Discussiones Mathematicae
Graph Theory 18 (1998) 85–89

NEW CLASSES OF CRITICAL KERNEL-IMPERFECT DIGRAPHS

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Abstract

A kernel of a digraph D is a subset $N \subseteq V(D)$ which is both independent and absorbing. When every induced subdigraph of D has a kernel, the digraph D is said to be kernel-perfect. We say that D is a critical kernel-imperfect digraph if D does not have a kernel but every proper induced subdigraph of D does have at least one. Although many classes of critical kernel-imperfect-digraphs have been constructed, all of them are digraphs such that the block-cutpoint tree of its asymmetrical part is a path. The aim of the paper is to construct critical kernel-imperfect digraphs of a special structure, a general method is developed which permits to build critical kernel-imperfect-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, any directed cactus (an asymmetrical digraph all of whose blocks are directed cycles) whose blocks are directed cycles of length at least 5 is the asymmetrical part of some critical kernel-imperfect-digraph.

Keywords: digraphs, kernel, kernel-perfect, critical kernel-imperfect, block-cutpoint tree.

1991 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts we refer the reader to [1]. If D is a digraph, $V(D)$ and $F(D)$ denote the set of vertices and arcs of D respectively. An arc $u_1u_2 \in F(D)$ is *asymmetrical* (resp. symmetrical) if $u_2u_1 \notin F(D)$ (resp. $u_2u_1 \in F(D)$). The asymmetrical part of D (resp. *symmetrical* part of D) which is denoted by Asym D (resp. sym D) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. If D_1 and D_2 are two digraphs not necessarily disjoint, we will denote by $D_1 \cup D_2$

the digraph $D_1 \cup D_2$ whose arcs are $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$ and whose vertices are $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$. A *kernel* N of D is an independent set of vertices such that for every $z \in V(D) - N$ there exists a zN-arc in D. A digraph D is called kernel-perfect (or KP-digraph) if every induced subdigraph of D has a kernel, and critical kernel-imperfect digraph (CKI-digraph) if D has no kernel and every proper induced subdigraph has a kernel. We will say that a digraph D is complete if its underlying graph is a complete graph. Thus every complete subdigraph $\mathbb{C}$ of a kernel-perfect digraph must have an absorbing vertex (i.e., a successor of all other vertices of $\mathbb{C}$). A digraph D is called a normal orientation of its underlying graph G if every complete subdigraph of D has a kernel. A graph G is called solvable if every one of its normal orientations is a kernel-perfect digraph.

Many papers have recently appeared which are devoted to construct CKI-digraphs (see [2, 3, 4, 6, 7]); however, all of those constructions allow only digraphs whose asymmetrical part has a path as its block-cutpoint tree. In this paper, we develop a general method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, we prove that any directed cactus all of whose blocks are directed cycles of length at least five is the asymmetrical part of some CKI-digraph.

Define the digraph $C = \overrightarrow{C}_n(j_1, j_2, \ldots, j_k)$ by

$$\begin{aligned} V(C) &= \{0, 1, \ldots, n-1\} \\ F(C) &= \{uv \mid v - u \equiv js \ (mod\, n) \text{ for } s = 1, \ldots, k\} \end{aligned}$$

In particular, we have the digraph

$$D = \overrightarrow{C_n}\left(1, \pm 2, \pm 3, \ldots, \pm \left[\frac{n}{2}\right]\right)$$

defined by

$$V(D) = \{0, 1, \ldots, n-1\}, \quad F(D) = \{uv | v - u \not\equiv -1 \ (\text{mod } n)\}.$$

Let G be a connected graph. Denote by $V(G)$ the set of vertices of G and $E(G)$ the set of edges of G; $B(G)$ and $c(G)$ the set of blocks and cutpoints of G, respectively. The block-cutpoint tree $b_c(G)$ of G is defined by $V(b_c(G)) = B(G) \cup c(G)$ and $E(b_c(G)) = \{[u, x] \mid u \in B(G), x \in c(G)$ and $x \in u\}$. The ordered bipartition $(B(G), c(G))$ is determined since $b_c(G)$ is connected and $B(G)$ contains all endpoints of $b_c(G)$. If α is a block-cutpoint tree, we will denote by α_b and α_c the block-part and the cutpoint-part of

$V(\alpha)$, respectively. We recall that a tree α is a block cutpoint tree if and only if the distance of any two distint endpoints is even (see [8, p. 36]).

