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MINIMAX THEOREMS WITHOUT CHANGELESS PROPORTION

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Abstract

The so-called minimax theorem means that if X and Y are two sets, and f and g are two real-valued functions defined on $X \times Y$, then under some conditions the following inequality holds:

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

We will extend the two functions version of minimax theorems without the usual condition: $f \leq g$. We replace it by a milder condition:

$$\sup_{x \in X} f(x,y) \leq \sup_{x \in X} g(x,y), \ \forall \ y \in Y.$$

However, we require some restrictions; such as, the functions f and g are *jointly upward*, and their upper sets are connected. On the other hand, by using some properties of multifunctions, we define X-*quasiconcave* sets, so that we can extend the two functions minimax theorem to the graph of the multifunction. In fact, we get the inequality:

$$\inf_{y \in T(X)} \sup_{x \in T^{-1}(y)} f(x,y) \leq \sup_{x \in X} \inf_{y \in T(x)} g(x,y),$$

where T is a multifunction from X to Y.

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1. INTRODUCTION AND PRELIMINARIES

In 1972, Terkelsen [8] gave a generalization of von Neumann's minimax theorem by mixing topological and algebraic conditions. As shown in [8], Terkelsen's minimax theorem is different from minimax theorems of Sion [9] and Ky Fan [3].

Let X and Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$. For $t \in (0,1)$, f is said to be *t-convex* on Y if for any y_1, y_2 in Y, there exists y_0 in Y such that for all x in X,

$$f(x,y_0) \le t \max\{f(x,y_1), f(x,y_2)\} + (1-t)\min\{f(x,y_1), f(x,y_2)\}.$$

Replacing the midpoint convexity in Terkelsen's minimax theorem [8] with *t-convexity*, a generalization of Terkelsen's result is given by Geraghty and Lin [4]. Simons [7] introduced the following upwardness concept which generalizes *t-convexity* and obtained a minimax theorem that includes the result of Geraghty-Lin [5].

A function $f : X \times Y \to \mathbb{R}$ is said to be *upward* on Y if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that for all x in X,

$$f(x,y_3) \le \max\{f(x,y_1), f(x,y_2)\},$$

and

$$|f(x,y_1) - f(x,y_2)| \ge \epsilon \Longrightarrow f(x,y_3) \le \max\{f(x,y_1), f(x,y_2)\} - \delta.$$

We shall say that f and g are *jointly upward* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that for all $x \in X$,

$$\max\{f(x,y_3), g(x,y_3)\} \le \max\{g(x,y_1), f(x,y_2)\},$$

and

$$\begin{aligned} |g(x,y_1) - f(x,y_2)| &\ge \epsilon \Longrightarrow \max\{f(x,y_3), g(x,y_3)\} \\ &\le \max\{g(x,y_1), f(x,y_2)\} - \delta. \end{aligned}$$

It is easy to see that when $f = g$ and the map $y \longrightarrow f(x,y)$ is convex, f and g are *jointly upward.* Lin and Yu [6] give a two functions version of Simon's minimax theorem with two jointly upward functions. In fact, they require the usual changeless proportion between two functions on the defining region:

$$f(x,y) \leq g(x,y), \ \forall \ (x,y) \in X \times Y.$$

In Section 2, we shall present a two functions version of minimax theorems without the above restriction and convexity; however, the results need merely some kind of connectedness. In Section 3, we shall restrict the feasible region to a multifunction, and define a class of X-*quasiconcave* sets as follows. Let $T : X \to Y$ be a multifunction. A set H_T is said to be X-*quasiconcave* to T if it consists of all the functions $g : X \times Y \to \mathbb{R}$ satisfying: for all $x_1, x_2 \in X$, there exist $x_3 \in X$ and $h \in H_T$ such that

$$h(x_3, y) \geq \max\{g(x_1,y), g(x_2,y)\}\,, \qquad \forall \ y \in T(X).$$

Clearly, any subset of $\{\ g : X \times Y \to \mathbb{R} : \ g(\cdot, y) \ is\ quasiconcave\ for\ each\ y\ \}$ is X-*quasiconcave.* In the sequel, we shall extend the two functions minimax theorem [6] to the graph of the multifunction under an X-*quasiconcave* property.

