Discussiones Mathematicae
Differential Inclusions, Control and Optimization 22 (2002) 125–149

IMPULSIVE PERTURBATION OF C_0-SEMIGROUPS AND STOCHASTIC EVOLUTION INCLUSIONS

N.U. Ahmed

School of Information Technology and Engineering
and Department of Mathematics
University of Ottawa
Ottawa, Ontario, Canada

Abstract

In this paper, we consider a class of infinite dimensional stochastic impulsive evolution inclusions. We prove existence of solutions and study properties of the solution set. It is also indicated how these results can be used in the study of control systems driven by vector measures.

Keywords: impulsive perturbations, C_0-semigroups, stochastic systems, differential inclusions, vector measures, impulsive controls.

2000 Mathematics Subject Classification: 34G25, 34K45, 49J27, 49J24, 93E03, 93C25, 93E20.

1. Introduction

In this paper, we consider a class of stochastic impulsive systems where the principal operator is the generator of a C_0-semigroup which is impulsively perturbed multiplicatively. A nonlinear additive term is driven by a vector measure. The diffusion term is given by a multi valued map. Symbolically, the system is described by the integral inclusion

$$dx(t) - Ax(t)d\beta(t) - F(t,x)d\mu(t) \in C(t,x)dW, \; x(0) = \xi, \tag{1}$$

where W is a cylindrical Brownian motion taking values in a Hilbert space U. We show that under certain assumptions on the pair $(A, \beta(\cdot))$, the nonlinear map F and the vector measure μ and the multivalued operator C, the stochastic inclusion has solutions. In other words, the solution set is nonempty. We also present some topological properties of the solution set. In a recent paper [1], we have studied the following system of evolution inclusions in general Banach spaces.

$$dx - Axd\beta \in F(t,x)dt, t \geq 0, x(0) = \xi \tag{2}$$

$$dx - Axd\beta - B(t,x)dt \in C(t,x)d\mu, t \geq 0, x(0) = \xi, \tag{3}$$

where F and C are multivalued maps. There, in the context of general Banach spaces, we proved the existence and regularity properties of solutions for such systems under mild assumptions on the operators and multivalued maps. We shall freely use the basic results of that paper [1], in particular the results related to the transition operator corresponding to the generator $(A, \beta(\cdot))$.

Here we assume that the operator A is the infinitesimal generator of a C_0-semigroup in the Hilbert space H; and F and C are suitable nonlinear operators and β is generally a nonnegative nondecreasing (except for A generating groups) scalar valued function of bounded variation on bounded intervals of $R_0 \equiv [0, \infty)$ and μ is a suitable vector measure on the sigma algebra of Borel subsets $\mathcal{B}_0$ of R_0. Here μ may also induce impulsive perturbation of the system. Thus these models are much more general and cover all classical models of impulsive systems as widely used in the literature [3, 7, 10, 11, 13, 14]. In fact, it also covers the models used to develop control theory in recent years like [2–4, 6, 7]. The admissible controls considered in [6] are vector measures and so may also be impulsive.

Although deterministic impulsive systems have been widely considered in the literature as mentioned above, they do not cover the models given by (2), (3). Here we consider stochastic evolution inclusions of the form (1) which generalize the models (2–3). To the knowledge of the author, it appears from the literature that no such work, involving infinite dimensional impulsive stochastic systems of the form (1), has been considered in the past. Systems of the form (1) may arise in population biology (at the molecular or environmental level) where the population may consist of multiple species of biological agents which interact with each other and migrate from one region to another. Under normal conditions, the migration coefficient is constant which may undergo abrupt changes due to sudden intrusion of toxic chemicals or foreign bodies (biological agents) in the medium creating a shock.

Examples of impulsive systems can be found in many engineering applications such as optical communication, pulsed radars, spacecraft antennas etc., see [1].

The rest of the paper is organized as follows. In Section 2, basic notations are introduced. In Section 3, we present some results from [1] on the basic evolution operator associated with the pair $(A, \beta(\cdot))$ and its properties. This is used to construct solutions to nonhomogeneous Cauchy problems like

$$dx(t) = Ax(t)d\beta(t) + f(t), t \geq 0, x(0) = \xi. \tag{4}$$

In Section 4, we consider the questions of existence, uniqueness, and regularity properties of solutions to stochastic evolution equations associated with the evolution inclusions. In Section 5, stochastic differential inclusions are considered proving the existence of a nonempty set of solutions. In the final Section 6, we briefly discuss some problems in the area of optimal control.

2. Some notations and terminologies

For any topological space $\mathcal{Z}$, $2^{\mathcal{Z}} \setminus \emptyset$ will denote the class of all nonempty subsets of $\mathcal{Z}$, and $c(\mathcal{Z})(cb(\mathcal{Z}), cc(\mathcal{Z}), cbc(\mathcal{Z}), ck(\mathcal{Z}))$ denote the class of nonempty closed (closed bounded, closed convex, closed bounded convex, compact convex) subsets of $\mathcal{Z}$.

Let $(\Omega, \mathcal{B})$ be an arbitrary measurable space and $\mathcal{Z}$ a Polish space. A multifunction $G : \Omega \longrightarrow 2^{\mathcal{Z}} \setminus \emptyset$ is said to be measurable (weakly measurable) if for every closed (open) set $C \subset \mathcal{Z}$ the set

$$G^{-1}(C) \equiv \{\omega \in \Omega : G(\omega) \cap C \neq \emptyset\} \in \mathcal{B}.$$

Let d be any metric induced by the topology of the Polish space $\mathcal{Z}$. It is known that measurability of the multifunction G is equivalent to the measurability of the function $\omega \to d(x, G(\omega))$ for every $x \in \mathcal{Z}$. Even more, it is also equivalent to the graph measurability of G in the sense that

$$\{(x, \omega) \in \mathcal{Z} \times \Omega : x \in G(\omega)\} \in \mathcal{B} \times \mathcal{B}(\mathcal{Z})$$

where $\mathcal{B}(\mathcal{Z})$ denotes the sigma algebra of Borel sets of $\mathcal{Z}$. Let X, Y be any two topological spaces and $G : X \longrightarrow c(Y)$ be a multifunction. G is said to be upper semicontinuous (USC) if for each set $C \in c(Y)$

$$G^{-1}(C) \equiv \{x \in X : G(x) \cap C \neq \emptyset\} \in c(X).$$

If Y is a metric space with metric d, we can introduce a metric d_H on $c(Y)$, called the Hausdorff metric, as follows:

$$d_H(K, L) \equiv \max\{\sup\{d(k, L), k \in K\}, \sup\{d(K, \ell), \ell \in L\}\},$$

where $d(x, K) \equiv \inf\{d(x, y), y \in K\}$ is the distance of x from the set K. If Y is a complete metric space, then $(c(Y), d_H)$ is also a complete metric space.

