chromatically unique graphs of order n are the graphs $X_i(n)$ for $i = 12, 13$ (see Figure 1) for each $n \geq 8$.

Corollary 5. *Each* χ*-equivalence class containing graphs* $X_{14}(n), X_{15}(n)$ *for* $n \geq 8$ *has exactly these two nonisomorphic elements.*

Corollary 6. *Each* χ*-equivalence class containing a graph* $X_7(n)$, $n \geq 7$, *consists of all graphs defined in the conjecture of Li and Whitehead* [5].

Table 2

$c=$	$h=$ or H_2	NB	$X_i,\ i=$
5	15	1	0
4	13	1	$0,1$
	14	2	0
3	11	1	$0,1$
	11	2	$0,1,2$
	12	3	$0,1$
	12	4	$0,1$
	13	5	0
	11	6	$0,1,3$
	12	7	$0,1$
	13	8	$0,1$
2	K_3	1	$5,6$
		2	$0,1,2,4$
		3	$0,1,2$
		4	$0,1,2$
		5	$0,1,2,8$
		6	0
		7	$0,1,2$
		8	$0,1$
		9	$0,1$
		10	0
		11	$0,1$
		12	$0,1$
		13	0
		14	$0,1,3$
		15	1
		16	0
		17	0
		18	0
		19	$0,1$
		20	0
	K_1+2K_2	1	0
		2	$0,1,2$
		3	$0,1$
		4	0
		5	0
	$2K_1+K_2$	1	$0,1$
		2	$0,1,2,3,7$
		3	$0,1,3$
		4	$0,3$
		5	$0,1$
1	6	1	$5,6,12,14$
		2	$5,6,13,14$
	7	1	$0,1,2,4,9$
		2	$0,1,2,4$
		3	$0,1,2,4$
		4	$0,1,2$
		5	$0,1,2$
		6	$0,1$
		7	5
		8	$5,6$
		9	$0,1,2,4,8$
		10	$0,1,2,4,8,11$
		11	$0,1,2$
		12	$0,1,2,8$
		13	$0,2,2,8$
		14	$0,1,2,8,10$

$c=$	$h=$ or H_2	NB	$X_i,\ i=$
1	8	1	0
		2	0
		3	0
		4	0
		5	0
		6	$0,1$
		7	$0,1$
		8	$0,1$
		9	$0,1$
		10	$0,1$
		11	$0,1$
		12	$0,1,2$
		13	$0,1$
		14	$0,1,2,4$
		15	$0,1,2$
		16	$0,1,2$
		17	$0,1,2$
		18	$0,1,2$
		19	$0,1,3$
		20	$0,1,2$
		21	$0,1$
		22	$0,1$
		23	$0,1,2,8$
		24	$0,1,2$
		25	$0,1,2,3,7$
		26	$0,1,3$
	9	1	0
		2	0
		3	0
		4	0
		5	0
		6	0
		7	0
		8	0
		9	$0,1$
		10	0
		11	1
		12	0
		13	0
		14	1
		15	$0,1$
		16	$0,1$
		17	$0,1$
		18	0
		19	0
		20	2
		21	2
		22	1
		23	$0,1$
		24	$0,1,3$
		25	$0,1,3$
	10	1	0
		2	0
		3	0
		4	0
		5	$0,1$
		6	1
	11	1	0

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