The following result was proved in [5, Theorems 3.2, 3.3 and 3.4]

Theorem 1.1. *Let D_1, D_2, H_1, H_2, D be digraphs, $v, u_i \in V(D_i)$, $i = 1, 2$ such that $u_i v \in F(sym\ (D_i))$, $V(D_1) \cap V(D_2) = \{v\}$, $H_i = D_i - u_i v - v u_i$ and $D = H_1 \cup H_2 + u_1 u_2 + u_2 u_1$. Suppose that H_1 and H_2 are KP-digraphs. Then*

(a) *D is a CKI-digraph iff D_1 and D_2 are CKI-digraphs. Moreover, if D is a CKI-digraph, then $D - u_1u_2 - u_2u_1$ is a KP-digraph.*

(b) *D is a KP-digraph iff at least one of D_1 and D_2 is a KP-digraph.*

Theorem 1.2 was proved in [5, Corollary 2.3]. Its second part is a direct consequence of the fact that for every CKI-digraph D, Asym D is strongly connected [5, Theorem 2.2].

Theorem 1.2. *$D = \overrightarrow{C_n}(1, \pm 2, \pm 3, \ldots, \pm[\frac{n}{2}])$ is a CKI-digraph for $n \geq 4$, and if $\emptyset \neq F_0 \subset F(Asym\ D)$, then $D - F_0$ is a KP-digraph.*

Lemma 1.1. *If $D = \overrightarrow{C_n}(1, \pm 2, \ldots, \pm[\frac{n}{2}])$ and, α is a subdigraph of sym D and α is a tree, then $D - F(\alpha)$ is a KP-digraph.*

We omit the proof of Lemma 1.1 since it is a direct consequence of the following result due to M. Blidia, P. Duchet and F. Maffray [2].

Theorem 1.3. *If the complement of the graph G is strongly perfect, then G is solvable.*

2. New Classes of Critical Kernel-Imperfect Digraphs

In this section, we develop a method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Also we construct CKI-digraphs whose asymmetrical part is a given directed cactus all of whose blocks are directed cycles of length at least five.

Theorem 2.1 *Let α be any block-cutpoint tree and $\mathcal{N} : \alpha_b \to \mathbb{N}$ a function satisfying $\mathcal{N}(u) \geq \max\{\delta_\alpha(u), 5\}$ ($\delta_\alpha(u)$ is the degree of u in α). Then there exists a CKI-digraph $D_{(\alpha,\mathcal{N})}$ and an isomorphism $h : V(\alpha) \to V(b_c(Asym\, D_{(\alpha,\mathcal{N})}))$ satisfying the following conditions:*

(i) *For each $u \in \alpha_b$, $h(u)$ is a directed cycle in $Asym\, D$ of length $\mathcal{N}(u)$.*

(ii) *$D - F_0$ is a KP-digraph for every non empty set $F_0 \subset F$ $(Asym\, D)$.*

(iii) *For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F$ (sym D) and $D - \{wz, zw\}$ is a KP-digraph.*

Proof. We proceed by induction on $n = |V(\alpha)|$. First let $n = 1$, $\mathcal{N}(u) = k \geq 5$. Take $D = D_{(\alpha,\mathcal{N})} = \overrightarrow{C}_k(1, \pm 2, \ldots, \pm[\frac{k}{2}])$. By Theorem 1.2 and Lemma 1.1 D satisfies conditions (ii) and (iii), condition (i) is trivial. Suppose the assertion of Theorem 2.1 holds for $n < s$ and consider any block-cutpoint tree α with $n = s$. Let u be any endpoint of α (therefore $u \in \alpha_b$) and call c the neighbour of u ($c \in \alpha_c$). Two cases are possible:

Case 1. If $\delta_\alpha(c) = 2$, call $u' \in \alpha_b$ the neighbour of c different from u. Notice that $\alpha - c$ decomposes into two connected components one (say α') containing u' and the other consisting of the single point u.