Let E be a Banach space and let $\mathcal{M}_c(J, E)$ denote the space of bounded countably additive vector measures on the sigma algebra $\mathcal{B}$ of subsets of the set $J \subset R_0 \equiv [0, \infty)$ with values in the Banach space E, furnished with the strong total variation norm. That is, for each $\mu \in \mathcal{M}_c(J, E)$, we write

$$|\mu|_v \equiv |\mu|(J) \equiv \sup_{\pi}\Big\{\sum_{\sigma \in \pi} \parallel \mu(\sigma) \parallel_E\Big\},$$

where the supremum is taken over all partitions π of the interval J into a finite number of disjoint members of $\mathcal{B}$. With respect to this topology, $\mathcal{M}_c(J, E)$ is a Banach space. For any $\Gamma \in \mathcal{B}$ define the variation of μ on Γ by

$$V(\mu)(\Gamma) \equiv V(\mu, \Gamma) \equiv |\mu|(\Gamma).$$

Since μ is countably additive and bounded, this defines a countably additive bounded positive measure on $\mathcal{B}$. In case $E = R$, the real line, we have the space of real valued signed measures. We denote this by simply $\mathcal{M}_c(J)$ in place of $\mathcal{M}_c(J, R)$. Clearly for $\nu \in \mathcal{M}_c(J)$, $V(\nu)$ is also a countably additive bounded positive measure. For uniformity of notation, we use λ to denote the Lebesgue measure. For any Banach space X, we let X^* denote the dual. Strong convergence of a sequence $\{\xi_n\} \in X$ to an element $\xi \in X$ is denoted by $\xi_n \xrightarrow{s} \xi$ and its weak convergence by $\xi_n \xrightarrow{w} \xi$. For any pair of Banach spaces X, Y, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators from X to Y.

We use $PWC(J, X)$ to denote the class of all piecewise continuous bounded functions with values in the Banach space X, and $PWC_r(J, X)$ $(PWC_\ell(J, X))$ are those elements of $PWC(J, X)$ which are continuous from the right (left) having left hand (right hand) limits. The space of all bounded X valued functions, denoted by $B(J,X)$ and furnished with the sup norm topology,

$$\| z \|_0 \equiv \sup\{\| z(t) \|_X, t \in J\},$$

is a Banach space and $PWC(J,X) \subset B(J,X)$, though it may not be a closed subspace.

3. BASIC EVOLUTION OPERATOR

We start with the Cauchy problem

$$dx(t) = Ax(t)d\beta(t), t \geq 0, x(0) = \xi. \tag{5}$$

Let D denote the collection of ordered sequence of discrete points from R_0 given by

$$D \equiv \{0 = t_0 < t_1 < t_2, \cdots t_n < t_{n+1}, \cdots n \in N_0\}$$

and let S denote the step function

$$S(t) = \begin{cases} 1 & \text{if } \ t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume that A is the infinitesimal generator of a C_0-semigroup of contraction $T(t), t \geq 0$, in a Banach space E and that the function β is given by

$$\beta(t) \equiv t + \sum_{k \geq 0} \alpha_k S(t - t_k), t \geq 0, t_k \in D, \tag{6}$$

where generally $\alpha_k \in R \cup \{+/-)\infty\}$, with $\alpha_0 = 0$. Define the intervals $\sigma_k \equiv [t_k, t_{k+1}); k \in N_0$ and note that

$$R_0 = \bigcup_{k \geq 0} \sigma_k.$$

In a recent paper [1], it was shown that under some reasonable assumptions the pair $(A, \beta(\cdot))$ generates an evolution operator. The following result is fundamental and can be found in [1].

Lemma 3.1. *Consider the system* (5) *and suppose A is the infinitesimal generator of a C_0-semigroup of contractions $T(t), t \geq 0$ in the Banach space X and the function β is given by the expression* (6) *where the coefficients*

$\{\alpha_k\}$ are nonnegative with $\alpha_0 = 0$. Then the system (5) has a unique mild solution $x(t)$, $t \in R_0$, which is right continuous having left hand limits. That is $x \in PWC_r(I, X)$.

Proof. See [1].
The evolution operator corresponding to the pair (A, β) was given in [1]. This is reproduced below. For arbitrary $t \in R_0$, define the following integer valued function

$$i(t) \equiv k, \text{ for } t \in \sigma_k, k \in N_0.$$

Using this notation one can express the evolution operator corresponding to the pair (A, β) [1] as

$$U_\beta(t,r) \equiv \sum_{\ell=0}^{i(t)} \Big(\prod_{k=\ell+1}^{i(t)} (I - \alpha_k A)^{-1} \Big) \chi_{\sigma_\ell}(r) T(t-r), \tag{7}$$

for any $t \in R_0$ and $0 \leq r < t$, where χ_σ denotes the indicator function of the set σ. In other words, for each $\xi \in X$, equation (5) has a unique solution

$$x(t) = U_\beta(t,0)\xi, t \geq 0.$$

From the expression (7), it is clear that for $r = 0$ all the terms except the one with $\ell = 0$ vanish and hence

$$U_\beta(t,0) = \Big(\prod_{k=1}^{i(t)} (I - \alpha_k A)^{-1} \Big) T(t). \tag{8}$$

Hence the (mild) solution to any nonhomogeneous equation of the form

$$dx = Axd\beta + fdt, x(0) = \xi \tag{9}$$

is given by

$$x(t) = U_\beta(t,0)\xi + \int_0^t U_\beta(t,r) f(r) dr, \text{ for } t \in R_0. \tag{10}$$

and that $x \in PWC_r(I, X)$.

For more general results see [1].

Remark. Note that for any $s \in R_0$, the solution to the problem (9), with f locally Lebesgue-Bochner integrable satisfying $f(r) = 0$ for almost all $r \geq s$, is given by

$$x(t) = U_\beta(t,s)x(s+), t \geq s.$$

The following Lemma summarizes the basic properties of the evolution operator $U_\beta(t,s)$.

Corollary 3.2. *Under the assumptions of Lemma* 3.1, *the evolution operator U_β satisfies the following properties:*

(P1): $t \to U_\beta(t,r), t > r$ *is continuous from the right in the strong operator topology in X; that is,* $s - \lim_{t\downarrow r} U_\beta(t,r)\xi = \xi, \xi \in X$.

(P2): $s - \lim_{t\uparrow\tau>r} U_\beta(t,r)\xi$ *exists* $\forall\ \xi \in X$ *and* $\tau > r$.

(P3): $\| U_\beta(t,s)\xi \| \leq \| \xi \|,\ \forall\ \xi \in X,$ *and* $0 \leq s \leq t < \infty$.

(P4): $r \to g_t(r) \equiv U_\beta(t,r)\xi, 0 \leq r \leq t,$ *is piecewise continuous having simple discontinuities at* $r \in \{t_k, t_k < t, k \geq 1\}$.

(P5): $U_\beta(t,s)U_\beta(s,r) = U_\beta(t,r)\ \forall\ 0 \leq r < s < t < \infty$.

Proof. See [1].
The following result holds for groups.

Theorem 3.3. *Consider the system* (5) *and suppose A is the infinitesimal generator of a C_0-group of contractions $T(t), t \in R$, in the Banach space X and the function β is given by the expression* (6) *where the coefficients $\{\alpha_k\} \in R$ with $\alpha_0 = 0$. Then for each $\xi \in X$, the system* (5) *has a unique mild solution $x(t), t \in R$, which is right continuous having left hand limits. The pair (A, β) generates a non expansive evolution operator $\{V_\beta(t,r), r, t \in R\}$ on X. Further, if $f \in L_1^{\ell oc}(R,X)$, then the Cauchy problem* (9) *has a unique mild solution $x \in PWC^{\ell oc}(R,X)$ given by*

$$x(t) = V_\beta(t,s)\xi + \int_s^t V_\beta(t,r)f(r)dr,\ \ s \leq t,\ \ s,t \in R.$$

Proof. See [1].