2. MINIMAX THEOREMS UNDER JOINTLY UPWARD PROPERTY

A lot of minimax theorems require the following changeless proportion between the two functions: $f(x,y) \leq g(x,y), \ \forall \ (x,y) \in X \times Y$. However, the condition is not necessary, in general. For example, let g, f be two real-valued functions defined on $[-1,1] \times [-1,1]$ by

$$g(x,y) = \begin{cases} (1-x)(1-y^2) & x \geq 0 \\ (1+x)(1-y^2) & x < 0 \end{cases}$$

and

$$f(x,y) = (1-x^2)(1-y^2).$$

Clearly, $g(x,y) \ngeq f(x,y)$. But we still have (see Example 2.3)

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = 0 \le 0 = \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

This motivates us to introduce the concept of *jointly upward functions.* Making use of the *jointly upward* property, we will give a two functions minimax theorem without the changeless proportion.

Theorem 2.1. *Let X be a nonempty compact subset of a topological space, and let Y be a nonempty set. Let f, g be two real-valued functions on $X\times Y$ satisfying the following properties:*

(0) $\sup_X f(x,y) \le \sup_X g(x,y)$, $\forall\ y\in Y$,

and for all $y_1, y_2 \in Y$

$$\sup_X \min\{g(x,y_1), f(x,y_2)\} \le \sup_X \min\{g(x,y_1), g(x,y_2)\};$$

(i) *$f(\cdot,y)$ and $g(\cdot,y)$ are upper semicontinuous on X for each $y\in Y$;*

(ii) *for any $y_1,\cdots,y_n \in Y$, and $\lambda\in\mathbb{R}$, the set $\bigcap_{i=1}^n\{x\in X; g(x,y_i)\ge\lambda\}$ is either connected or empty; and for each $y\in Y$, $\{x\in X; f(x,y)\ge\lambda\}$ is either connected or empty;*

(iii) *f and g are jointly upward.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

We need the following lemma.

Lemma 2.2. *Assume the conditions of Theorem 2.1. If for any $y_1, y_2\in Y$ and $\beta\in\mathbb{R}$ $\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta$, then there exists y_0 such that $\sup_X f(x,y_0) < \beta$.*

Proof. Choose $\epsilon > 0$ such that

$$\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta - 2\epsilon < \beta\,. \tag{1}$$

Suppose that $\sup_X f(x,y) \ge \beta$ for all $y\in Y$. Let $I(y) = \sup_{x\in X} f(x,y)$, $J(y) = \sup_{x\in X} g(x,y)$, $y\in Y$. Then $\inf_Y J(y) \ge \inf_Y I(y) \ge \beta$. Let

$$A_f(y) = \{x \in X; f(x,y) \geq \beta - 2\epsilon\},$$

$$A_g(y) = \{x \in X; g(x,y) \geq \beta - 2\epsilon\}.$$

Then $A_f(y)$ and $A_g(y)$ are nonempty closed subsets of X for all y in Y by condition (i). Choose $\delta > 0$ obtained by condition (iii). Next, we shall find $v_1 \in Y$ satisfying either (a) or (b):

(a) $A_g(v_1) \subset A_g(y_1)$ and for all $y \in Y$, $A_f(y) \subset A_g(v_1)$ implies $I(y) > J(v_1) - \delta$,

(b) $A_f(v_1) \subset A_g(y_1)$ and for all $y \in Y$, $A_g(y) \subset A_f(v_1)$ implies $J(y) > I(v_1) - \delta$.

Indeed, if for all y in Y,

$$A_f(y) \subset A_g(y_1) \Longrightarrow I(y) > J(y_1) - \delta,$$

then we take $v_1 = y_1$; otherwise, there exists $y^1 \in Y$ such that $A_f(y^1) \subset A_g(y_1)$ and $I(y^1) \leq J(y_1) - \delta$. If for all y in Y,

$$A_g(y) \subset A_f(y^1) \Longrightarrow J(y) > I(y^1) - \delta,$$

then we take $v_1 = y^1$; otherwise, there exists $y^2 \in Y$ such that $A_g(y^2) \subset A_f(y^1) \subset A_g(y_1)$ and $J(y^2) \leq I(y^1) - \delta \leq J(y_1) - 2\delta$.
Suppose we have chosen y^{n-1}. If n is odd, and for all y in Y,

$$A_f(y) \subset A_g(y^{n-1}) \Longrightarrow I(y) > J(y^{n-1}) - \delta,$$

then we take $v_1 = y^{n-1}$; otherwise, there exists $y^n \in Y$ such that $A_f(y^n) \subset A_g(y^{n-1})$ and $I(y^n) \leq J(y^{n-1}) - \delta \leq J(y_1) - n\delta$. On the other hand, if n is even, and for all y in Y,