Case 2. If $\delta_\alpha(c) > 2$, take $\alpha' = \alpha - u$. In both cases α' is a block-cutpoint tree. Take the restriction $\mathcal{N}' = \mathcal{N}/\alpha'_b$. Then we obtain by induction a CKI-digraph $D' = D_{(\alpha',\mathcal{N}')}$ and an isomorphism $h' : V(\alpha') \to V(b_c \text{ Asym}\, D')$ satisfying (i), (ii) and (iii). Choose $t \in V(D')$ as follows: In Case 1, $h'(u')$ induces a directed cycle $\overrightarrow{\gamma}'_u$ in Asym D' of length $\mathcal{N}'(u') = \mathcal{N}(u') \geq \delta_\alpha(u') = \delta_{\alpha'}(u') + 1$.

Therefore $\overrightarrow{\gamma}_{u'}$ contains some vertex t which is not a cutpoint of D'.

In Case 2, let $t = h'(c)$ be any cutpoint of D' which corresponds to c. Take any isomorphic copy D'' of $\overrightarrow{C}_{\mathcal{N}(u)}$ $(1, \pm 2, \ldots, \pm[\frac{1}{2}\mathcal{N}(u)])$ such that $V(D'') \cap V(D') = \{t\}$ and choose $t_1 \in V(D')$ such that $tt_1 \in F(\text{sym } D')$ and $D' - \{tt_1, t_1t\}$ is a KP-digraph (condition (iii)). Choose also $t_2 \in V(D'')$ such that $tt_2 \in F(\text{sym } D'')$ and $D'' - \{tt_2, t_2t\}$ is a KP-digraph (Theorem 1.2). By Theorem 1.1 (a) $D = (D' - \{tt_1, t_1t\}) \cup (D'' - \{tt_2, t_2t\}) + t_1t_2 + t_2t_1$ is a CKI-digraph. Extend h' to $h : V(\alpha) \to V(b_c \text{ (Asym}\, D))$ in an obvious way. Condition (i) is obvious, condition (ii) follows from Theorem 1.1 (b), and by the fact that for every CKI-digraph D, Asym D is strongly connected [5, Theorem 7.2]. Finally, condition (iii) follows from the induction hypothesis and Theorem 1.1 (b) in case $z \neq t_1, t_2, t$; in case $z = t_1, t_2$, from the fact that $D - \{t_1t_2, t_2t_1\}$ is a KP-digraph (Theorem 1.1 (a)) and, in case $z = t$, by taking $w \in V(D'')$ $w \neq t_2$, such that $tw \in F(\text{sym } D'')$. By Lemma 1.1, $D'' - \{tt_2, t_2t\}$, $D'' - \{tw, wt\}$ and $D'' - \{tt_2, t_2t, tw, wt\}$ are KP-digraphs. Applying Theorem 1.1 (b), $D - \{tw, wt\}$ is a KP-digraph and the proof is complete. ■

Theorem 2.2. *Let H be an asymmetrical digraph each one of whose blocks is a directed cycle of length at least five. Then there exists a critical kernel-imperfect-digraph D satisfying the following properties:*

(i) *$Asym\,D$ is isomorphic to H.*

(ii) *$D - F_0$ is a KP-digraph for every non empty set $F_0 \subset F\ (Asym\ D)$.*

(iii) *For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F\ (sym\ D)$ and $D - \{wz, zw\}$ is a KP-digraph.*

The proof of Theorem 2.2 is similar to that of Theorem 2.1.

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Received 22 April 1997
Revised 3 November 1997