4. STOCHASTIC EVOLUTION EQUATIONS

Let $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, t \geq 0, P)$ denote a complete filtered probability space with $\mathcal{F}_t, t \geq 0$, denoting an increasing family of complete sub sigma algebras of the sigma algebra $\mathcal{F}$. For any $\mathcal{F}$-measurable random variable z, we use the standard notation Ez to denote the integral of z with respect to the probability measure P, that is,

$$Ez = \int_{\Omega} z(\omega)P(d\omega).$$

Let U, H be any pair of separable Hilbert spaces and let $\mathcal{L}_{HS}(U,H)$ denote the space of Hilbert-Schmidt operators from U to H furnished with the scalar product and the associated norm

$$< B, C > \equiv Tr(BC^*), \text{ and } \parallel B \parallel_{HS} = \sqrt{Tr(BB^*)}$$

respectively. It is easy to show that $Y \equiv \mathcal{L}_{HS}(U,H)$ is a separable Hilbert space. Assuming that Y is furnished with its topological Borel field $\mathcal{B}(Y)$, we have a measurable space $(Y, \mathcal{B}(Y))$. We consider random variables $\{\sigma\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values from $Y = \mathcal{L}_{HS}(U,H)$ In fact, we are more interested in stochastic processes taking values from the separable Hilbert space Y.

All the random processes considered in this paper are assumed to be adapted to the filtration $\mathcal{F}_t, t \geq 0$. Let $J \equiv [0, a]$ denote a finite interval and $\mathcal{P}$ the σ-algebra of progressively measurable subsets of the set $J \times \Omega$. Let $L_2(\mathcal{P}, Y) \equiv L_2(\mathcal{P}, \mathcal{L}_{HS}(U,H))$ denote the class of progressively measurable random processes taking values from the space of Hilbert-Schmidt operators $Y \equiv \mathcal{L}_{HS}(U,H)$ with square integrable Hilbert-Schmidt norms. For convenience of presentation and to emphasize time, we denote this by

$$M_{2,2}(J, \mathcal{L}_{HS}(U,H)) \equiv L_2(\mathcal{P}, \mathcal{L}_{HS}(U,H))$$

or briefly as

$$M_{2,2}(J, Y) \equiv L_2(\mathcal{P}, Y).$$

Its topology is induced by the scalar product

$$< K, L > \equiv E \int_J Tr(K(t)L^*(t))dt,$$

where L^* denotes the adjoint of the operator L. Clearly, the norm is given by

$$\parallel K \parallel \equiv \left(\int_J E\{\parallel K(t) \parallel_{HS}^2\} dt \right)^{1/2}.$$

With respect to this norm topology, it is again a Hilbert space. We shall also use the notation $M_{\infty,2}(J,H)$ for the space $L_\infty(J,L_2(\Omega,H))$ of all progressively measurable random processes with values in H having essentially bounded second moments. This is furnished with the norm topology

$$\| x \| \equiv ess - sup\Big\{\sqrt{E \| x(t) \|_H^2}, t \in J\Big\}.$$

With respect to this topology $M_{\infty,2}(J,H)$ is a Banach space. For initial states we choose the Hilbert space $L_2(\mathcal{F}_0,H)$ and denote this by $M_2(H)$. Note that this consists of all H-valued $\mathcal{F}_0$-measurable random variables having finite second moments. Since $\mathcal{F}_0$ is complete, this is a closed subspace of the Hilbert space $L_2(\mathcal{F},H)$ and hence a Hilbert space. For the study of the differential inclusion (1), we need some basic results.

Before we can consider the Differential Inclusion (1), we consider the stochastic differential equation,

$$dx(t) = Ax(t)d\beta(t) + F(t,x)d\mu(t) + L(t)dW,\ x(0) = \xi, \tag{11}$$

where L is a suitable operator valued random process to be defined shortly. Let H, U be any pair of separable Hilbert spaces as introduced above and E an arbitrary Hilbert space. The pair $(A,\beta(\cdot))$ is as described in Section 3 and μ is any countably additive E valued vector measure of bounded total variation. The process $W \equiv \{W(t), t \geq 0\}$ with $P(W(0) = 0) = 1$, is a cylindrical Brownian motion with values in U. First, we shall prove the following result.

Theorem 4.1. *Suppose the pair (A,β) satisfies the assumptions of Lemma* 3.1 *with X replaced by H. Let $F : J \times H \longrightarrow \mathcal{L}(E,H)$ be measurable in t on J and Lipschitz in x on E and that there exists a $K \in L_2^+(J,|\mu|)$, so that the following growth and Lipschitz conditions hold*

$$\| F(t,x) \|_{\mathcal{L}(E,H)} \leq K(t)(1 + |x|_H) \tag{12}$$

$$\| F(t,x) - F(t,y) \|_{\mathcal{L}(E,H)} \leq K(t)\ |x - y|_H. \tag{13}$$

Then, for every $\xi \in M_2(H)$, and every $L \in M_{2,2}(J,\mathcal{L}_{HS}(U,H))$, independent of $\{\xi\}$, equation (11) *has a unique mild solution $x \in M_{\infty,2}(J,H)$.*

Proof. First, we must prove an apriori bound. Let $x \in M_{\infty,2}(J,H)$ be a mild solution, given that one exists. Then by the variation of parameters formula, x must be a solution to the integral equation

$$(14)\quad \begin{aligned} x(t) &= U_\beta(t,0)\xi + \int_0^t U_\beta(t,s)F(s,x(s))d\mu \\ &+ \int_0^t U_\beta(t,s)L(s)dW(s), t \in J. \end{aligned}$$

Define

$$(15)\quad z_L(t) \equiv \int_0^t U_\beta(t,s)L(s)dW(s), t \in J.$$

Now it follows from (14) that

$$(16)\quad |x(t)|_H^2 \le 3\Big\{|\xi|_H^2 + |z_L(t)|_H^2 + 2|\mu|(J)\int_0^t K^2(s)(1+|x(s)|_H^2)d|\mu|\Big\},$$

P-a.s for all $t \in J$. Clearly, taking expectation, it follows from this expression that

$$(17)\quad \begin{aligned} &(1+E|x(t)|_H^2) \\ &\le 3\left\{(1+E|\xi|_H^2) + E|z_L(t)|_H^2 + 2|\mu|(J)\int_0^t K^2(s)(1+E|x(s)|_H^2)d|\mu|\right\}. \end{aligned}$$

For convenience of notation, we define

$$(18)\quad \varphi(t) \equiv \left(1+E|x(t)|_H^2\right), c = 6|\mu|(J),$$

$$(19)\quad g(t) \equiv 3\left\{(1+E|\xi|_H^2) + E|z_L(t)|_H^2\right\}.$$

Substituting (18) and (19) into (17) we obtain

$$(20)\quad \varphi(t) \le g(t) + c\int_0^t K^2(s)\varphi(s)d|\mu|(s).$$

This is a generalized Gronwall type inequality. It follows from a result of the author [2, Lemma 5, p] that

$$(21)\quad \varphi(t) \le g(t) + \int_0^t c\ \exp\Big\{c\int_s^t K^2(r)d|\mu|(r)\Big\}g(s)K^2(s)d|\mu|(s),\ t \in J.$$