$$A_g(y) \subset A_f(y^{n-1}) \Longrightarrow J(y) > I(y^{n-1}) - \delta,$$

then we take $v_1 = y^{n-1}$; otherwise, there exists $y^n \in Y$ such that $A_g(y^n) \subset A_f(y^{n-1})$ and $J(y^n) \leq I(y^{n-1}) - \delta \leq J(y_1) - n\delta$. Since $\inf_{y\in Y} J(y) > \inf_{y\in Y} I(y) > -\infty$, this process must stop at some m. Let $v_1 = y^m$. Thus, if m is even, then we have

$$A_g(v_1) \subset A_f(y^{m-1}) \subset A_g(y^{m-2}) \subset \cdots \subset A_g(y_1)$$

and for all $y \in Y$,

$$A_f(y) \subset A_g(v_1) \Longrightarrow I(y) > J(v_1) - \delta.$$

If m is odd, then we have

$$A_f(v_1) \subset A_g(y^{m-1}) \subset A_f(y^{m-2}) \subset \cdots \subset A_g(y_1)$$

and for all $y \in Y$,

$$A_g(y) \subset A_f(v_1) \Longrightarrow J(y) > I(v_1) - \delta.$$

(I) Under case (a), we first notice that there exists $v_2 \in Y$ such that $A_f(v_2) \subset A_f(y_2)$ and for all $y \in Y$, $A_f(y) \subset A_f(v_2)$ implies $I(y) > I(v_2) - \delta$. Also, by condition (iii), there exists an element y_3 in Y such that

$$f(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \leq \max\{g(x, v_1), f(x, v_2)\}, \tag{2}$$

and for all $x \in X$,

$$\begin{aligned} |g(x, v_1) - f(x, v_2)| \ \geq\ & \epsilon \Longrightarrow f(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \\ \leq\ & \max\{g(x, v_1), f(x, v_2)\} - \delta. \end{aligned} \tag{3}$$

By (2),

$$A_f(y_3) \subset A_g(v_1) \cup A_f(v_2). \tag{4}$$

Next, we want to show that $A_f(y_3) \cap A_f(v_2) \neq \emptyset$ and $A_f(y_3) \cap A_g(v_1) \neq \emptyset$. Choose $x \in X$, with $f(x, y_3) > \beta - \epsilon$. Then $x \in A_f(y_3)$.

If $f(x, v_2) \geq \beta - 2\epsilon$ then $x \in A_f(v_2)$, so $A_f(y_3) \cap A_f(v_2) \neq \emptyset$. If $f(x, v_2) < \beta - 2\epsilon < \beta - \epsilon$, we must have $g(x, v_1) > \beta - \epsilon$ by (2).

Then $|g(x, v_1) - f(x, v_2)| > \epsilon$ by (3) and

$$f(x, y_3) \leq \max\{g(x, v_1), f(x, v_2)\} - \delta \leq g(x, v_1) - \delta \leq J(v_1) - \delta.$$

Since this inequality holds for all $x \in A \equiv \{x; f(x, y_3) > \beta - \epsilon\}$, we have

$$I(y_3) = \sup_{x \in X} f(x, y_3) = \sup_{x \in A} f(x, y_3) \leq J(v_1) - \delta.$$

Hence $A_f(y_3) \not\subseteq A_g(v_1)$ by the choice of v_1.

By (4),

$$A_f(y_3) \cap A_f(v_2) \neq \emptyset. \tag{5}$$

Similarly, we can show that

$$A_f(y_3) \cap A_g(v_1) \neq \emptyset. \tag{6}$$

Observe that $A_f(y_3)$ is nonempty and connected by condition (ii). Then from (4), (5) and (6), it follows that $A_g(v_1) \cap A_f(v_2) \neq \emptyset$, and hence $A_g(y_1) \cap A_f(y_2) \neq \emptyset$.

Let $x_0 \in A_g(y_1) \cap A_f(y_2)$. Thus, $\min\{g(x_0, y_1), f(x_0, y_2)\} \geq \beta - 2\epsilon$.

This contradicts

$$\sup_X \min\{g(x, y_1), f(x, y_2)\} \leq \sup_X \min\{g(x, y_1), g(x, y_2)\} < \beta - 2\epsilon.$$

Therefore, there exists y_0 in Y such that $\sup_X f(x, y_0) < \beta$.