Define

$$\hat{g} \equiv ess - sup\{g(t), t \in J\}.$$

Note that this is finite. Indeed, since U_β is non-expansive, one can verify that

$$(22)\qquad \begin{aligned} E|z_L(t)|_H^2 &\leq \int_0^t E \parallel L(s) \parallel_{HS}^2 ds \\ &\leq \parallel L \parallel_{M_{2,2}(J,\mathcal{L}_{HS}(U,H))}^2 \quad \forall t \in J, \end{aligned}$$

and therefore $z_L \in M_{\infty,2}(J,H)$ and the essential bound of g follows. Using this, it follows from the inequality (21) that

$$(23)\qquad \begin{aligned} &\sup\{\varphi(t), t \in J\} \\ &\leq \hat{g}\Big\{1 + c\int_J |K(r)|^2 d|\mu|(r) \ \exp\Big\{c\int_J |K(r)|^2 d|\mu|(r)\Big\}\Big\}. \end{aligned}$$

Since by our basic assumption $K \in L_2^+(J,|\mu|)$, the right hand expression of the above inequality is finite. Hence, it follows from this inequality that there exists a positive constant $\tilde{c}$, independent of $\{\xi, L\}$, such that

$$(24)\qquad \begin{aligned} &\sup\{E|x(t)|_H^2,\ t \in J\} \\ &\leq \tilde{c}\Big\{1 + E|\xi|_H^2 + \int_J E \parallel L(r) \parallel_{HS}^2 dr\Big\} < \infty. \end{aligned}$$

Thus we have shown that if x is any solution to the equation (11), it has finite second moment and that it is bounded above by the expression (24) determined by the norms of the data $\{\xi, L\}$. More precisely, the map

$$(\xi, L) \longrightarrow x$$

is bounded from $M_2(H) \times M_{2,2}(J, \mathcal{L}_{HS}(U,H))$ to $M_{\infty,2}(J,H)$. For the existence, let $(\xi, L) \in M_2(H) \times M_{2,2}(J, \mathcal{L}_{HS}(U,H))$ be given. Define the operator G by

$$(25)\qquad Gx(t) \equiv U_\beta(t,0)\xi + z_L(t) + \int_0^t U_\beta(t,s)F(s,x(s))d\mu(s), t \in J.$$

We show that G maps $M_{\infty,2}(J,H)$ into itself. Define

$$\nu(\sigma) \equiv \int_\sigma |K(r)|^2 d|\mu|(r), \sigma \in \mathcal{B}_a,$$

where $\mathcal{B}_a$ denotes the sigma algebra of Borel subsets of the interval J. Since μ is a countably additive bounded vector measure with values in the Hilbert space E, it is clear that ν is a countably additive bounded positive measure. Carrying out similar computations as done above, it is easy to verify that there exists a positive constant c such that

$$\begin{aligned} &\| Gx \|_{M_{\infty,2}(J,H)} \\ (26) \quad &\leq c\left\{E|\xi|^2 + \sup\left\{E|z_l(t)|_H^2, t \in J\right\} + \nu(J)\left(1 + \| x \|^2_{M\infty,2(J,H)}\right)\right\}. \end{aligned}$$

This proves that G maps $M_{\infty,2}(J,H)$ into itself. We show that G has a unique fixed point in $M_{\infty,2}(J,H)$. For $x, y \in M_{\infty,2}(J,H)$ and $t \in J$, define

$$(27) \qquad d_t(x,y) \equiv ess - sup\left\{\sqrt{E|x(s) - y(s)|_H^2}, 0 \leq s \leq t\right\}$$

and set $d_a(x,y) = d(x,y)$. This defines a metric on $M_{\infty,2}(J,H)$ and since $M_{\infty,2}(J,H)$ is a Banach space, the space$M_{\infty,2}(J,H)$, furnished with this metric, is a complete metric space. Now using the expression (25), it is easy to verify that

$$(28) \qquad d_t^2(Gx, Gy) \leq |\mu|_v \int_0^t |K(s)|^2 d_s^2(x,y) d|\mu|(s), t \in J.$$

Define

$$(29) \qquad \gamma(t) \equiv \int_0^t d\nu(s) = \nu([0,t]) = \int_0^t |K(s)|^2 d|\mu|(s).$$

Since ν is a countably additive bounded positive measure on J, the function γ as defined is a monotone nondecreasing function of bounded total variation on J. Using this notation, we may rewrite (28) in the form,

$$(30) \qquad d_t^2(Gx, Gy) \leq |\mu|_v \int_0^t d_s^2(x,y) d\gamma(s), t \in J.$$

By repeated substitution of (30) into itself, one can verify that

$$(31) \qquad d_t^2(G^n x, G^n y) \leq ((|\mu|_v)^n \gamma(t)^n / \Gamma(n)) d_t^2(x,y), t \in J.$$

Since γ is a monotone nondecreasing function on J, it follows from this that

$$(32) \qquad d(G^n x, G^n y) \leq ((|\mu|_v)^{n/2} \gamma(a)^{n/2} / \sqrt{\Gamma(n)}) d(x,y).$$

It is clear from this expression that, for n sufficiently large, the n-th power (iterate) of the operator G is a contraction. Thus by Banach fixed point theorem G^n and hence G itself has a unique fixed point in the metric space $M_{\infty,2}(J,H)$. This completes the proof. ∎

Remark. It is clear that the solution to equation (11) is certainly not continuous. However, if all the jumps of β are zero and the vector measure μ is absolutely continuous with respect to Lebesgue measure, that is, E has RNP with respect to Lebesgue measure and that μ is λ continuous, then the solution to (11) has continuous trajectories almost surely. Since $L \in M_{2,2}(J,\mathcal{L}(U,H))$, this follows from the classical results of Da Prato and Zabczyk [5, Theorem 7.4].

Remark. If β is as given and the vector measure μ has RND (Radon-Nikodym derivative) with respect to Lebesgue measure, then the solution $x \in PWC(J,H)$ almost surely.

Corollary 4.2. *Suppose the the assumptions of Theorem* 4.1 *hold. Then the map*

$$(\xi, L) \longrightarrow x$$

is Lipschitz continuous from $M_2(H) \times M_{2,2}(J,\mathcal{L}_{HS}(U,H))$ *to* $M_{\infty,2}(J,H)$.

Proof. Let $\xi_1,\xi_2 \in M_2(H)$ and $L_1,L_2 \in M_{2,2}(J,\mathcal{L}_{HS}(U,H))$ and let $x_1,x_2 \in M_{\infty,2}(J,H)$ denote the unique solutions to (11) corresponding to the pairs $(\xi_1,L_1),(\xi_2,L_2)$ respectively. Clearly, the existence and uniqueness follows from Theorem 4.1 and both x_1 and x_2 must satisfy the integral equation (14). Define

$$e(t) \equiv E|x_1(t) - x_2(t)|_H^2 \tag{33}$$

$$h(t) \equiv E|\xi_1 - \xi_2|_H^2 + E\int_0^t \| L_1(s) - L_2(s) \|_{HS}^2 \, ds, t \in J. \tag{34}$$

Now using (14), it is easy to verify that there exists a positive constant c such that

$$e(t) \le c\left\{h(t) + |\mu|_v \int_0^t |K(s)|^2 e(s) d|\mu|(s)\right\}, \; t \in J. \tag{35}$$

Using again the generalized inequality as utilized in Theorem 4.1, and noting that h is a nondecreasing function of its argument and that $K \in L_2^+(J,|\mu|)$, we can show that

$$e(t) \leq ch(t)\Big\{1+\Big(\int_0^t c|\mu|_v |K(s)|^2 d|\mu|\Big)\exp\Big(\int_J c|\mu|_v |K(s)|^2 d|\mu|(s)\Big)\Big\}, \tag{36}$$

for $t \in J$.

Clearly, it follows from this inequality that there exists a constant $\tilde{c}$ dependent on $c, K, |\mu|$ but independent of $\{\xi_i, L_i, i = 1, 2\}$ such that

$$e(t) \leq \tilde{c}h(t), \forall t \in I. \tag{37}$$

Thus from (33), (34) and (37), we obtain

$$\begin{aligned} &ess - sup\{E|x_1(t) - x_2(t)|_H^2,\ t \in J\} \\ &\leq\ \tilde{c}\Big\{E|\xi_1 - \xi_2|_H^2 + E\int_J \parallel L_1(s) - L_2(s) \parallel_{HS}^2 ds\Big\}. \end{aligned} \tag{38}$$

This proves the Lipschitz continuity as stated in the corollary. ■

5. Stochastic differential inclusions

Often, deterministic systems governed by parabolic (or hyperbolic) variational inequalities, systems with uncertain parameters, systems with discontinuous vector fields, and control systems can be modeled as differential inclusions. The same remark applies to stochastic systems as well. Stochastic differential inclusions of the classical type, like

$$dx \in Axdt + F(t, x)dt + C(t, x)dW,$$

were studied by the author in [5] where the existence of solutions in an appropriate weak sense was established under different situations. For example, cases like F multivalued and C single valued, F single valued and C multivalued and both multivalued were considered under the assumptions that the multivalued maps are weakly inward and α-condensing where α denotes the Kuratowski's measure of non compactness. Also nonlinear systems with A monotone hemicontinuous with respect to the so called Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ were covered. Here we consider the Differential Inclusion (1) given by

$$dx \in Axd\beta + F(t, x)d\mu + C(t, x)dW. \tag{39}$$

Clearly this model is significantly different from the classical ones and in fact generalizes them.

For convenience of notation we shall use Y to denote the Hilbert space $\mathcal{L}_{HS}(U, H)$ with the scalar product as defined at the beginning of Section 4. Since we have assumed that both U and H are separable Hilbert spaces, it is clear that $Y \equiv \mathcal{L}_{HS}(U, H)$ is also a separable Hilbert space and so a Polish space or a complete separable metric space induced by the Hilbert-Schmidt norm. Note that according to our earlier notation, $L_2(\mathcal{P}, Y) = M_{2,2}(J, Y)$. Let $cb(Y)$ denote the class of nonempty closed bounded subsets of Y and d_H denote the Hausdorff metric on it. It is easy to verify that $cb(Y)$, furnished with this metric, is a complete separable metric space and hence a Polish space.

The multi-valued diffusion C is a map

$$C : J \times H \longrightarrow cb(Y). \tag{40}$$

We need the following notion of solution, for stochastic differential inclusions as introduced in [5]. By a solution, of course, we always mean a mild solution.

Definition 5.1. An element $x \in M_{\infty,2}(J, H)$ is a (mild) solution of the evolution inclusion (39), if there exists an $L \in M_{2,2}(J, Y)$ such that x is a (mild) solution to the evolution equation

$$dx = Axd\beta + F(t, x)d\mu + L(t)dW, t \geq 0, x(0) = \xi, \tag{41}$$

and

$$L(t) \in C(t, x(t)) \ a.e. \ P - a.s. \tag{42}$$

For fixed $\xi \in M_2(H)$, let N denote the solution map $L \longrightarrow x(L)(\cdot) \equiv N(L)(\cdot)$ corresponding to the system (41). It follows from Theorem 4.1, that equation (41) has a unique solution for each given $L \in M_{2,2}(J, Y)$. Thus it is evident from the definition that if the pair $\{L, x\}$ satisfies the relations (41) and (42), and N is the solution map as introduced above, L must satisfy the following inclusion relation

$$L(t) \in C(t, N(L)(t)), a.e. \ t \in I, P - a.s.$$

In other words, the question of existence of a solution to the stochastic evolution inclusion (39) is equivalent to the question of existence of a fixed point of the multivalued map $\hat{C}$ in the Hilbert space $L_2(\mathcal{P}, Y) \equiv M_{2,2}(J, H)$, where

$$\hat{C}(L) \equiv \{\Gamma \in L_2(\mathcal{P}, Y) : \Gamma(t) \in C(t, N(L)(t)) \ a.e., P - a.s.\} \tag{43}$$

Theorem 5.2. *Suppose* 0 *is not an atom of* μ, *the pair* (A, β) *satisfies the assumptions of Lemma* 3.1, F *satisfies the assumptions of Theorem* 4.1, *and the multifunction* C *satisfies the following assumptions:*

(C1): $C : J \times H \longrightarrow cb(Y)$, *measurable in* t *on* J *for each fixed* $x \in H$, *and, for almost all* $t \in J$, *it is (USC) upper semicontinuous on* H,

(C2): *there exists an* $\ell_0 \in L_2^+(J)$ *such that*

$$\inf\{\| L \|_Y, L \in C(t, e)\} \leq \ell_0(t)(1 + |e|_H), t \in J,$$

(C3): *there exists an* $\ell \in L_2^+(J)$ *such that*

$$d_H(C(t, x), C(t, y)) \leq \ell(t)|x - y|_H, \ \forall \ x, y \in H, t \in J.$$

Then for each $\xi \in M_2(H)$, *the evolution inclusion* (39) *has at least one solution* $x \in M_{\infty,2}(J, H)$.

Proof. For clarity, we let $\lambda \times P$ denote the product measure on the product sigma algebra $B_J \times \mathcal{F}$ with λ being the Lebesgue measure and P the probability measure. Let Π denote the restriction of the product measure on the sigma field $\mathcal{P}$ of progressively measurable subsets of the set $B_J \times \mathcal{F}$. It follows from Theorem 4.1 that, for each $L \in M_{2,2}(J, Y)$, the evolution equation,

$$dx = Axd\beta + F(t, x)d\mu + L(t)dW, x(0) = \xi, \tag{44}$$

has a unique (mild) solution $x_L(\cdot) \equiv x(L) \in M_{\infty,2}(J, H)$. Let N denote the map $L \longrightarrow x(L)$ from $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$. Since $x \in M_{\infty,2}(J, H)$, it is clearly measurable with respect to the sigma algebra of $\mathcal{F}_t$ progressively subsets $\mathcal{P}$ of the set $J \times \Omega$.

By assumption (C1), C is measurable in t on J for each $x \in H$ and it is usc on H for almost all $t \in J$. Therefore the multivalued process

$$t \longrightarrow C(t, x(t)) = C(t, N(L)(t))$$

is measurable with respect to the sigma algebra of progressively measurable sets $\mathcal{P}$, having nonempty closed values.