(II) Under case (b), we notice that there exists $v_2 \in Y$ such that $A_g(v_2) \subset A_g(y_2)$ and for all $y \in Y$, $A_g(y) \subset A_g(v_2)$ implies $J(y) > J(v_2) - \delta$. Also, by condition (iii), there exists an element y_3 in Y such that

$$g(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \leq \max\{f(x, v_1), g(x, v_2)\}, \tag{7}$$

and for all $x \in X$

$$\begin{aligned} |f(x, v_1) - g(x, v_2)| &\geq \epsilon \Longrightarrow g(x, y_3) \leq \max\{f(x, y_3), g(x, y_3)\} \\ &\qquad \leq \max\{f(x, v_1), g(x, v_2)\} - \delta. \end{aligned} \tag{8}$$

By (7),

$$A_g(y_3) \subset A_f(v_1) \cup A_g(v_2). \tag{9}$$

Next, we want to show that $A_g(y_3) \cap A_g(v_2) \neq \emptyset$ and $A_g(y_3) \cap A_f(v_1) \neq \emptyset$. Choose $x \in X$, with $g(x, y_3) > \beta - \epsilon$. Then $x \in A_g(y_3)$.

If $g(x, v_2) \geq \beta - 2\epsilon$ then $x \in A_g(v_2)$, so $A_g(y_3) \cap A_g(v_2) \neq \emptyset$. If $g(x, v_2) < \beta - 2\epsilon < \beta - \epsilon$, we must have $f(x, v_1) > \beta - \epsilon$ by (7).

Then $|f(x, v_1) - g(x, v_2)| > \epsilon$ by (8) and

$$g(x, y_3) \leq \max\{f(x, v_1), g(x, v_2)\} - \delta \leq f(x, v_1) - \delta \leq I(v_1) - \delta.$$

Since this inequality holds for all $x \in B \equiv \{x; g(x, y_3) > \beta - \epsilon\}$, then

$$J(y_3) = \sup_{x \in X} g(x, y_3) = \sup_{x \in B} g(x, y_3) \leq I(v_1) - \delta.$$

Hence $A_g(y_3) \not\subseteq A_f(v_1)$ by the choice of v_1.

By (9),

$$A_g(y_3) \cap A_g(v_2) \neq \emptyset. \tag{10}$$

Similarly, we can show that

$$A_g(y_3) \cap A_f(v_1) \neq \emptyset. \tag{11}$$

By condition (ii), $A_g(y_3)$ is nonempty and connected. Then from (9), (10) and (11), it follows that $A_f(v_1) \cap A_g(v_2) \neq \emptyset$ and hence $A_g(y_1) \cap A_g(y_2) \neq \emptyset$. Let $x_0 \in A_g(y_1) \cap A_g(y_2)$. Thus,

$$\min\{g(x_0, y_1), g(x_0, y_2)\} \geq \beta - 2\epsilon.$$

This contradicts (1).
Therefore, there exists y_0 in Y such that $\sup_x f(x, y_0) < \beta$. ■

***Proof of Theorem* 2.1.** If $\sup_X \inf_Y g(x, y) = +\infty$, then the assertion clearly holds; so we assume that $\sup_X \inf_Y g(x, y) < +\infty$. Let α be a real number such that $\sup_X \inf_Y g(x, y) < \alpha$. Choose β such that

$$\sup_X \inf_Y g(x, y) < \beta < \alpha. \tag{12}$$

For each $y \in Y$, let $L_g(y) = \{x \in X; g(x,y) < \beta\}$. Then each $L_g(y)$ is open, and by (12),

$$X = \bigcup_{y \in Y} L_g(y).$$

Since X is compact, $X = \bigcup_{i=1}^{n} L_g(y_i)$ for some $y_1, \cdots, y_n \in Y$, and this implies that

$$\sup_X \min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta.$$

We want to show, by induction on n, that there exists $y_0 \in Y$ such that $\sup_X f(x,y_0) < \alpha$. For $n = 1$, it follows by condition (0), that

$$\sup_X f(x,y_0) \le \sup_X g(x,y_0) = \sup_X g(x,y_1) < \beta < \alpha.$$

For $n = 2$, since

$$\sup_X \min\{g(x,y_1), g(x,y_2)\} < \beta,$$

then by Lemma 2.2, there exists $y_0 \in Y$ such that $\sup_X f(x,y_0) < \alpha$.