Hence by the well known Yankov-Von Neumann-Auman selection theorem, see [6, Theorem 2.14, p. 154], it has Π-measurable selections, in other words, progressively measurable selections. Since the multifunction as defined above is measurable, it is graph measurable. From the assumption (C2), it follows that the process q given by

$$q(t) \equiv \ell_0(t)(1 + |x(t)|_H)$$

belongs to $L_2(\mathcal{P}, R_0)$. Thus by virtue of assumption (C2), it follows from [6, Lemma 3.2, p. 175] that it has $L_2(\mathcal{P}, Y) = M_{2,2}(J, Y)$ selections. That is,

$$\begin{aligned} \hat{C}(L) \;&\equiv\; \{\Gamma \in L_2(\mathcal{P}, Y) : \Gamma(t) \in C(t, N(L)(t)) \; \Pi - a.e.\} \\ &\equiv\; \mathcal{S}^2_{C(\cdot, N(L)(\cdot))} \neq \emptyset. \end{aligned} \tag{45}$$

Since $C(t, e) \in cb(Y)$, $\hat{C}(L)$ is also closed and bounded in $L_2(\mathcal{P}, Y)$. In other words,

$$\hat{C} : M_{2,2}(J, Y) \longrightarrow cb(M_{2,2}(J, Y)).$$

We must show that $\hat{C}$ has a fixed point in $L_2(\mathcal{P}, Y) = M_{2,2}(J, Y)$. We prove this by showing that there is a Hausdorff metric on $cb(M_{2,2}(J, Y))$ with respect to which $\hat{C}$ is Lipschitz with Lipschitz constant less than one and then use the generalized Banach fixed point theorem for multivalued maps [15] to complete the proof.

Define

$$\gamma(t) \equiv \int_0^t \ell^2(s)ds \;\; t \in J. \tag{46}$$

Let $L_0(\mathcal{P}, Y)$ denote the space of progressively measurable random processes with values in Y and let X_0 denote the Hilbert space $M_{2,2}(J, Y) = L_2(\mathcal{P}, Y)$ and define, for $\delta > 0$, the normed vector space

$$X_\delta \equiv \left\{ \Gamma \in L_0(\mathcal{P}, Y) : \parallel \Gamma \parallel_\delta^2 \; \equiv E \int_J \parallel \Gamma(t) \parallel_Y^2 e^{-2\delta\gamma(t)} dt < \infty \right\}. \tag{47}$$

In fact, this is a scalar product space with the scalar product given by

$$(K, L)_\delta \equiv E \int_J Tr(K(t)L^*(t)) e^{-2\delta\gamma(t)} dt.$$

With respect to the norm topology, as defined above, X_δ is also a Hilbert space and one can easily verify that the two spaces X_0 and X_δ are topologically equivalent. This follows from the simple fact that the two norms are equivalent, that is, there exists a positive constant c such that

$$\| K \|_{X_\delta} \leq \| K \|_{X_0} \leq c \| K \|_{X_\delta}$$

for any $\delta > 0$.

Let $L_1, L_2 \in X_0$ with the corresponding solutions to equation (41) given by $x_1 = x(L_1) = N(L_1)$ and $x_2 = x(L_2) = N(L_2)$ respectively. It follows from Corollary 4.2, expression (37), that

$$E|x_1(t) - x_2(t)|_H^2 \leq \hat{c}h(t), t \in J, \tag{48}$$

where now, unlike in (37), h is given by

$$h(t) \equiv E \int_0^t \| L_1(s) - L_2(s) \|_Y^2 \, ds, t \in J. \tag{49}$$

Let $\varepsilon > 0$ and $\Gamma_1 \in \hat{C}(L_1)$, that is, $\Gamma_1(t) \in C(t, x_1(t))$ $\Pi - a.e.$ Since the multifunction C is Lipschitz in the second argument with respect to the Hausdorff metric d_H on $cb(Y)$, there exists a $\Gamma_2 \in \hat{C}(L_2)$, that is, $\Gamma_2(t) \in C(t, x_2(t))$ $\Pi - a.e.$ such that

$$\begin{aligned} &E \| \Gamma_1(t) - \Gamma_2(t) \|_Y^2 \leq \ell^2(t) E\{|x_1(t) - x_2(t)|_H^2\} + \varepsilon \\ &\leq \tilde{c}\, \ell^2(t) h(t) + \varepsilon, \ t \in J. \end{aligned} \tag{50}$$

Multiplying (50) by $\exp\{-2\delta\gamma(t)\}$ and integrating with respect to the Lebesgue measure and then using integration by parts we find that

$$\begin{aligned} &\int_J E\left\{\| \Gamma_1(t) - \Gamma_2(t) \|_Y^2\} \exp\{-2\delta\gamma(t)\right\} dt \\ &\leq (\tilde{c}/2\delta) \int_J E\left\{\| L_1(t) - L_2(t) \|_Y^2\right\} \exp\{-2\delta\gamma(t)\} dt + \varepsilon\lambda(J). \end{aligned} \tag{51}$$

Denoting the Hausdorff metric on $cb(X_\delta)$ by $D_{H,\delta}$, it follows from (51) that

$$D_{H,\delta}(\hat{C}(L_1), \hat{C}(L_2)) \leq \sqrt{(\tilde{c}/2\delta)} \| L_1 - L_2 \|_\delta + \sqrt{\varepsilon\lambda(J)}. \tag{52}$$

Since $\varepsilon > 0$ is arbitrary and J is a finite interval it follows from the above expression that, for $\delta > (\tilde{c}/2)$, the multifunction $\hat{C}$ is a contraction on X_δ

and hence by Banach fixed point theorem [Zeidler, 15, Theorem 9A, p. 449] it has at least one fixed point $\Gamma^o \in X_\delta$. Since $X_\delta \cong X_0$, this is also a fixed point of $\hat{C}$ in X_0. Hence $x^o \equiv N(\Gamma^o)$ is a (mild) solution of the evolution inclusion (39). This completes the proof. ∎

Remark. It is the author's conjecture that the conclusion of theorem 5.2 may also hold under local Lipschitz condition for F.

Solution Set. In general, differential inclusions posses many solutions. Hence it is natural to consider the solution set. Consider the system (41) with given $\xi \in M_2(H)$. Let $Fix(\hat{C})$ denote the set of fixed points of the multifunction $\hat{C}$ mapping $L_2(\mathcal{P}, Y)$ to $2^{L_2(\mathcal{P},Y)}$. That is,

$$Fix(\hat{C}) \equiv \{L \in L_2(\mathcal{P}, Y) : L \in \hat{C}(L)\}. \tag{53}$$

It is clear from Theorem 5.2, that $Fix(\hat{C}) \neq \emptyset$. Let X_ξ denote the set of solutions of the evolution inclusion (41) corresponding to the initial state $\xi \in M_2(H)$. Define the linear operator $\mathcal{K}$ mapping $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$ by

$$\mathcal{K}(L)(t) \equiv \int_0^t U_\beta(t, s)L(s)dW(s), t \in J.$$

Clearly, this is a bounded linear operator from $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$ and for each $L \in M_{2,2}(J, Y)$ we have

$$E|\mathcal{K}(L)(t)|_H^2 = \int_0^t E \parallel U_\beta(t, s)L(s) \parallel_Y^2 ds, \quad t \in J.$$

It follows from this that $\mathcal{K}$ is an isometric map from $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$. Using the properties of the transition operator one can also verify that the operator $\mathcal{K}$ is also injective. The following result has important applications in optimal control.

Corollary 5.3. *Suppose the assumptions of Theorem* 5.2 *hold and that the linear operator* $\mathcal{K}$ *maps every closed subset of* $M_{2,2}(J, Y)$ *into a closed subset of* $M_{\infty,2}(J, H)$. *Then, the solution set* X_ξ *to the evolution inclusion* (41), *corresponding to a fixed initial state* $\xi \in M_2(H)$, *is a nonempty sequentially closed subset of* $M_{\infty,2}(J, H)$.

Proof. By definition

$$X_\xi = \{N(L) : L \in Fix(\hat{C})\}.$$

Let $\{x_n\} \in X_\xi$ and suppose $x_n \xrightarrow{s} x^*$ in $M_{\infty,2}(J,H)$. We show that $x^* \in X_\xi$. Define the operator Υ by

$$\Upsilon(x)(t) \equiv x(t) - U_\beta(t,0)x_0 - \int_0^t U_\beta(t,s)F(s,x(s))d\mu, \quad t \in J. \tag{54}$$

The reader can easily verify that the operator Υ maps $M_{\infty,2}(J,H)$ into itself and that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} & E\left\{|\Upsilon(x_n)(t) - \Upsilon(x^*)(t)|_H^2\right\} \\ & \leq c_1 \; ess-sup\left\{E|x_n(s) - x^*(s)|_H^2, 0 \leq s \leq t\right\}, \; t \in J. \end{aligned} \tag{55}$$

where

$$c_1 \equiv 4\left(1 + |\mu|_v \int_J |K(s)|^2 d|\mu|(s)\right).$$

This shows that

$$\Upsilon(x_n) \xrightarrow{s} \Upsilon(x^*)$$

in the Banach space $M_{\infty,2}(J,H)$. In other words, the operator Υ maps every strongly convergent sequence into a strongly convergent sequence of the Banach space $M_{\infty,2}(J.H)$. Since $x_n \in X_\xi$, it follows from the definition of solution (definition 5.1) that corresponding to the sequence $\{x_n\}$ there exists a sequence $\{L_n\} \in M_{2,2}(J,Y)$ such that

$$x_n = N(L_n) \;\text{ and }\; L_n \in Fix(\hat{C}) \;\; \forall n \in N_0.$$

Clearly, this is equivalent to

$$\Upsilon(x_n) = \mathcal{K}(L_n) \;\text{ and }\; L_n \in Fix(\hat{C}) \; \forall n \in N_0.$$

Since $\Upsilon(x_n) \xrightarrow{s} \Upsilon(x^*) \equiv z^*$, it is clear that $\mathcal{K}(L_n)$ must also converge strongly to the same element $z^* \in M_{\infty,2}(J,H)$. Since $L_n \in Fix(\hat{C})$ for each integer n, it is clear that

$$\mathcal{K}(L_n) \in \mathcal{K}(Fix(\hat{C})), \; \forall \; n \in N_0.$$

Assuming for the moment that $Fix(\hat{C})$ is a closed subset of $M_{2,2}(J,Y)$, it follows from our hypothesis that $\mathcal{K}(Fix(\hat{C}))$ is a closed subset of $M_{\infty,2}(J,H)$. Thus the limit z^* must belong to the set $\mathcal{K}(Fix(\hat{C}))$ and hence there exists an $L_0 \in Fix(\hat{C})$ such that $z^* = \mathcal{K}(L_0)$. This means that $\Upsilon(x^*) = \mathcal{K}(L_0)$ and hence $x^* = N(L_0)$ with $L_0 \in \hat{C}(L_0)$. Thus $x^* \in X_\xi$ and hence X_ξ

is sequentially closed. Thus we must prove that $Fix(\hat{C})$ is a sequentially closed subset of $M_{2,2}(J,Y)$. Let $L_n \in Fix(\hat{C})$ and suppose $L_n \stackrel{s}{\longrightarrow} L_0$ in $M_{2,2}(J,Y) = L_2(\mathcal{P},Y)$. We must show that $L_0 \in Fix(\hat{C})$. This will follow if the multifunction $\hat{C}$ is continuous in the Hausdorff metric and it has closed values, that is, $\hat{C}(L) \in c(L_2(\mathcal{P},Y))$. We have already seen in Theorem 5.2 that $\hat{C}$ is continuous in the Hausdorff metric on $L_2(\mathcal{P},Y)$. Clearly, this implies its upper semicontinuity on $L_2(\mathcal{P},Y)$. Hence for any $\varepsilon > 0$ there exists an integer $n(\varepsilon)$ such that $L_n \in \hat{C}^{\varepsilon}(L_0)$ for all $n \geq n(\varepsilon)$, where $\hat{C}^{\varepsilon}(L_0) = \hat{C}(L_0) + \varepsilon B_1(L_2(\mathcal{P},Y))$ is the ε neighborhood of $\hat{C}(L_0)$. Hence $L_0 \in \hat{C}^{\varepsilon}(L_0)$. Thus if $\hat{C}$ is closed valued, we will have $L_0 \in \hat{C}(L_0)$. In other words, $Fix(\hat{C})$ is a closed subset of $L_2(\mathcal{P},Y)$. Thus it suffices to verify that $\hat{C}$ has closed values. Let $L_0 \in L_2(\mathcal{P},Y)$ be arbitrary and suppose $\{K_n\} \in \hat{C}(L_0)$ and that $K_n \stackrel{s}{\longrightarrow} K_0$ in $L_2(\mathcal{P},Y)$. We show that $K_0 \in \hat{C}(L_0)$. Clearly, the strong convergence implies that there exists a subsequence of the sequence $\{K_n\}$, relabeled as the original sequence, such that

$$K_n(t) \stackrel{s}{\longrightarrow} K_0(t), \text{ in } Y, \ \Pi \ a.e.$$

Since $K_n(t) \in C(t, N(L_0)(t))$ Π *a.e.* and $C(t, N(L_0)(t)) \in cb(Y)$ Π *a.e.*, we have

$$K_0(t) \in C(t, N(L_0)(t)) \ \Pi \ a.e.$$

Hence $K_0 \in \hat{C}(L_0)$. This shows that $\hat{C}$ has closed values. This completes the proof. ■

Remark. The assumption that the linear operator $\mathcal{K}$ maps closed subsets of $M_{2,2}(J,Y)$ into closed subsets of $M_{\infty,2}(J,H)$ holds if the range of the operator $\mathcal{K}$ is closed.
Another assumption under which the Corollary is valid is as follows. The multifunction

$$\Gamma_t(s) \equiv \{U_\beta(t,s)L(s), L \in Fix(\hat{C})\}, \ s \leq t, \ t \in J$$

is $\mathcal{F}_t$ measurable with values from the class of nonempty closed subsets of $M_{2,2}(J_t,Y)$ where $J_t \equiv [0,t], t \in J$. In this case one invokes the theory of measurable selections.

Remark. Even though the solution set X_ξ is closed, as stated in Corollary 5.3, it may not be bounded. For boundedness we need an additional condition on the multifunction C. This is given in the following theorem.

Theorem 5.4. *Consider the system* (41) *and suppose the assumptions of Theorem* 5.2 *hold. Further suppose that* C *satisfies the following growth condition: there exists a* $\zeta \in L_2^+(J)$ *such that*

$$\sup\{\| L \|_Y : L \in C(t,x)\} \leq \zeta(t)(1 + |x|_H) \ \forall\, x \in H.$$

Then the solution set X_ξ to the evolution inclusion (41) is a closed bounded subset of $M_{\infty,2}(J,H)$.

Proof. The first part follows from Corollary 5.3. Boundedness follows from similar computations as those of the single valued case and so omitted.

6. Optimal control problem

The results of this paper can be easily extended to include the controlled system,

$$dx \in Axd\beta + F(t,x)d\mu + G(t,x)du + C(t,x)dW,\ x(0) = \xi, \tag{56}$$

where u belongs to a suitable class of vector measures representing controls. The operators $\{A, \beta, F, C\}$ are as in the previous sections. Let V be another Hilbert space, or, in general, a reflexive Banach space. The operator G is a single valued map mapping $J \times H$ to $\mathcal{L}(V,H)$ satisfying similar properties with respect to the vector measure $u \in \mathcal{M}_c(J,V)$ as those of F with respect to the vector measure μ.