For $n > 2$, let $A = \{x \in X;\ g(x,y_i) \ge \beta,\ i = 1, \ldots, n-1\}$. By upper semicontinuity of $g(\cdot, y_i)$ and by the compactness of X, the set A is compact in X. For $x \in A$, since $g(x,y_i) \ge \beta$ for $i = 1, \cdots, n-1$ and $\min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta$, we have

$$\min\{g(x,y_1), \cdots, g(x,y_n)\} = g(x,y_n).$$

Then

$$\begin{aligned} \sup_A g(x,y_n) &= \sup_A \min\{g(x,y_1), \cdots, g(x,y_n)\} \\ &\le \sup_X \min\{g(x,y_1), \cdots, g(x,y_n)\} < \beta. \end{aligned}$$

Notice that $\sup_X \min\{g(x,y_1), g(x,y_n)\} < \beta$. Thus, by applying Lemma 2.2 again, we obtain $y_0 \in Y$ such that $\sup_X f(x,y_0) < \alpha$. ■

Next, we give an example of minimax inequality without $f \le g$.

Example 2.3. Let g, f be two real-valued functions defined on $[-1,1]\times[-1,1]$ by

$$g(x,y)=\begin{cases}(1-x)(1-y^2) & x\geq 0\\ (1+x)(1-y^2) & x<0\end{cases}$$

$$f(x,y)=(1-x^2)(1-y^2).$$

We will check the conditions of Theorem 2.1:

(0) Since $\sup_X g(x,y)=1-y^2=\sup_X f(x,y)$, then

$$\sup_X \min\{g(x,y_1),f(x,y_2)\}=\min\{1-y_1^2,1-y_2^2\}=\sup_X \min\{g(x,y_1,g(x,y_2)\}.$$

(i) Since $f(\cdot,y)$ and $g(\cdot,y)$ are continuous on X for each $y\in Y$, we have $f(\cdot,y)$ and $g(\cdot,y)$ are upper semicontinuous on X for each $y\in Y$.

(ii) for any $y\in Y$

$$\{x\in X; g(x,y)\geq\lambda\}=\begin{cases}\left[\frac{\lambda}{1-y^2}-1,1-\frac{\lambda}{1-y^2}\right] & \text{if } \lambda\in[0,1-y^2]\\ \emptyset \quad \text{if } \lambda\in(-\infty,0)\cup(1-y^2,\infty)\,. & \end{cases}$$

Hence, for any $y_1,\cdots,y_n\in Y$, and $\hat{y}=\max\{|y_1|,|y_2|,\cdots,|y_n|\}$, the set

$$\bigcap_{i=1}^{n}\{x\in X; g(x,y_i)\geq\lambda\}=\begin{cases}\left[\frac{\lambda}{1-\hat{y}^2}-1,1-\frac{\lambda}{1-\hat{y}^2}\right] & \text{if } \lambda\in[0,1-\hat{y}^2]\\ \emptyset \quad \text{if } \lambda\in(-\infty,0)\cup(1-\hat{y}^2,\infty) & \end{cases}$$

is either connected or empty for any $\lambda\in\mathbb{R}$.

For all $y\in Y$,

$$\{x\in X; f(x,y)\geq\lambda\}=\begin{cases}\left[-\sqrt{1-\frac{\lambda}{1-y^2}},\sqrt{1-\frac{\lambda}{1-y^2}}\right] & \text{if } \lambda\in[0,1-y^2]\\ \emptyset \quad \text{if } \lambda\in(-\infty,0)\cup(1-y^2,\infty) & \end{cases}$$

is either connected or empty for any $\lambda\in\mathbb{R}$.

(iii) Clearly, $\max\{f(x,y), g(x,y)\} = f(x,y)$, $\forall (x,y) \in X \times Y$.

If $|g(x,y_1) - f(x,y_2)| \geq \epsilon$, we take $\delta = \epsilon$, $k = \min\{1 - y_1^2, 1 - y_2^2\}$ and $y_3 = \sqrt{1 - \frac{k^2}{3}}$. Then we have $f(x,y_3) = (1 - x^2)\frac{k^2}{3}$.

(a) If $g(x,y_1) \geq f(x,y_2) + \epsilon$, then

$$g(x,y_1) - \delta \geq f(x,y_2) = (1-x^2)(1-y_2^2) \geq (1-x^2)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3).$$

(b) If $f(x,y_2) \geq g(x,y_1) + \epsilon$, then

$$f(x,y_2) - \delta \geq g(x,y_1) = (1-x)(1-y_1^2) \geq (1-x)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3);$$

for $x \geq 0$ and

$$f(x,y_2) - \delta \geq g(x,y_1) = (1+x)(1-y_1^2) \geq (1+x)k \geq (1-x^2)\frac{k^2}{3} = f(x,y_3).$$

for $x < 0$.

That is, f and g are jointly upward. Thus, by Theorem 2.1, we have

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

In fact, $\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} g(x,y) = 0$.