We are interested in control problems for this system. As in classical stochastic control problems, it is natural to consider admissible controls to be only those which are non anticipative with respect to the filtration $\mathcal{F}_t, t \geq 0$, or simply $\mathcal{F}_t$ adapted. In general, there controls can be chosen from the class of progressively measurable stochastic processes. Since our controls are V-valued measures defined on $\mathcal{B}_J$ it is necessary to clarify what is meant by non anticipating. We assume that the Banach space V is furnished with its topological Borel field $\mathcal{B}_V$ so that $(V, \mathcal{B}_V)$ is a measurable space. We introduce the following definition.

Definition 6.1. A random vector valued measure u defined by the mapping

$$u : \mathcal{B}_J \times \Omega \longrightarrow V$$

is said to be $\mathcal{F}_t$-progressively measurable if for every $t \in J$ and every set $\varpi \in \mathcal{B}_{[0,t)}$, the random variable $\omega \longrightarrow u(\varpi, \omega) = u(\varpi)(\omega)$ is $\mathcal{F}_t$ measurable. That is,

$$\{\omega \in \Omega : u(\varpi)(\omega) \in \Gamma\} \in \mathcal{F}_t$$

for every $\Gamma \in \mathcal{B}_V$.

We denote this class of vector measures by $\mathcal{M}_0$. For $1 \leq p \leq \infty$, let $L_p(\Omega, \mathcal{M}_c(J, V))$ denote the Banach space of random vector measures with the norm topology given by

$$\| u \|_p \equiv (E|u|_v^p)^{1/p}$$

where $|u|_v$ denotes the total variation norm. The variation norm is given by

$$|u|_v \equiv |u|(J) \equiv \sup_{\pi}\Big\{\sum_{\sigma\in\pi} \| u(\sigma) \|_V\Big\} < \infty,$$

where the supremum is taken over all partitions π of the interval J into a finite number of disjoint members of $\mathcal{B}_J$. Let ν be a countably additive bounded positive measure on J. For admissible controls, one may choose the family

$$\begin{aligned} \mathcal{U}_{ad} \equiv \{u \in \mathcal{M}_0 \cap L_\infty(\Omega, \mathcal{M}_c(J, V)) : |u|(\sigma) \\ \leq \nu(\sigma)\ P - a.s,\ \forall\ \sigma \in \mathcal{B}_J\} \end{aligned} \tag{57}$$

The following result follows as corollary of Theorem 5.1.

Corollary 6.2. *Consider the system* (56) *with admissible controls* $\mathcal{U}_{ad}$ *as described above and suppose all the assumptions of Theorem* 5.1 *hold and there exists a* $g \in L_2^+(J, \nu)$ *such that*

$$\| G(t, x) \|_{\mathcal{L}(V,H)} \leq g(t)(1 + |x|_H) \tag{58}$$

$$\| G(t, x) - G(t, y) \|_{\mathcal{L}(V,H)} \leq g(t)\ |x - y|_H. \tag{59}$$

Then, for every $\xi \in M_2(H)$ and $u \in \mathcal{U}_{ad}$, the system has a nonempty set of solutions X_u and that it is a closed bounded subset of the Banach space $M_{\infty,2}(J, H)$.

For suitable f, φ and Ψ, the natural cost functional for a control problem may be given by,

$$J_0(u) \equiv \sup\Big\{E\Big(\int_J f(t, x(t))dt + \varphi(x(T)) + \Psi(|u|_v)\Big), x \in X_u\Big\}, \tag{60}$$

which is an appropriate measure of maximum risk or cost. The problem here is to find a control $u^o \in \mathcal{U}_{ad}$ that minimizes the maximum risk, that is,

$$J_0(u^o) \leq J_0(u) \ \forall \ u \in \mathcal{U}_{ad}.$$

In general, the function f is measurable in t on J and continuous in x on H satisfying

$$h_0(t) \leq f(t,x) \leq h(t)[1 + |x|_H^2], \tag{61}$$

for some $h_0 \in L_1(J)$ and $h \in L_1^+(J)$ and φ is required to satisfy

$$\alpha_0 \leq \varphi(x) \leq \alpha_1 + \alpha_2 |x|_H^2, \alpha_0, \alpha_1 \in R, \quad \alpha_2 \geq 0. \tag{62}$$

The function Ψ is a nonnegative and nondecreasing function defined on the positive half of the real line, $[0, \infty]$, signifying a measure of cost of control used. In view of the choice of the admissible controls $\mathcal{U}_{ad}$ and the assumption (61–62), the control problem (56), (60) is well defined.

The questions of existence of optimal controls and also necessary conditions of optimality are of fundamental importance in control theory. At this point of time these are open problems. However, for deterministic problems involving impulsive systems some results have been proved recently in [6].

There are other interesting and more difficult questions also. Is it possible to develop HJBI (Hamilton-Jacobi-Bellman-Issac) equation for the problem (56), (60) and construct feedback control laws?

Acknowledgment

This work was partially supported by the National Science and Engineering Research Council of Canada under grant no A7109.

REFERENCES

[1] N.U. Ahmed, *Impulsive Perturbation of C_0 semigroups and Evolution Inclusions*, Nonlinear Functional Analysis and Applications, (to appear).

[2] N.U. Ahmed, *Vector measures for optimal control of impulsive systems in Banach spaces*, Nonlinear Functional Analysis and Applications **5** (2) (2000), 95–106.

[3] N.U. Ahmed, *Some remarks on the dynamics of impulsive systems in Banach spaces*, Dynamics of Continuous, Discrete and Impulsive Systems **8** (2001), 261–274.

[4] N.U. Ahmed, *State dependent vector measures as feedback controls for impulsive systems in Banach spaces*, Dynamics of Continuous, Discrete and Impulsive Systems **8** (2001), 251–261.

[5] N.U. Ahmed, *Existence of solutions of nonlinear stochastic differential inclusions on Banach spaces*, Proc. World Congress of Nonlinear Analysis' 92, (ed: V. Lakshmikantham), (1992), 1699–1712.

[6] N.U. Ahmed, *Existence of optimal controls for a general class of impulsive systems on Banach spaces*, SIAM Journal Contr. and Optim. (to appear).

[7] N.U. Ahmed, *Measure solutions impulsive evolutions differential inclusions and optimal control*, Nonlinear Analysis **47** (2001), 13–23.

[8] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge, England 1992.

[9] S. Hu and N.S. Papageorgiou, Handbook of multivalued analysis, Kluwer Academic Publishers, Dordrecht, Boston, London 1997.

[10] V. Lakshmikantham, D.D. Bainov and P.S. Simenov, Theory of impulsive differential equations, World Scientific, Singapore, London 1999.

[11] J.H. Liu, *Nonlinear impulsive evolution equations, dynamics of continuous*, Discrete and Impulsive Systems **6** (1999), 77–85.

[12] J. Motyl, *On the solution of stochastic differential inclusions*, J. Math. Anal. and Appl. **192** (1995), 117–132.

[13] A.M. Samoilenk and N.A. Perestyuk, Impulsive differential equations, World Scientific, Singapore 1995.

[14] T. Yang, Impulsive control theory, Springer-Verlag, Berlin 2001.

[15] E. Zeidler, Nonlinear functional analysis and its applications, Vol. 1, Fixed Point Theorems, Springer-Verlag New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest.

Received 8 April 2002