Corollary 2.4 ([6])**.** *Let X be a nonempty compact topological space and let Y be a nonempty set. Let f and g be two real-valued functions on $X \times Y$ satisfying the following properties:*

(0) *$f(x,y) \leq g(x,y)$ for all $(x,y) \in X \times Y$;*

(i) *$f(\cdot,y)$ and $g(\cdot,y)$ are upper semicontinuous on X for each $y \in Y$;*

(ii) *for any $y_1, \cdots, y_n \in Y$, and $\lambda \in \mathbb{R}$, the set $\bigcap_{i=1}^n \{x \in X; g(x,y_i) \geq \lambda\}$ is either connected or empty;*

(iii) *f and g are jointly upward.*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

When $f = g$ are upward or t-convex, then f and g are jointly upward. Thus, we have

Corollary 2.5 ([4], [7]). *Let X be a nonempty compact topological space and let Y be a nonempty set. Let f be a real-valued function defined on $X \times Y$ such that $\inf_Y \sup_X f(x,y) > -\infty$ and satisfying the following properties:*

(i) *$f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$;*

(ii) *for any $y_1, \cdots, y_n \in Y$, and $\lambda \in \mathbb{R}$, the set $\bigcap_{i=1}^n \{x \in X; f(x, y_i) \geq \lambda\}$ is either connected or empty;*

(iii) *f is upward (or t-convex on Y for some $t \in (0,1)$).*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

3. Minimax theorems under the graph of a multifunction

In some feasible region, the two functions version of minimax theorems does not hold again. However, by restricting to a proper region, the minimax theorem is done. For instance, define the set $A = \{(x,y) \in [0,1] \times [0,1]; 0 \leq y \leq \frac{1}{2}x\} \backslash \{(0,0)\}$, and let g be a real-valued function on $[0,1] \times [0,1]$, defined by

$$g(x,y) = \begin{cases} 0 & \text{if} \quad (x,y) \in A \cup \{(0,1)\} \\ 1 & \text{otherwise}. \end{cases}$$

It is easy to see that

$$\inf_{y \in Y} \sup_{x \in X} g(x,y) = 1 > 0 = \sup_{x \in X} \inf_{y \in Y} g(x,y),$$

where $X = Y = [0,1]$. If T is a multifunction on $[0,1]$, defined by

$$T(x) = \begin{cases} \left[x, x + \frac{1}{2}\right] & \text{if} \quad x < \frac{1}{2} \\ [x, 1] & \text{if} \quad x \geq \frac{1}{2} \end{cases},$$

then we have (see Example 3.4)

$$\inf_{y \in Y} \sup_{x \in T^{-1}(y)} g(x,y) = 1 = \sup_{x \in X} \inf_{y \in T(x)} g(x,y).$$

This motivates us to define an *X-quasiconcave* set.

Theorem 3.1. *Let X be a nonempty compact convex set of a Hausdorff topological vector space, and let Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, H_T an X-quasiconcave set of T, and let f and g be real-valued functions on $X \times Y$ satisfying the following properties:*

(0) $\sup_X f(x,y) \leq \sup_X g(x,y)$ *for all* $y \in Y$ *and* $g \in H_T$*;*

(i) T *is upper semicontinuous on* X*;*

(ii) *for each* $x \in X$, $y \in T(X)$ *and* $g \in H_T$, $g(x,\cdot)$ *is lower semicontinous on* $T(X)$ *and* $g(\cdot,y)$ *is quasiconcave on* X*.*

Then for any $\lambda \in \mathbb{R}$, the following alternative holds:

Either

(A) $\sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y) \geq \lambda,$

or

(B) *there exists* $y_0 \in T(X)$ *such that* $f(x,y_0) \leq \lambda$ *for all* $x \in T^{-1}(y_0)$*.*

Proof. Fix $\lambda \in \mathbb{R}$. We define

$$U_g(y) = \{x \in X; g(x,y) > \lambda\},$$

and

$$V_g(x) = \{y \in Y; g(x,y) > \lambda\}.$$

Assume that (B) does not hold, i.e.,

$$\forall\, y \in T(X) \quad \exists\, x_0 \in T^{-1}(y) \quad f(x_0,y) > \lambda. \tag{13}$$

We show that (A) holds in this case. For $g \in H_T$, let $S_g : X \to X$ be defined as $S_g(x) = \bigcap_{y\in T(x)} U_g(y)$. We shall prove that some map S_g has a fixed point. First we show that

(α) There exists some $g \in H_T$ such that $S_g(x) \neq \emptyset, \quad \forall x \in X$.

Let

$$A = \{V_\varphi(z) \cap T(X); z \in X, \varphi \in H_T\}.$$

Obviously, the family A is a nonempty partially ordered set with respect to inclusion relation of subsets of Y. It is not difficult to check that any totally ordered subset of A has an upper bound. By Zorn's Lemma, there exists a maximal element $V_g(\hat{x}) \cap T(X)$ of A, i.e., $\hat{x} \in X$ and $g \in H_T$ satisfying that for any $z \in X$, $\varphi \in H_T$

$$\begin{array}{ll} \text{if} & V_g(\hat{x}) \cap T(X) \subset V_\varphi(z) \cap T(X) \\ \text{then} & V_\varphi(z) \cap T(X) = V_g(\hat{x}) \cap T(X). \end{array} \tag{14}$$

Suppose $V_g(\hat{x}) \cap T(X) \neq T(X)$.

Let $y \in T(X) \setminus V_g(\hat{x})$, by (13) we have some $x_0 \in T^{-1}(y)$ such that $f(x_0, y) > \lambda$.

By (0), we deduce

$$\sup_X g(x, y) \geq \sup_X f(x, y) \geq f(x_0, y) > \lambda.$$

This implies that there exists some $x_1 \in X$ such that $g(x_1, y) > \lambda$.

Since H_T is X-quasiconcave, for g, $\hat{x}$ and x_1 there exists $h \in H_T$, $x_3 \in X$ such that

$$h(x_3, y) \geq \max\{g(\hat{x}, y), g(x_1, y)\} > \lambda, \quad \forall\, y \in T(X).$$

Thus, we have $y \in V_h(x_3)$ and hence $V_g(\hat{x}) \cap T(X) \subsetneq V_h(x_3) \cap T(X)$, which is a contradiction with (14).

This shows that $V_g(\hat{x}) \cap T(X) = T(X)$ and hence $T(X) \subset V_g(\hat{x})$. In other words, for all $y \in T(X)$, $g(\hat{x}, y) > \lambda$.

It follows that $\hat{x} \in \cap_{y \in T(X)} U_g(y) \subset S_g(x)$, $\forall\, x \in X$. Hence, $S_g(x) \neq \emptyset$, $\forall\, x \in X$.

(β) For each $x \in X$, $S_g(x)$ is convex.
Since $g(\cdot, y)$ is quasiconcave by (ii) for each $y \in T(X)$, $U_g(y)$ is convex, Hence $S_g(x)$ is also convex.

(γ) For each $z \in X$, $S_g^{-1}(z) = \{x \in X; g(z, y) > \lambda,\ \forall\, y \in T(x)\}$ is open.
Since $g(z, \cdot)$ is l.s.c, $V_g(z)$ is open. It follows from upper semicontinuity of T that

$$S_g^{-1}(z) = \{x \in X; T(x) \subset V_g(z)\} = T^+(V_g(z))$$

is open for each $z \in X$.

We already have some $g \in H_T$ satisfying (α), (β) and (γ). By Browder's Fixed Point Theorem [1], there exists some $x_1 \in X$ such that $x_1 \in S_g(x_1)$. That is,

$$g(x_1, y) > \lambda,\ \forall\, y \in T(x_1) \Longrightarrow \inf_{y \in T(x_1)} g(x_1, y) \geq \lambda \Longrightarrow \sup_{x \in X} \inf_{y \in T(x)} g(x, y) \geq \lambda.$$

We conclude that

$$\sup_{g \in H_T} \sup_{x \in X} \inf_{y \in T(x)} g(x, y) \geq \lambda. \qquad \blacksquare$$

Theorem 3.2. *Let X be a nonempty compact and convex subset of a Hausdorff topological vector space, and let Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, H_T an X-quasiconcave set for T, and let f and g be real-valued functions on $X \times Y$ satisfying the following properties:*

- (0) $\sup_X f(x, y) \leq \sup_X g(x, y)$ *for all* $y \in Y$ *and* $g \in H_T$*;*
- (i) *T is upper semicontinuous on X;*
- (ii) *for each $x \in X$, $y \in T(X)$ and $g \in H_T$, $g(x, \cdot)$ is lower semicontinous on $T(X)$ and $g(\cdot, y)$ is quasiconcave on X.*

Then

$$\inf_{y \in T(X)} \sup_{x \in T^{-1}(y)} f(x, y) \leq \sup_{g \in H_T} \sup_{x \in X} \inf_{y \in T(x)} g(x, y).$$

Proof. Let $\lambda \in \mathbb{R}$ be such that

$$\sup_{g \in H_T} \sup_{x \in X} \inf_{y \in T(x)} g(x, y) < \lambda.$$

Thus, by Theorem 3.1, there exists $y_0 \in T(X)$ such that $f(x, y_0) \leq \lambda$ for all $x \in T^{-1}(y_0)$. This implies that $\sup_{x \in T^{-1}(y_0)} f(x, y_0) \leq \lambda$ and hence $\inf_{y \in T(X)} \sup_{x \in T^{-1}(y)} f(x, y) \leq \lambda$. It follows that

$$\inf_{y \in T(X)} \sup_{x \in T^{-1}(y)} f(x, y) \leq \sup_{g \in H_T} \sup_{x \in X} \inf_{y \in T(x)} g(x, y). \qquad \blacksquare$$

Corollary 3.3 *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, and Y be a nonempty set. Let $T : X \to Y$ be a multifunction having nonempty images, and f, g be real-valued functions on $X \times Y$ satisfying the following properties:*

(0) $\sup_X f(x,y) \le \sup_X g(x,y)$ *for all* $y \in Y$*;*

(i) T *is upper semicontinuous on* X*;*

(ii) *for each* $x \in X$*,* $y \in T(X)$ *and* $g(x,\cdot)$ *is lower semicontinous on* $T(X)$ *and* $g(\cdot,y)$ *is quasiconcave on* X*.*

Then

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

Proof. Let $H_T = \{g\}$, then by (iii) H_T is X-quasiconcave to T. By Theorem 3.1 and the equality

$$\sup_{g\in H_T} \sup_{x\in X} \inf_{y\in T(x)} g(x,y) = \sup_{x\in X} \inf_{y\in T(x)} g(x,y),$$

we have

$$\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

■

Finally, we show by an example (where $f = g$) that $\inf_{y\in Y} \sup_{x\in X} f \le \sup_{x\in X} \inf_{y\in Y} g$ does not hold, but under some multifunction T, the inequality $\inf_{y\in T(X)} \sup_{x\in T^{-1}(y)} f \le \sup_{x\in X} \inf_{y\in T(x)} g$ holds.

Example 3.4 Let the set A be given by $A = \{(x,y) \in [0,1]\times[0,1]; 0 \le y \le \frac{1}{2}x\}\backslash\{(0,0)\}$, and let g be a real-valued function defined on $[0,1]\times[0,1]$ by

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A \cup \{(0,1)\} \\ 1 & \text{otherwise}. \end{cases}$$

If T is a multifunction on $[0,1]$ defined by

$$T(x) = \begin{cases} \left[x, x+\frac{1}{2}\right] & \text{if } x < \frac{1}{2} \\ [x,1] & \text{if } x \ge \frac{1}{2}, \end{cases}$$

then we have

$$\inf_{y\in Y} \sup_{x\in X} g(x,y) = 1 > 0 = \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

We will check the conditions of Corollary 3.3.

(i) Clearly, T is u.s.c on X.

(ii) Let $L^{\beta}(x) = \{y; g(x,y) \le \beta\}$.

If $\beta \ge 1$, then $L^{\beta}(x) = [0,1]$,

when $0 \le \beta < 1$, then

$$L^{\beta}(x) = \begin{cases} \left[0, \frac{1}{2}x\right], & \text{if } \ 0 < x \le 1 \\ \{(0,1)\}, & \text{if } \ x = 0\,. \end{cases}$$

If $\beta < 0$, then $L^{\beta}(x) = \emptyset$.

Hence, $g(x,\cdot)$ is lower semicontinuous on Y.

(iii) Let $U^{\beta}(y) = \{x; g(x,y) \ge \beta\}$.

If $\beta > 1$, then $U^{\beta}(y) = \emptyset$,

when $0 < \beta \le 1$, then

$$U^{\beta}(y) = \begin{cases} \{(0,0)\}, & \text{if } \ y = 0 \\ [0,2y), & \text{if } \ 0 < y \le \frac{1}{2} \\ [0,1], & \text{if } \ \frac{1}{2} < y < 1 \\ (0,1], & \text{if } \ y = 1\,. \end{cases}$$

If $\beta \le 0$, then $U^{\beta}(y) = [0,1]$.

Hence, $g(\cdot,y)$ is quasiconcave on X. By Corollary 3.3, we have

$$\inf_{y\in Y} \sup_{x\in T^{-1}(y)} g(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y).$$

In fact, $\inf_{y\in Y} \sup_{x\in T^{-1}(y)} g(x,y) \le \sup_{x\in X} \inf_{y\in T(x)} g(x,y) = 0$.

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